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T-Equivalences: The Metric Behavior Revisited

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Abstract: Since the notion of T-equivalence, where T is a t-norm, was introduced as a fuzzy generalization of the notion of crisp equivalence relation, many researchers have worked in the study of the metric behavior of such fuzzy relations. Concretely, a few techniques to induce metrics from T-equivalences, and vice versa, have been developed. In several fields of computer science and artificial intelligence, a generalization of pseudo-metric, known as partial pseudo-metrics, have shown to be useful. Recently, Bukatin, Kopperman and Matthews have stated that the notion of partial pseudo-metric and a type of generalized T-equivalence are linked. Inspired by the preceding fact, in this paper, we state a concrete relationship between partial pseudo-metrics and the aforesaid generalized T-equivalences. Specifically, a method for constructing partial pseudo-metrics from the new type of T-equivalences and, reciprocally, for constructing the generalized T-equivalences from partial pseudo-metrics are provided. However, important differences between the new approach and the classical one are established. Special interest is paid to the case in which the minimum, drastic, and Łukasiewicz t-norms are under consideration.

Keywords: continuous t-norm; Archimedean t-norm; additive generator; T-equivalence; T-equality; partial pseudo-metric

1. Introduction

The notion of T-equivalence (or T-indistinguishability operator), where T is a t-norm, was introduced in fuzzy logic by Trillas with the aim of giving the minimal number of properties that a fuzzy relation should satisfy in order to be considered as a graded equivalence relation (for a detailed discussion of the topic, we refer the reader to [1,2]).

From now on, we make the assumption that the reader is familiar with the theory of triangular norms. One of the best bibliographical references on this topic is [3].

According to [2] (see also [1]), a T-equivalence on a (nonempty) set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$ the following axioms:

(E1) $E(x,x) = 1$,
(E2) $E(x,y) = E(y,x)$,
(E3) $T(E(x,y), E(y,z)) \le E(x,z)$.

Following [1], a T-equivalence E is said to be a T-equality when, for all $x, y \in X$, the following condition is fulfilled:

(E1') $E(x,y) = 1 \Leftrightarrow x = y$.

On account of [2] (see also [1]), given $x, y \in X$, the numerical value $E(x, y)$ can be understood as the degree of indistinguishability (or similarity) between x and y. Clearly, the condition $E(x, x) = 1$ expresses that an element x is completely indistinguishable from itself. Moreover, condition $E(x, y) = E(y, x)$ states that the degree of indistinguishability from x to y is exactly the same as the degree of indistinguishability from y to x. Furthermore, the third axiom expresses that x and z are indistinguishable whenever x and y are indistinguishable and, in addition, y and z are also indistinguishable. This last condition manifests the importance of T-equivalences. Indeed, the T-transitivity provides, by means of the degree of indistinguishability from x to y and from y to z, a threshold to the degree of indistinguishability from x to z. Thus, it allows us to introduce equivalence relations (note that, for each t-norm T, we obtain a new fuzzy equivalence relation) that avoid the Poincaré paradox which disclaims the transitivity in the physical universe (let us recall that Poincaré's paradox states that in the physical universe there are objects A, B, C such that A and B are indistinguishable, B and C are also indistinguishable, but A and C are totally distinguishable). Indeed, axiom $(E3)$ (above) states that x and z cannot be very distinguishable provided that x and y are indistinguishable and y and z are indistinguishable.

In the study of indistinguishability, the relationship between metrics (distinguishability relations) and T-equivalences have been tackled by many authors (see [1,3–11]).

With the aim of exposing the aforesaid relationship, let us recall the well-known notion of pseudo-metric. According to [12], a pseudo-metric on a (nonempty) set X is a function $d : X \times X \to [0, \infty[$ such that for all $x, y, z \in X$:

(d1) $d(x, x) = 0$,
(d2) $d(x, y) = d(y, x)$,
(d3) $d(x, z) \leq d(x, y) + d(y, z)$.

The notion of metric can be retrieved from the notion of pseudo-metric. Indeed, a metric d on X is a pseudo-metric which fulfills the next stronger version of axiom (d1) for all $x, y \in X$:

(d1') $d(x, y) = 0 \Leftrightarrow x = y$.

A (pseudo-)metric d is called a (pseudo-)ultrametric when it satisfies the following stronger version of axiom (d3):

(d3') $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Notice that actually the preceding introduced notions can take the value ∞ and, thus, they match up with extended (pseudo-)metrics and extended (pseudo-)ultrametric in the sense of [12].

In the aforementioned references, the following concrete correspondences between pseudo-metrics and T-equivalences were explored and stated.

Let us remember that given a t-norm T an additive generator $f_T : [0, 1] \to [0, \infty]$ of T is a strictly decreasing function which is also right-continuos at 0 and satisfies $f_T(1) = 0$, such that $f_T(x) + f_T(y) \in Ran(f_T) \cup [f_T(0), \infty]$, and $T(x, y) = f_T^{(-1)}(f_T(x) + f_T(y))$ for all $x, y \in [0, 1]$, where $f^{(-1)}$ denotes the pseudo-inverse of f, i.e., $f^{(-1)}(y) = \sup\{x \in [0, 1] : f(x) > y\}$. Observe that, if the additive generator is continuous, then $f_T^{(-1)}(x) = f^{-1}(\min\{f_T(0), x\})$ for all $x \in [0, \infty]$, where f^{-1} stands for the ordinary inverse of f. We refer the reader to [3] for a fuller treatment of the topic.

In order to introduce the first type of aforesaid relationship, let us consider a t-norm T that admits an additive generator f_T, a T-equivalence E on a nonempty set X and a function $d_{E,f_T} : X \times X \to [0, \infty]$ defined by

$$d_{E,f_T}(x, y) = f_T(E(x, y))$$

for all $x, y \in X$. Then, it seems natural to wonder whether the function d_{E,f_T} is always a pseudo-metric. An affirmative answer to the posed question was provided by the next result and, thus, a technique for constructing pseudo-metrics from T-equivalences was developed (see, for instance, [4,5]).

Theorem 1. *Let X be a nonempty set whose cardinality is greater than 2. Let T and T^* be two t-norms such that T^* admits an additive generator f_{T^*}. Then, the next conditions are equivalent:*

1. *T^* is weaker than T (i.e., $T^*(x,y) \leq T(x,y)$ for all $x,y \in X$),*
2. *The function $d_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a pseudo-metric on X for any T-equivalence E on X.*
3. *The function $d_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a metric on X for any T-equality E on X.*

Observe that in the statement of the preceding result the cardinality of the set X has been assumed to be greater than 2. This is due to the fact that the function $d_{E,f}$ is always a pseudo-metric (metric) on X for every function $f : [0,1] \to [0,\infty]$ such that $f(1) = 0$ ($f(x) = 0 \Leftrightarrow x = 1$), where $d_{E,f}(x,y) = f(E(x,y))$ for all $x,y \in X$. Consequently, $d_{E,f_{T^*}}$ is always a pseudo-metric on X for every t-norm T^* and any additive generator of T^* whenever the cardinality of X is lower than 3. Clearly, in order to show the equivalence between 2 and 3, it is not necessary to assume such a constraint about the cardinality.

Since every t-norm T is weaker than T_{Min}, the next surprising result which was deduced from Theorem 1 and can be found, for instance, in [4].

Corollary 1. *Let X be a nonempty set and let E be a T_{Min}-equivalence (T_{Min}-equality) on X. If f_T is an additive generator of a t-norm T, then the function d_{E,f_T} is a pseudo-metric (metric) on X.*

The drastic t-norm T_D is the weakest t-norm the following result, which can be found in [4], can also be deduced from Theorem 1.

Corollary 2. *Let X be a nonempty set and let E be a T-equivalence (T-equality) on X. If f_{T_D} is an additive generator of the drastic t-norm T_D, then the function $d_{E,f_{T_D}}$ is a pseudo-metric (metric) on X.*

With the aim of introducing the second type of relationship between pseudo-metrics and T-equivalences that have been explored in the literature, let us now consider a pseudo-metric d on a nonempty set X and a t-norm T that admits an additive generator f_T. Then, one can wonder naturally if the fuzzy binary relation $E_{d,f_T^{(-1)}}$ is a T-equivalence on X, where

$$E_{d,f_T^{(-1)}}(x,y) = f_T^{(-1)}(d(x,y))$$

for all $x,y \in X$. The next result answered affirmatively the question raised when the T-norm is, in addition, continuous and, hence, it gave the following technique to generate T-equivalences from pseudo-metrics (see, for instance, [4,10]).

Theorem 2. *Let T be a t-norm which admits an additive generator f_T. If T is continuous, then the following assertions hold:*

1. *If d is a pseudo-metric on X, then the fuzzy relation $E_{d,f_T^{(-1)}}$ is a T-equivalence on X.*
2. *If d is a metric on X, then the fuzzy relation $E_{d,f_T^{(-1)}}$ is a T-equality on X.*

Notice that the converse of the assertions in the statement of the preceding result are not true in general. As an illustrative example, take the function $d : [0,1] \times [0,1] \to [0,\infty]$ defined by $d(x,y) = x \cdot y$ for all $x,y \in [0,1]$. It is clear that $E_{d,f_{T_D}^{(-1)}}$ is a T_D-equivalence on $[0,1]$, but d is not a pseudo-metric on $[0,1]$ because $d(1,1) = 1$.

In view of Theorem 2, the question naturally arises about the necessity of the t-norm continuity. However, it was proved that such an assumption cannot be taken out in Example 3 of [4].

Notice that the Łukasiewicz t-norm T_L admits the function $f_{T_L} : [0,1] \to [0,\infty]$, given by $f_{T_L}(x) = 1 - x$, as additive generator. Then, in the particular case in which the t-norms T and T^* are taken as

$T = T^* = T_L$, Theorems 1 and 2 give the next celebrated result which can be found in [6]. Observe that in this case the constraint about the cardinality is not necessary.

Corollary 3. *Let X be a nonempty set and let E be a fuzzy relation on X. Then, the next conditions are equivalent:*

1. *E is a T_L-equivalence (T_L-equality) on X.*
2. *The function $1 - E$ (i.e., $d_{E,f_{T_L}}$) is a pseudo-metric (metric) on X.*

It must be stressed that the minimum t-norm T_{Min} does not admit an additive generator. However, a more explicit, than the given by Corollary 1, and a well-known relationship between T_{Min}-equivalences and pseudo-metrics can directly be stated. Such a relationship was given in [11] (see, also, [1]).

Corollary 4. *Let X be a nonempty set and let E be a fuzzy relation on X. Then, the next conditions are equivalent:*

1. *E is a T_{Min}-equivalence (T_{Min}-equality).*
2. *The function $1 - E$ is a pseudo-ultrametric (ultrametric) on X.*

In 1994, Matthews introduced a generalization of the metric notion which have shown to be a useful tool for developing quantitative models in computer science [13–21]. Concretely, such a notion is nowadays known as partial pseudo-metric. Let us recall, according to [12,15], that, given a non-empty set X, a partial pseudo-metric on X is a function $p : X \times X \to [0, \infty]$ that fulfills for all x, y, z the axioms below:

(p1) $p(x, x) \leq p(x, y)$,
(p2) $p(x, y) = p(y, x)$,
(p3) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Similar to the metric case, the partial pseudo-metrics can take the value ∞. Thus, such a notion actually coincides with the counterpart to an extended pseudo-metric.

Note that a partial pseudo-metric satisfies all axioms of pseudo-metrics with exception of $(d1)$. Indeed, given $x \in X$, a partial pseudo-metric p on X does not satisfy in general that $p(x, x) = 0$ such as the next example shows.

Example 1. *Consider the function $p_{\max} : [0, \infty] \times [0, \infty] \to [0, \infty]$ defined by $p_{\max}(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty]$. It is not hard to verify that $p_{\max}$ is a partial pseudo-metric. In addition, $p_{\max}(x, x) = x \neq 0$ for all $x \in]0, \infty]$.*

According to [12,15], a partial metric on a nonempty set X is a partial pseudo-metric p that satisfies for all $x, y \in X$ the axioms below:

(p0) $p(x, x) = p(x, y) = p(y, y) \Leftrightarrow x = y$.

Clearly, the partial pseudo-metric $p_{\max}$ is exactly a partial metric on $[0, \infty]$.

It is obvious that a (pseudo-)metric on a nonempty set X is a partial (pseudo-)metric p such that $p(x, x) = 0$ for all $x \in X$.

Following [22], a partial pseudo-ultrametric on a nonempty set X is a partial pseudo-metric p on X which satisfies for all x, y, z the following:

(p3') $p(x, y) \leq max\{p(x, z), p(z, y)\}$.

When a partial pseudo-ultrametric p on a nonempty set X satisfies, in addition, the axiom $(p0)$ then it is called partial ultrametric.

An instance of partial ultrametric is provided by the next example.

Example 2. *Consider the set of all finite and infinite sequences Σ^∞ over a nonempty alphabet Σ. Given $v \in \Sigma^\infty$, denote by $\mathrm{l}(v)$ the length of v. Thus, $\mathrm{l}(v) \in \mathbb{N} \cup \{\infty\}$ for all $v \in \Sigma^\infty$. Moreover, if $\Sigma_F = \{v \in \Sigma^\infty : \mathrm{l}(v) \in \mathbb{N}\}$ and $\Sigma_\infty = \{v \in \Sigma^\infty : \mathrm{l}(v) = \infty\}$, then $\Sigma^\infty = \Sigma_F \cup \Sigma_\infty$. Consider the function $p_\Sigma : \Sigma^\infty \times \Sigma^\infty \to [0, \infty]$ defined by*

$$p_\Sigma(v, w) = 2^{-\mathrm{l}(v,w)}$$

for all $v, w \in \Sigma^\infty$, where $\mathrm{l}(x, y)$ denotes the longest common prefix of x and y (of course, if x and y have not a common prefix, then $l(x, y) = 0$). We have adopted the convention that $2^{-\infty} = 0$. It is not hard to check that p_Σ is a partial (pseudo-)ultrametric on Σ^∞.

Recently, it has been showed that the notions of T-equivalence and partial pseudo-metric are very closely linked. In particular, on the one hand, Demirci ([23]) and, on the other hand, Bukatin, Kopperman, and Matthews ([22]) have elucidated that the logical counterpart for partial pseudo-metrics is, in some sense, a generalized T-equivalence that we have called partial T-equivalence.

Let us recall that, according to [24] (see also [22,25]), a partial T-equivalence E on a (nonempty) set X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying for all $x, y, z \in X$ the following axioms:

(PE1) $E(x, y) \leq E(x, x)$,
(PE2) $E(x, y) = E(y, x)$,
(PE3) $T(E(x, y), E(y, y) \to_T E(y, z)) \leq E(x, z)$,

where $\to_T$ denotes the residual implication generated from the t-norm T. Observe that, in general, the t-norm is assumed to be left-continuous in order to guarantee that the residuation property is fulfilled (see [3] and Proposition 2.5.2 of [26]).

Moreover, a partial T-equivalence E will be called a partial T-equality provided that

(PE0) $E(x, y) = E(x, x) = E(y, y) \Leftrightarrow x = y$.

Notice that the notion of partial T-equivalence retrieves as a particular case the classical notion of T-equivalence whenever the partial T-equivalence E on X satisfies, in addition that $E(x, x) = 1$ for all $x \in X$. Indeed, the axiom $(PE3)$ is transformed into the axiom $(E3)$ because $T(E(x, y), E(y, z)) = T(E(x, y), 1 \to_T E(y, z)) = T(E(x, y), E(y, y) \to_T E(y, z)) \leq E(x, z)$. However, there are partial T-equivalences that are not T-equivalences like the next example proves.

Example 3. *Consider the set of all finite and infinite sequences Σ^∞ over a nonempty alphabet Σ. Denote by $\mathrm{l}(v)$ the length of $v \in \Sigma^\infty$ as in Example 2. Define the fuzzy binary relation E_Σ on Σ^∞, where*

$$E_\Sigma(u, v) = 1 - 2^{-\mathrm{l}(v,w)}$$

for all $u, v \in \Sigma^\infty$. The convention that $2^{-\infty} = 0$ has been assumed. A straightforward computation shows that E_Σ is a partial T_{Min}-equivalence. Note that E_Σ is not a Min-equivalence because $E_\Sigma(u, u) = 1 - \frac{1}{2^{\mathrm{l}(u)}}$ for all $u \in \Sigma^\infty$. Moreover,

$$E_\Sigma(u, u) = 1 \Leftrightarrow \mathrm{l}(u) = \infty.$$

Taking into account the exposed relationship between, on the one hand, T-equivalences and partial T-equivalences and, on the other hand, between pseudo-metrics and T-equivalences, the purpose of the present work is to explore more in depth the duality correspondence between partial pseudo-metrics and partial T-equivalences. With this aim, we state a method for constructing partial pseudo-metrics from partial T-equivalences in the spirit of Theorem 1. Reciprocally, a method for constructing partial T-equivalences from partial pseudo-metrics is provided extending to the new context the technique given by Theorem 2. However, important differences between the new framework and the classical one are also established. Special interest is paid to the case in which the minimum, drastic, and Łukasiewicz t-norms are under consideration. Hence, counterparts to Corollaries 1, 3, and 4 are obtained in this new approach. Nevertheless, we show that Corollary 2 cannot be extended to the new framework. Finally, conclusions and future work are exposed.

2. Generating Partial Pseudo-Metrics from Partial T-Equivalences

In this section, we provide a method for constructing partial pseudo-metrics from partial T-equivalences. Such a method is studied in the case in which the triangular norm is continuous, since the t-norm under consideration in the notion of partial T-equivalence is left-continuous and we are interested, motivated by Theorem 1, in those t-norms that admits an additive generator. Let us recall that, when a triangular norm T admits an additive generator, the left-continuity of T is equivalent to its continuity (see Proposition 3.26 of [3]) and thus the t-norm T is Archimedean and continuous, i.e., T admits the following representation:

$$T(x,y) = f_T^{-1}\left(\min\{f_T(0), f_T(x) + f_T(y)\}\right)$$

for all $x, y \in [0,1]$. Note that for continuous and Arquimedean t-norms $f_T^{(-1)} = f_T^{-1}$. In this particular case, the residual implication $\to_T$ is given as follows: $x \to_T y = f_T^{-1}(\max\{f_T(y) - f_T(x), 0\})$ for all $x, y \in [0,1]$ (see Theorem 2.5.21 of [26]).

The next result gives the aforesaid method in order to induce a partial pseudo-metric from a partial T-equivalence. Let us stress that, if the set X in which the partial T-equivalences are defined has cardinality lower than 3, then the function $p_{E,f}$ is always a partial pseudo-metric (partial metric) on X for every monotone (strictly monotone) function $f : [0,1] \to [0,\infty]$, where $p_{E,f}(x,y) = f(E(x,y))$ for all $x, y \in X$.

Proposition 1. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If E is a partial T-equivalence on X, then the function $p_{E,f_T} : X \times X \to [0,\infty]$ is a partial pseudo-metric, where*

$$p_{E,f_T}(x,y) = f_T(E(x,y)),$$

for all $x, y \in X$.

Proof. Let us see that p_{E,f_T} verifies the axioms of a partial pseudo-metric. First, we show that p_{E,f_T} satisfies $(p1)$. Since f_T is strictly decreasing and $E(x,y) \leq E(x,x)$ for all $x, y \in X$, we have that

$$p_{E,f_T}(x,x) = f_T(E(x,x)) \leq f_T(E(x,y)) = p_{E,f_T}(x,y).$$

Next, we show that p_{E,f_T} satisfies $(p2)$. However, this is straightforward because of the symmetry of E. Indeed,

$$p_{E,f_T}(x,y) = f_T(E(x,y)) = f_T(E(y,x)) = p_{E,f_T}(y,x).$$

Now, we show that p_{E,f_T} satisfies $(p3)$. Since E is a partial T-equivalence, we have that $T(E(x,y), E(y,y) \to_T E(y,z)) \leq E(x,z)$ for all $x, y, z \in X$. Moreover, the fact that f_T is an additive generator of T gives that

$$T(E(x,y), E(y,y) \to_T E(y,z)) =$$

$$f_T^{-1}\left(\min\{f_T(0), f_T(E(x,y)) + f_T\left(E(y,y) \to_T E(y,z)\right)\}\right)$$

and

$$f_T\left(E(y,y) \to_T E(y,z)\right) =$$

$$f_T\left(f_T^{-1}(\max\{f_T(E(y,z)) - f_T(E(y,y)), 0\})\right) =$$

$$f_T(E(y,z)) - f_T(E(y,y)).$$

Hence, we obtain that

$$T(E(x,y), E(y,y) \to_T E(y,z)) =$$

$$f_T^{-1}\left(\min\{f_T(0), f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y))\}\right).$$

Next, we distinguish two possible cases:

Case 1. $f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y)) \in Range(f_T)$. Then, we have that

$$f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y))) \leq f_T(0).$$

It follows that

$$f_T^{-1}\left(f_T(E(x,y)) + f_T\left(E(y,y) \to_T E(y,z)\right)\right) = T(E(x,y), E(y,y) \to_T E(y,z)),$$

since

$$T(x,y) = f_T^{-1}\left(\min\{f_T(0), f_T(x) + f_T(y)\}\right)$$

for all $x, y \in [0,1]$.

The fact that E is a partial T-equivalence on X gives, in addition, that $T(E(x,y), E(y,y) \to_T E(y,z)) \leq E(x,z)$. Thus,

$$f_T^{-1}\left(f_T(E(x,y)) + f_T\left(E(y,y) \to_T E(y,z)\right)\right) \leq E(x,z).$$

Since f_T is strictly decreasing, we deduce that

$$f_T(E(x,y)) + f_T\left(E(y,y) \to_T E(y,z)\right) =$$

$$f_T\left(f_T^{-1}\left(f_T(E(x,y)) + f_T\left(E(y,y) \to_T E(y,z)\right)\right)\right) \geq f_T(E(x,z)).$$

Since $x \to_T y = f_T^{-1}(\max\{f_T(y) - f_T(x), 0\})$ for all $x, y \in [0,1]$, f_T is strictly decreasing, and $E(y,z) \leq E(y,y)$, we deduce that

$$f_T\left(E(y,y) \to_T E(y,z)\right) = f_T(E(y,z)) - f_T(E(y,y)),$$

whence we obtain that

$$f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y)) \geq f_T(E(x,z)).$$

Case 2. $f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y)) \notin Range(f_T)$. Then, the fact that f is an additive generator of T gives that

$$f_T(E(x,y)) + f_T(E(y,z)) - f_T(E(y,y)) > f_T(0) \geq f_T(E(x,z)).$$

Therefore, p_{E,f_T} satisfies $(p3)$ as claimed. $\square$

An example of partial pseudo-metric constructed by means of the technique given by Proposition 1 is the following one (see also Example 6).

Example 4. *Consider again the set of all finite and infinite sequences Σ^{∞} over a nonempty alphabet Σ and fuzzy binary relation E_{Σ} on Σ^{∞} introduced in Example 3. It is clear that E_{Σ} is a partial T_L-equivalence on Σ^{∞}. Proposition 1 guarantees that the function p_{E,f_T} is a partial pseudo-metric on Σ^{∞} with $p_{E,f_T}(x,y) = 1 - E_{\Sigma}(u,v)$ for all $u,v \in \Sigma^{\infty}$. Observe that the partial pseudo-metric p_{E,f_T} coincides with the partial pseudo-metric p_{Σ} introduced in Example 2.*

When the partial T-equivalence is exactly a T-equivalence, we obtain the next result from Proposition 1.

Corollary 5. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If E is a T-equivalence on X, then the function $p_{E,f_T} : X \times X \to [0,\infty]$ is a pseudo-metric, where*

$$p_{E,f_T}(x,y) = f_T(E(x,y)),$$

for all $x,y \in X$.

Proof. Proposition 1 gives that the function p_{E,f_T} is a partial pseudo-metric on X. Moreover, $p_{E,f_T}(x,x) = f_T(E(x,x)) = f_T(1) = 0$. $\square$

In the light of the preceding result and taking into account that, in the statement of Theorem 1, the continuity of the t-norm is not required, it seems natural to ask if actually one can remove the continuity from the statement of Proposition 1. The next example gives a negative answer to the posed question and, thus, shows a difference between the new approach and the classical one.

Example 5. *Consider the fuzzy relation $E : [0,1] \times [0,1] \to [0,1]$ given by*

$$E(x,y) = \begin{cases} 1 & \text{if } x = y \\ \max\{x,y\} & \text{otherwise.} \end{cases}$$

It is not hard to check that E is a T_D-equivalence and, hence, a partial T_D-equivalence, where T_D is the drastic t-norm. Nevertheless, the function $p_{E,f_{T_D}}$ is not a partial pseudo-metric on $[0,1]$ because it does not satisfy the axiom $(p3)$. Indeed,

$$f_{T_D}(\frac{1}{2}) = p_{E,f_{T_D}}(\frac{1}{4},\frac{1}{2}) > p_{E,f_{T_D}}(\frac{1}{4},1) + p_{E,f_{T_D}}(1,\frac{1}{2}) - p_{E,f_{T_D}}(1,1) = f_{T_D}(1) = 0,$$

where

$$f_{T_D}(x) = \begin{cases} 2 - x & \text{if } x \in [0,1[\\ 0 & \text{if } x = 1. \end{cases}$$

When the partial T-equivalence is exactly a partial T-equality, we obtain the following refined version of the technique given by Proposition 1.

Proposition 2. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If E is a partial T-equality on X, then the function $p_{E,f_T} : X \times X \to [0,\infty]$ is a partial metric, where*

$$p_{E,f_T}(x,y) = f_T(E(x,y)),$$

for all $x, y \in X$.

Proof. It is only necessary to prove that the function p_{E,f_T} fulfills the axiom $(p0)$. Since f_T is strictly decreasing, we deduce that $E(x,x) = E(y,y) = E(x,y) \Leftrightarrow p_{E,f_T}(x,x) = p_{E,f_T}(y,y) = p_{E,f_T}(x,y)$. Thus, we obtain, from the fact that E is a T-equality, that $p_{E,f_T}(x,x) = p_{E,f_T}(y,y) = p_{E,f_T}(x,y) \Leftrightarrow x = y$. $\square$

In the particular case in which the partial T-equality is exactly a T-equality, we obtain the next result from Proposition 2.

Corollary 6. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If E is a T-equality on X, then the function $p_{E,f_T} : X \times X \to [0,\infty]$ is a metric, where*

$$p_{E,f_T}(x,y) = f_T(E(x,y))$$

for all $x, y \in X$.

Proof. Corollary 5 gives that the function p_{E,f_T} is a pseudo-metric on X. Moreover, taking into account that f_T is strictly decreasing and that $f_T(1) = 0$, we have that $0 = p_{E,f_T}(x,y) \Leftrightarrow 0 = f_T(E(x,y)) \Leftrightarrow E(x,y) = 1 \Leftrightarrow x = y$. Consequently, p_{E,f_T} is a metric on X. $\square$

The next example gives an instance of a partial metric generated by means of the methodology provided in Proposition 2, which is not a metric.

Example 6. *Let E be the fuzzy relation on $[0,1]$ defined by*

$$E(x,y) = \min\{x,y\}$$

for all $x, y \in X$. Consider the additive generator f_{T_P} of the product t-norm given by $f_{T_P}(x) = -\log(x)$ for all $x \in [0,1]$, where we assume that $\log(0) = \infty$. Since E is a partial T_P-equality, Proposition 2 yields that the function $p_{E,f_{T_P}}$ is a partial metric on $[0,1]$, where

$$p_{E,f_{T_P}}(x,y) = -\log(\min\{x,y\})$$

for all $x, y \in [0,1]$.

Next, we focus our effort on trying to obtain a version of Theorem 1 in our new framework. Recall that, if the set X in which the partial T-equivalences are defined has cardinality lower than 3, then the function p_{E,f_T} is always a partial pseudo-metric (partial metric) on X.

Theorem 3. *Let X be a nonempty set. Assume that T and T^* are two continuous t-norms and that T^* admits an additive generator f_{T^*}. Then, the following assertions are equivalent:*

1. *The function $p_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a partial pseudo-metric on X for the partial T-equivalence E on X.*
2. *The function $p_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a partial metric on X for the partial T-equality E on X.*

Proof. If the cardinality of the set X is lower than 3, then both assertions are equivalent. Assume that the cardinality of the set X is greater than 3. Next, we show the equivalence between both assertions.

1. $\Rightarrow$ 2. Assume that E is a partial T-equality on X. Then, the function $p_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a partial pseudo-metric on X. We only need to show that $p_{E,f_{T^*}}$ satisfies axiom $(p0)$ when the partial T-equivalence E is a partial T-equality on X. Since f_{T^*} is strictly decreasing, we deduce that

$E(x,x) = E(y,y) = E(x,y) \Leftrightarrow p_{E,f_{T^*}}(x,x) = p_{E,f_{T^*}}(y,y) = p_{E,f_{T^*}}(x,y)$. The fact that E is a T-equality yields that

$$p_{E,f_T}(x,x) = p_{E,f_T}(y,y) = p_{E,f_T}(x,y) \Leftrightarrow x = y.$$

Thus, $p_{E,f_{T^*}}$ is a partial metric on X.

2. $\Rightarrow$ 1. For the purpose of contradiction, we assume that there exists a t-norm T, a non-empty set X, and a partial T-equivalence E on X such that the function $p_{E,f_{T^*}}$ is not a partial pseudo-metric on X. It follows that $p_{E,f_{T^*}}$ does not satisfy either condition $(p1)$ or condition $(p3)$.

First, we suppose that $(p1)$ is not fulfilled. Then, there exists $x, y \in X$ such that $p_{E,f_{T^*}}(x,y) < p_{E,f_{T^*}}(x,x)$, whence we obtain that $f_{T^*}(E(x,y)) < f_{T^*}(E(x,x))$. However, the fact that f_{T^*} is strictly decreasing provides that $E(x,x) < E(x,y)$, which is a contradiction because E is a partial T-equivalence and, thus, $E(x,y) \leq E(x,x)$.

Next, we suppose that $(p3)$ is not fulfilled. Then, there exists $x, y, z \in X$ such that $p_{E,f_{T^*}}(x,y) > p_{E,f_{T^*}}(x,z) + p_{E,f_{T^*}}(z,y) - p_{E,f_{T^*}}(z,z)$. Next, we show that $\max\{E(x,y), E(x,z), E(y,z)\} < 1$. We differentiate the following three possible cases:

Case 1. $E(x,y) = 1$. Then, we have that

$$0 = p_{E,f_{T^*}}(x,y) > p_{E,f_{T^*}}(x,z) + p_{E,f_{T^*}}(z,y) - p_{E,f_{T^*}}(z,z) \geq 0,$$

which is not possible.

Case 2. $E(x,z) = 1$. Then, $E(z,z) = 1$. Thus, $p_{E,f_{T^*}}(x,z) = p_{E,f_{T^*}}(z,z) = 0$ and $p_{E,f_{T^*}}(x,y) > p_{E,f_{T^*}}(z,y)$. Since f_{T^*} is strictly decreasing, we have that $E(x,y) < E(z,y)$. Hence, $E(z,y) = 1 \rightarrow_T E(z,y) = E(z,z) \rightarrow_T E(z,y)$. The fact that E is a partial T-equivalence yields that

$$T(E(x,z), E(z,z) \rightarrow_T E(z,y)) \leq E(x,y).$$

Thus, $E(z,y) = T(1, 1 \rightarrow_T E(z,y)) \leq E(x,y) < E(z,y)$, which is a contradiction.

Case 3. $E(y,z) = 1$. The conclusion can obtain the following similar arguments to those given in Case 2.

Since $\max\{E(x,y), E(x,z), E(y,z)\} < 1$, we define a binary fuzzy relation E' on $Y = \{x, y, z\}$ as follows: $E'(w,v) = E(v,w)$ for all $w, v \in Y$ with $w \neq v$ and $E'(w,w) = 1$ for all $w \in Y \setminus \{z\}$ and $E'(z,z) = E(z,z)$. It is clear that $E'(w,v) = E'(w,w) = E'(v,v) \Leftrightarrow w = v$ for all $w, v \in Y$. Since $x \rightarrow_T y \leq z \rightarrow_T y$ for all $z \leq x$ for all $x, y \in X$, $E(v,v) \leq E'(v,v)$ and $E(w,v) \leq E'(w,v)$ for all $w, v \in Y$, a straightforward computation shows that

$$T(E'(w,v), E'(v,v) \rightarrow_T E'(v,u)) \leq T(E(w,v), E(v,v) \rightarrow_T E(v,u)) \leq$$
$$E(w,u) \leq E'(w,u)$$

for all $w, v, u \in Y$. Therefore, E' is a partial T-equality on Y. By hypothesis, we have that $p_{E',f_{T^*}}$ is a partial metric on Y. Then,

$$p_{E',f_{T^*}}(x,y) \leq p_{E',f_{T^*}}(x,z) + p_{E',f_{T^*}}(z,y) - p_{E',f_{T^*}}(z,z)$$

for all $x, y, z \in Y$. However, this contradicts the fact that

$$p_{E,f_{T^*}}(x,z) + p_{E,f_{T^*}}(z,y) - p_{E,f_{T^*}}(z,z) < p_{E,f_{T^*}}(x,y)$$

□

The next result establishes what is the relationship between the t-norms T and T^* when the function $p_{E,f_{T^*}}$ is a partial (pseudo-)metric on X for any partial T-equivalence (T-equality) E on X. Notice that, in this direction, an important difference appears with respect to what occurs in the classical case (see Theorem 1).

Theorem 4. *Let X be a nonempty set with cardinality greater than 2. Assume that T and T^* are two continuous t-norms and that T^* admits an additive generator f_{T^*}. If the function $p_{E,f_{T^*}} : X \times X \to [0,\infty]$ is a partial metric on X for the partial T-equality E on X, then T^* is weaker than T.*

Proof. Suppose that E is a partial T-equality on X. Let $a,b \in [0,1]$. Our aim is to show that $T^*(a,b) \leq T(a,b)$. With this aim, consider three different elements $x,y,z \in X$ and the fuzzy binary relation E on X given by:

$$E(u,v) = E(v,u) = \begin{cases} T(a,b) & \text{if } u = x \text{ and } v = y \\ a & \text{if } u = x \text{ and } v = z \\ b & \text{if } u = y \text{ and } v = z \\ 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}.$$

Notice that we can assume that $a,b \in [0,1[$ because, if $\max\{a,b\} = 1$, then we have immediately that $T^*(a,b) = T(a,b)$. A straightforward computation yields that E is partial T-equality on X. By assumption, we have that $p_{E,f_{T^*}}$ is a partial metric on X. Thus,

$$p_{E,f_{T^*}}(x,y) \leq p_{E,f_{T^*}}(x,z) + p_{E,f_{T^*}}(z,y) - p_{E,f_{T^*}}(z,z)$$

for all $x,y,z \in X$, whence we obtain that

$$\begin{aligned} & T(a,b) = E(x,y) = f_{T^*}^{(-1)}\left(f_{T^*}(E(x,y))\right) \geq \\ & f_{T^*}^{(-1)}\left(f_{T^*}(E(x,z)) + f_{T^*}(E(z,y)) - f_{T^*}(E(z,z))\right) = \\ & f_{T^*}^{(-1)}\left(f_{T^*}(E(x,z)) + f_{T^*}\left(E(z,z) \to_{T^*} E(z,y)\right)\right) = \\ & T^*(E(x,z), E(z,z) \to_{T^*} E(z,y)) = \\ & T^*(E(x,z), E(y,z)) = T^*(a,b). \end{aligned}$$

□

Contrary to the classical metric case, the converse of the preceding implication is not true in general such as the next example shows.

Example 7. *Let us recall that the family $\left(T_\lambda^{MT}\right)_{\lambda\in[0,1]}$ of Mayor-Torrens t-norms are given by*

$$T_\lambda^{MT}(x,y) = \begin{cases} \max\{x+y-\lambda, 0\}, & \textit{if } \lambda \in]0,1] \textit{ and } (x,y) \in [0,\lambda]^2 \\ \min\{x,y\}, & \textit{elsewhere} \end{cases}.$$

According to [3,27], each t-norm T_λ^{MT} is continuous.

Next, consider the fuzzy binary relation E on a set $X = \{x, y, z\}$ such that $E(x, x) = E(y, y) = E(z, z) = 0.89$, $E(x, y) = E(y, x) = 0$, and $E(x, z) = E(z, x) = E(y, z) = E(z, y) = 0.45$. It is not hard to check that E is a $T_{0.9}^{MT}$-equivalence. Clearly, T_L is weaker than $T_{0.9}^{MT}$, i.e., $T_L(x, y) \leq T_{0.9}^{MT}(x, y)$ for all $x, y \in [0, 1]$. Nevertheless, $p_{E,f_{T_L}}(x, y) = 1 - E(x, y)$ is not a partial pseudo-metric on X. In fact, $p_{E,f_{T_L}}(x, y) > p_{E,f_{T_L}}(x, z) + p_{E,f_{T_L}}(z, y) - p_{E,f_{T_L}}(z, z)$, since $0.89 = E(x, y) + E(z, z) < E(x, z) + E(z, y) = 0.45 + 0.45 = 0.90$.

Combining Theorems 1 and 4, the following result can be stated.

Corollary 7. *Let X be a nonempty set with cardinality greater than 2. Assume that T and T^* are two continuous t-norms and that T^* admits an additive generator f_{T^*}. Then, each of the following assertions implies its successor.*

1. *The function $p_{E,f_{T^*}} : X \times X \to [0, \infty]$ is a partial pseudo-metric on X for any partial T-equivalence E on X.*
2. *The function $p_{E,f_{T^*}} : X \times X \to [0, \infty]$ is a partial metric on X for any partial T-equality E on X.*
3. *T^* is weaker than T.*
4. *The function $p_{E,f_{T^*}} : X \times X \to [0, \infty]$ is a pseudo-metric on X for any T-equivalence E on X.*
5. *The function $p_{E,f_{T^*}} : X \times X \to [0, \infty]$ is a metric on X for any T-equality E on X.*

The fact that every t-norm T is weaker than T_{Min} allows us to deduce, from Proposition 1, the next result which extends Corollary 1 to our context.

Corollary 8. *Let X be a nonempty set and let E be a partial T_{Min}-equivalence (partial T_{Min}-equality) on X. If f_T is the additive generator of any continuous t-norm T, then the function p_{E,f_T} is a partial pseudo-metric (partial metric) on X.*

Proof. Since E a partial T_{Min}-equivalence, we have that $T_{Min}(E(x, y), E(z, y)) \leq E(x, y)$ for all $x, y, z \in X$. It follows that

$$T(E(x, z), E(z, z) \to_T E(z, y)) \leq T(E(x, z), E(z, y)) \leq T_{Min}(E(x, y), E(z, y)) \leq E(x, y).$$

Therefore, E is a partial T-equivalence. Thus, by Proposition 1, we have that p_{E,f_T} is a partial pseudo-metric on X. Similar arguments allow us to show that p_{E,f_T} is a partial metric on X whenever E is a T_{Min}-equality on X but now applying Proposition 2.
□

We end this section pointing out another difference between the new approach and the classical one. Indeed, observe that Example 5 shows that a direct extension of Corollary 2 to our context fails to hold.

3. Generating Partial T-Equivalences from Partial Pseudo-Metrics

In the following, we show that the converse of Proposition 1 is fulfilled. Thus, we provide a method for constructing partial T-equivalences from partial pseudo-metrics. Moreover, we show that the classical method, given by Theorem 2, for generating T-equivalences from pseudo-metrics is retrieved as a particular case.

Theorem 5. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If $p : X \times X \longrightarrow [0, \infty]$ is a partial pseudo-metric, then the binary fuzzy relation $E_{p,f_T} : X \times X \to [0, 1]$ given by*

$$E_{p,f_T}(x, y) = f_T^{(-1)}(p(x, y)) \text{ for all } x, y \in X$$

is a partial T-equivalence on X.

Proof. We first show that E_{p,f_T} satisfies axiom (PE1). Since p is a partial pseudo-metric, we have that $p(x,x) \leq p(x,y)$ for all $x,y \in X$. The fact that the function $f_T^{(-1)}$ is decreasing gives that $f_T^{(-1)}(p(x,x)) \geq f_T^{(-1)}(p(x,y))$ and, hence, $E_{p,f_T}(x,x) \geq E_{p,f_T}(x,y)$ for all $x,y \in X$.

The axiom (PE2) holds trivially. Indeed, $E_{p,f_T}(x,y) = f_T^{-1}(p(x,y)) = f_T^{-1}(p(y,x)) = E_{p,f_T}(y,x)$ for all $x,y \in X$.

Next, we prove that E_{p,f_T} satisfies axiom (PE3):

$$T(E_{p,f_T}(x,z), E_{p,f_T}(z,z) \rightarrow_T E_{p,f_T}(y,z)) =$$

$$f_T^{-1}(min\{f_T(0), f_T(E_p(x,z)) + f_T(E_{p,f_T}(z,z) \rightarrow_T E_{p,f_T}(y,z))\}) =$$

$$f_T^{-1}(min\{f_T(0), f_T(E_p(x,z)) + f_T(E_{p,f_T}(y,z)) - f_T(E_{p,f_T}(z,z))\}) =$$

$$f_T^{-1}(min\{f_T(0), p(x,z) + p(y,z) - p(z,z)\})$$

Since p is a partial pseudo-metric on X, we have that

$$p(x,y) \leq p(x,z) + p(y,z) - p(z,z)$$

for all $x,y,z \in X$. From the fact that $f_T^{(-1)}$ is decreasing and the preceding inequality, we deduce that

$$f_T^{(-1)}(p(x,z) + p(z,y) - p(z,z)) \leq f_T^{(-1)}(p(x,y)).$$

It follows that

$$f_T^{-1}(min\{f_T(0), p(x,z) + p(z,y) - p(z,z)\}) \leq E_{p,f_T}(x,y).$$

Therefore,

$$T(E_{p,f_T}(x,z), E_{p,f_T}(z,z) \rightarrow_T E_{p,f_T}(z,y)) \leq E_{p,f_T}(x,y)$$

for all $x,y,z \in X$. □

Combining Proposition 1 and Theorem 5, we obtain the following nice characterization of partial T-equivalences.

Corollary 9. *Let X be a nonempty set, let E be a fuzzy binary relation on X, and let T be a continuous t-norm which admits an additive generator f_T. Then, the following assertions hold:*

1. *E is a partial T-equivalence on X.*
2. *The function p_{E,f_T} is a partial pseudo-metric on X.*

Of course, when we consider the Łukasiewicz t-norm T_L in the preceding result, we immediately get the next result that extends the celebrated Corollary 3 to the partial T_L-equivalences.

Corollary 10. *Let X be a nonempty set and let E be a fuzzy binary relation on X. Then, the following assertions hold:*

1. *E is a T_L-equivalence on X.*
2. *The function $1 - E$ is a partial pseudo-metric on X.*

Although the minimum t-norm T_{Min} is continuous, it does not admit an additive generator. Nonetheless, the next explicit correspondence between partial T_{Min}-equivalences and partial pseudo-metrics can directly be stated. Observe that such a relationship extends Corollary 4.

Corollary 11. *Let X be a nonempty set and let E be a fuzzy binary relation on X. Then, the following assertions are equivalent:*

1. *E is a partial T_{Min}-equivalence (T_{Min}-equality).*
2. *The function $1-E$ is a partial pseudo-ultrametric (partial ultrametric) on X.*

Proof. 1. $\Rightarrow$ 2. Taking $T = T_L$ in Corollary 8, we deduce that $1-E$ is a partial pseudo-metric on X. Next, we show that, in addition, $1-E$ is a partial pseudo-ultrametric on X. Since E is a partial T_{Min}-equivalence, we have for all $x,y,z \in X$ that

$$\min\{E(x,y),E(y,z)\} = T_{Min}(E(x,y),E(y,y) \to_{T_{Min}} E(y,z)) \leq E(x,z).$$

Moreover, it is clear that the following sequence of inequalities are equivalent for all $x,y,z \in X$:

$$\begin{aligned} \min\{E(x,y),E(y,z)\} &\leq E(x,z) \\ -\min\{E(x,y),E(y,z)\} &\geq -E(x,z) \\ 1-\min\{E(x,y),E(y,z)\} &\geq 1-E(x,z) \\ \max\{1-E(x,y),1-E(y,z)\} &\geq 1-E(x,z) \end{aligned}$$

2. $\Rightarrow$ 1. The fact that $1-E$ is a partial pseudo-ultrametric on X gives that it is a partial pseudo-metric on X. By Corollary 9, we deduce that E is a partial T_L-equivalence on X. Thus, E satisfies axioms (PE1) and (PE2). It remains to prove that

$$T_{Min}(E(x,y),E(y,y) \to_{T_{Min}} E(y,z)) \leq E(x,z).$$

We have that $\max\{1-E(x,y),1-E(y,z)\} \geq 1-E(x,z)$, since $1-E$ is a partial pseudo-ultrametric on X. Since the preceding inequality is equivalent to $\min\{E(x,y),E(y,z)\} \leq E(x,z)$, we conclude that

$$T_{Min}(E(x,y),E(y,y) \to_{T_{Min}} E(y,z)) \leq E(x,z).$$

Finally, $1-E$ is a partial ultrametric if and only if $1-E$ is a T_{Min}-equality because of $1-E(x,y) = 1-E(x,x) = 1-E(y,y) \Leftrightarrow E(x,y) = E(x,x) = E(y,y) \Leftrightarrow x = y$. $\square$

It must be stressed that Corollary 11 ensures that the partial T_{Min}-equivalence introduced in Example 3 is the logic counterpart of the partial ultrametric p_Σ introduced in Example 2 such as it was suggested in [22].

We get Theorem 2 from Theorem 5 when the partial pseudo-metric is a pseudo-metric.

Corollary 12. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. Then, the following assertions hold:*

1. *If $p : X \times X \longrightarrow [0,\infty]$ is a pseudo-metric, then the binary fuzzy relation $E_{p,f_T} : X \times X \to [0,1]$, given by*
$$E_{p,f_T}(x,y) = f_T^{(-1)}(p(x,y)) \text{ for all } x,y \in X,$$
is a T-equivalence on X.
2. *If $p : X \times X \longrightarrow [0,\infty]$ is a metric, then the binary fuzzy relation $E_{p,f_T} : X \times X \to [0,1]$, given by*
$$E_{p,f_T}(x,y) = f_T^{(-1)}(p(x,y)) \text{ for all } x,y \in X,$$
is a T-equality on X.

Proof. 1. By Theorem 5, we have that the binary fuzzy relation E_{p,f_T} is a partial T-equivalence. We only need to show that $E_{p,f_T}(x,x) = 1$ for all $x \in X$. Certainly, we have that

$$E_{p,f_T}(x,x) = f_T^{(-1)}(p(x,x)) = f_T^{-1}(\min\{f_T(0), 0\}) = f_T^{-1}(0) = 1.$$

2. By assertion 1. we have that E_{p,f_T} is a T-equivalence on X. In order to show that it is a T-equality, we assume that there exist $x, y \in X$ such that $E_{p,f_T}(x,y) = 1$. Then, $f_T^{(-1)}(p(x,y)) = 1$. Thus, $f_T^{-1}(\min\{f_T(0), p(x,y)\}) = 1$. Clearly, it must hold that $p(x,y) < f_T(0)$ because otherwise we have that $f_T^{-1}(f_T(0)) = 0$, which contradicts that $E_{p,f_T}(x,y) = 1$—in the case when $1 = f_T^{-1}(\min\{f_T(0), p(x,y)\}) = f_T^{-1}(p(x,y))$, whence we deduce that $p(x,y) = 0$. Since p is a metric on X, we conclude that $x = y$. Thus, E_{p,f_T} is a T-equality on X. □

In view of Theorem 5 and Corollary 12, it seems natural to wonder whether the partial T-equivalence $E_{p,f_T}(x,y)$ is exactly a partial T-equality when we consider a partial pseudo-metric p, which is, in addition, a partial metric. However, the next result provides a negative answer to that question when considering the restriction to $[0,\infty[$ of the partial metric.

Example 8. *Consider the restriction of the partial metric p_{max}, introduced in Example 1, to $[0,\infty[\times[0,\infty[$. Denote such a restriction again by $p_{\max}$. Now consider the Łukasiewicz t-norm T_L. It is clear that $p_{max}(3,2) > f_{T_L}(0) = 1$. Theorem 5 gives that $E_{p_{max},f_{T_L}}$ is a partial T_L-equivalence on $[0,\infty[$. Nevertheless,*

$$E_{p_{max},f_{T_L}}(3,2) = E_{p_{max},f_{T_L}}(3,3) = E_{p_{max},f_{T_L}}(2,2) = 0.$$

Thus, $E_{p_{max},f_{T_L}}$ is a partial T_L-equivalence on $[0,\infty[$, but it is not a T_L-equality.

Taking into account Example 8, we give a sufficient condition in order to guarantee that the induced partial T-equivalence is exactly a partial T-equality when one counts with a partial metric.

Proposition 3. *Let X be a nonempty set and let T be a continuous t-norm which admits an additive generator f_T. If $p : X \times X \longrightarrow [0,\infty]$ is a partial metric such that $p(x,y) \leq f_T(0)$ for all $x, y \in X$, then the binary fuzzy relation $E_{p,f_T} : X \times X \to [0,1]$ given by*

$$E_{p,f_T}(x,y) = f_T^{(-1)}(p(x,y))$$

is a partial T-equality on X.

Proof. By Theorem 5, we obtain that E_{p,f_T} is a partial T-equivalence on X. Next, we show that it is, in addition, a partial T-equality. Since $p(x,y) \leq f_T(0)$ for all $x, y \in X$, we obtain that $f_T^{(-1)}(p(x,y)) = f_T^{-1}(p(x,y))$ for all $x, y \in X$. Assume that, given $x, y, z \in X$, we have that

$$E_{p,f_T}(x,y) = E_{p,f_T}(x,x) = E_{p,f_T}(y,y).$$

Thus, $f_T^{-1}(p(x,x)) = f_T^{-1}(p(x,y)) = f_T^{-1}(p(y,y))$. Suppose that $p(x,x) < p(x,y)$. Then, $f_T^{-1}(p(x,x)) > f_T^{-1}(p(x,y))$, which is a contradiction. Thus, $p(x,y) = p(x,x)$. A similar reasoning runs for proving that $p(x,y) = p(y,y)$. The fact that p is a partial metric on X provides that $x = y = z$ and thus that E_{p,f_T} is a partial T-equality. □

Notice that, as an instance of partial pseudo-metric that satisfies the "boundness" condition introduced in Proposition 3, we have the partial pseudo-metric p_Σ introduced in Example 2 when the Łukasiewicz t-norm is under consideration. Indeed, p_Σ fulfills that $p_\Sigma(u,v) = 2^{-l(u,v)} \leq f_{T_L}(0) = 1$ for all $u, v \in \Sigma^\infty$. Furthermore, the restriction of the partial metric $p_{\max}$, introduced in Example 1, to $[0,1]$ provides another instance of partial pseudo-metric which fulfills the boundness condition when the

Łukasiewicz t-norm is also considered. Indeed, $p_{\max}(x,y) \leq f_{T_L}(0) = 1$. Thus, Proposition 3 gives that $E_{p_{\max},f_{T_L}}(x,y) = 1 - \max\{x,y\}$ is a partial T_L-equality on $[0,1]$.

Observe that, in assertion 2 in the statement of Corollary 12, we do not need to assume the condition "$p(x,y) \leq f_T(0)$ for all $x,y \in X$" because we always have that $E_{p,f_T}(x,y) = f_T^{-1}(\min\{f_T(0), p(x,y)\}) = 1 \Leftrightarrow 0 = p(x,y) < f_T(0)$.

From Proposition 3, we can derive the following result whose easy proof we omit. In order to state such a result, let us recall that, according to Proposition 3.29 and Corollary 3.30 of [3], a continuous t-norm which admits that an additive generator f_T is strict provided that $f_T(0) = \infty$.

Corollary 13. *Let X be a nonempty set and let T be a strict continuous t-norm which admits an additive generator f_T. If $p : X \times X \longrightarrow [0,\infty]$ is a partial metric, then the binary fuzzy relation $E_{p,f_T} : X \times X \to [0,1]$ given by*

$$E_{p,f_T}(x,y) = f_T^{(-1)}(p(x,y))$$

is a partial T-equality on X.

Inspired by the classical pseudo-metric case (see Section 1), we wonder if the continuity of the t-norm can be deleted in the statement of Theorem 5. We end the paper with an example which gives a negative response to such a question.

Example 9. *Let us consider the partial metric p_+ on $[0,\infty[$ defined by*

$$p_+(x,y) = \begin{cases} x+y & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Then, $p_+(\frac{1}{2},0) = \frac{1}{2}$, $p_+(0,1) = 1$ and $p_+(\frac{1}{2},1) = \frac{3}{2}$. Consider, in addition, the drastic t-norm T_D that is not continuous and admits an additive generator. It is clear that the fuzzy binary relation $E_{p_+,f_{T_D}}$ on $[0,\infty[$ induced by the technique stated in Theorem 5 is given by

$$E_{p_+,f_{T_D}}(x,y) = f_{T_D}^{(-1)}(p_+(x,y)) = \begin{cases} 0 & \text{if } 2 \leq x+y \\ 2-x-y & \text{if } 1 < x+y < 2 \\ 1 & \text{if } 0 \leq x+y \leq 1 \end{cases}$$

However, it is clear that $E_{p_+,f_{T_D}}$ is not a T_D-equivalence on $[0,\infty[$, since

$$1 = T_D(1,1) = T_D(E_{p_+,f_{T_D}}(\tfrac{1}{2},0), E_{p_+,f_{T_D}}(0,0) \to_{T_D} E_{p_+,f_{T_D}}(0,1))$$

$$> E_{p_+,f_{T_D}}(\tfrac{1}{2},1) = \tfrac{1}{2}.$$

4. Conclusions

The notion of T-equivalence, where T is a t-norm, was introduced as a fuzzy generalization of the notion of crisp equivalence relation. Since then, many efforts have been devoted to the study of the metric behavior of such fuzzy relations. Concretely, a few techniques to induce metrics from T-equivalences, and, vice versa, have been deeply developed (see Theorems 1 and 2). Motivated by the fact that partial pseudo-metrics, a generalization of of pseudo-metrics that have been shown to be useful in computer science and artificial intelligence, have been suggested as the logical counterpart of T-equivalences, we have studied the duality correspondence between partial pseudo-metrics and partial T-equivalences. Concretely, a method for constructing partial pseudo-metrics from partial T-equivalences have been provided in the spirit of the classical case (see Propositions 1 and 2 and Theorem 3). However, in contrast to the aforementioned case, in the new approach, the continuity of the t-norm has been shown to be crucial. Moreover, a method for constructing partial T-equivalences from partial pseudo-metrics is also provided in such a way that the classical one is retrieved as a particular case (see Theorem 5 and Corollaries 9, 10 and 11). Important differences between the new framework and the classical one have been shown (see Theorem 4 and Proposition 3).

As future work, we consider the possibility of exploring those conditions that allow us to induce partial pseudo-metrics from partial T-equivalences according to the technique stated in Proposition 1 when the t-norms involved are not left-continuous. With this aim, we will try to take advantage of the method to generate implications with respect to non left-continuous t-norms given by Biba and Hliněná in [28].

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