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# Stability of the Fréchet Equation in Quasi-Banach Spaces

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**Abstract:** We investigate the Hyers–Ulam stability of the well-known Fréchet functional equation that comes from a characterization of inner product spaces. We also show its hyperstability on a restricted domain. We work in the framework of quasi-Banach spaces. In the proof, a fixed point theorem due to Dung and Hang, which is an extension of a fixed point theorem in Banach spaces, plays a main role.

**Keywords:** Hyers–Ulam stability; hyperstability; Fréchet equation; quasi-Banach space; fixed point theorem

**MSC:** 39B52; 39B82; 47H10

## 1. Introduction

About eighty years ago, Ulam [1] raised a problem of finding conditions under which there exists an exact additive map near an approximate additive map. An answer to the problem between Banach spaces was given by Hyers [2]. After that, many authors have studied the stability problems. We refer to [3–7] for more information.

One of the most important outcomes of the stability of functional equations is the following theorem.

**Theorem 1.** *Let $X, Y$ be two Banach spaces and $f : X \to Y$ be a mapping. Consider the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \le c(\|x\|^p + \|y\|^p), \tag{1}$$

*where $c > 0$ and $p \neq 1$ are real constants. Then the following statements hold.*

(i) *If $p \ge 0$ and (1) holds for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that*

$$\|f(x) - T(x)\| \le \frac{c}{|1 - 2^{p-1}|}\|x\|^p \text{ for all } x \in X.$$

(ii) *If $p < 0$ and (1) holds for all $x, y \in X \setminus \{0\}$, then $f$ is additive.*

The case $p = 0$ is reduced to the stability by Hyers [2]. The case $0 < p < 1$ is due to Aoki [8] (see also [9]). Gajda [10] showed the stability of the Cauchy functional equation for $p > 1$. Statement (ii) was proved first by Lee [11] and Brzdęk [12] showed it on a restricted domain.

Let $G$ be an additive abelian group and let $Y$ be a linear space. We say that $f : G \to Y$ satisfies the *Fréchet equation* if

$$f(x+y) + f(y+z) + f(x+z) = f(x+y+z) + f(x) + f(y) + f(z), \quad x, y, z \in G. \tag{2}$$

The above equation was introduced by the classical equality

$$\|x+y\|^2 + \|y+z\|^2 + \|x+z\|^2 = \|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2, \quad x, y, z \in Y \tag{3}$$

in real or complex inner product spaces $Y$. In 1935, Fréchet [13] proved that in a normed space $Y$, (3) is equivalent to the fact that $Y$ is an inner product space.

Recall that a map $q : G \to Y$ is said to be *quadratic* if it satisfies

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \qquad x, y \in G.$$

It is known that every solution of (2) is of the form $f = a + q$, where $a : G \to Y$ is an additive mapping and $q : G \to Y$ is a quadratic mapping. (see, e.g., [14]). The stability of (2) in Banach spaces has been investigated by many authors (see, e.g., [15–22]). In particular, Bahyrycz et al. [15], Brzdęk et al. [16] and Malejki [21] have studied the generalized Fréchet functional equations with constant coefficients using a fixed point theorem in metric spaces by Brzdęk et al. [23].

In recent studies of the stability of functional equations, fixed point theorems play important roles. Dung and Hang [24] generalized the fixed point theorem of Brzdęk et al. [23] in metric spaces to $b$-metric spaces, and hence to quasi-Banach spaces. By using that fixed point theorem, they obtained a hyperstability of general linear equations. For more information on the stability of functional equations and fixed point theorems, we refer to [25,26].

Several authors have studied the stability of many functional equations in quasi-Banach spaces (see, e.g., [24,27–32]).

The purpose of this paper is to obtain the (hyper)stability of (2) by using the fixed point theorem of Dung and Hang [24].

This paper is organized as follows.

In Section 2, we consider the hyperstability of (2) on a restricted domain. More precisely, let $X$ be a nonempty subset of a quasi-normed linear space and $Y$ be a quasi-Banach space. We say that a function $f : X \to Y$ satisfies the *Fréchet equation on* $X$ if

$$f(x+y) + f(y+z) + f(x+z) = f(x+y+z) + f(x) + f(y) + f(z)$$

for all $x, y, z \in X$ such that $x+y+z, x+y, y+z, x+z \in X$. We will show that Fréchet equation on $X$ is hyperstable; that is, if $f : X \to Y$ satisfies

$$\|f(x+y) + f(y+z) + f(x+z) - f(x+y+z) - f(x) - f(y) - f(z)\| \le c(\|x\|^p + \|y\|^p + \|z\|^p) \tag{4}$$

for all $x, y, z$ in some set $X$, $p < 0$ and $c \ge 0$, then $f$ must satisfy the Fréchet equation on $X$.

In Section 3, we consider the Hyers–Ulam stability results of (2) in quasi-Banach spaces. Especially, we investigate (4) for various $p \ge 0$.

In Section 4, we show that the Fréchet equation is not stable for $p = 1, 2$.

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{R}_+ := [0, \infty)$ and $A^B$ denotes the family of all functions mapping a set $B \ne \emptyset$ into a set $A \ne \emptyset$.

We recall some relevant notions of quasi-Banach spaces:

**Definition 1.** *Let $X$ be a nonempty set, $\kappa \ge 1$ and $d : X \times X \to \mathbb{R}_+$ be a function such that for all $x, y, z \in X$,*

1. $d(x,y) = 0$ *if and only if* $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,z) \le \kappa(d(x,y) + d(y,z))$.

*Then*

1. *$d$ is called a b-metric on $X$ and $(X, d, \kappa)$ is called a b-metric space.*

2. *The sequence $\{x_n\}$ is convergent to $x$ in $(X, d, \kappa)$ if $\lim_{n\to\infty} d(x_n, x) = 0$.*
3. *The sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.*
4. *The space $(X, d, \kappa)$ is said to be complete if each Cauchy sequence is convergent.*

**Definition 2.** *Let $X$ be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\kappa \geq 1$ and $\|\cdot\| : X \times X \to \mathbb{R}_+$ be a function such that for all $x, y, z \in X$ and all $a \in \mathbb{K}$,*

1. *$\|x\| = 0$ if and only if $x = 0$.*
2. *$|ax\| = |a|\|x\|$.*
3. *$\|x + y\| \leq \kappa(\|x\| + \|y\|)$.*

*Then $\|\cdot\|$ is called a quasi-norm on $X$ and $(X, \|\cdot\|, \kappa)$ is called a quasi-normed space.*

Note that if $(X, \|\cdot\|, \kappa)$ is a quasi-normed space, letting $d(x, y) = \|x - y\|$ for $x, y \in X$, $(X, d, \kappa)$ becomes a $b$-metric space. Complete quasi-normed spaces are called *quasi-Banach spaces*.

A quasi-norm $\|\cdot\|$ is called a $p$-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X.$$

In this case, we call the quasi-Banach space a *$p$-Banach space*. It is well-known that each quasi-norm is equivalent to some $p$-norm (see [33]). Since working with $p$-norms is much easier than working with quasi-norms, authors often restrict their attention to $p$-norms when they study the stability of functional equations between quasi-Banach spaces. However we will investigate the stability in quasi-Banach spaces with quasi-norms.

One of the most important class of quasi-Banach spaces is the class of $L^p(\mu)$ for $0 < p < 1$ with the usual quasi-norm

$$\|f\|_p = \left( \int |f|^p d\mu \right)^{\frac{1}{p}}.$$

In this case,

$$\|f + g\|_p \leq 2^{\frac{1}{p}-1} \left( \|f\|_p + \|g\|_p \right), \quad f, g \in L^p(\mu).$$

Hence, taking a particular case of $L^p(\mu)$, we have the following example.

**Example 1.** *For $(x_1, x_2) \in \mathbb{R}^2$, define the quasi-norm of $(x_1, x_2)$ by*

$$\|(x_1, x_2)\| = \left( \sqrt{|x_1|} + \sqrt{|x_2|} \right)^2.$$

*Then $(\mathbb{R}^2, \|\cdot\|, 2)$ is a quasi-Banach space.*

The following lemma can be seen easily from 3 of Definition 2.

**Lemma 1** ([31])**.** *Let $(X, \|\cdot\|, \kappa)$ be a quasi-normed space and $x_1, \ldots, x_{2n+1} \in X$. Then*

$$\left\| \sum_{j=1}^{2n} x_j \right\| \leq \kappa^n \sum_{j=1}^{2n} \|x_j\|, \qquad \left\| \sum_{j=1}^{2n+1} x_j \right\| \leq \kappa^{n+1} \sum_{j=1}^{2n+1} \|x_j\|.$$

## 2. Hyperstability of (2) on a Restricted Domain

The following theorem, which is a generalization of the outcome of [23], is the main tool in proving the results of this paper.

**Theorem 2** ([24], Corollary 2.2)**.** *Suppose that*

1. *$X$ is a nonempty set, $(Y, \|\cdot\|, \kappa)$ is a quasi-Banach space, and $J : Y^X \to Y^X$ is a given function.*
2. *There exist $f_1, \dots, f_n : X \to X$ and $L_1, \dots, L_n : X \to \mathbb{R}_+$ such that for every $\xi, \mu \in Y^X$, $x \in X$,*

$$\|J\xi(x) - J\mu(x)\| \leq \sum_{i=1}^{n} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|. \tag{5}$$

3. *There exist $\epsilon : X \to \mathbb{R}_+$ and $\phi : X \to Y$ such that for all $x \in X$,*

$$\|J\phi(x) - \phi(x)\| \leq \epsilon(x). \tag{6}$$

4. *For every $x \in X$ and $\theta = \log_{2\kappa} 2$,*

$$\epsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \epsilon)^\theta (x) < \infty, \tag{7}$$

*where*

$$\Lambda\delta(x) = \sum_{i=1}^{n} L_i(x)\delta(f_i(x)) \tag{8}$$

*for all $\delta \in \mathbb{R}_+^X$ and $x \in X$.*

*Then we have*

1. *For every $x \in X$, the limit*

$$\lim_{n\to\infty} J^n\phi(x) = \psi(x), \tag{9}$$

*exists and the function $\psi : X \to Y$ so defined is a fixed point of $J$ satisfying*

$$\|\phi(x) - \psi(x)\|^\theta \leq 4\epsilon^*(x) \tag{10}$$

*for all $x \in X$.*

2. *For every $x \in X$, if*

$$\epsilon^*(x) \leq \left( M \sum_{n=0}^{\infty} (\Lambda^n \epsilon)(x) \right)^\theta < \infty \tag{11}$$

*for some positive real number $M$, then the fixed point of $J$ satisfying (10) is unique.*

Now we state the main result of this section. Note that the domain of the mapping $f$ is a subset of a quasi-normed space that is not necessarily a commutative group. We adapt some ideas from [34,35]. Throughout this section, we denote $X := X_0 \setminus \{0\}$ for a subset $(0 \in) X_0$ of a quasi-Banach space.

**Theorem 3.** *Assume that $X_0$ is a nonempty subset of a quasi-normed space such that $0 \in X_0 = -X_0$ and there exists $n_0 \in \mathbb{N}$ with $nx \in X_0$ for all $x \in X_0$ and for all $n \geq n_0$. Let $(Y, \|\cdot\|, \kappa)$ be a quasi-Banach space, $p < 0$ and $c \geq 0$. If $f : X_0 \to Y$ is a mapping that satisfies $f(0) = 0$ and*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \leq c(\|x\|^p + \|y\|^p + \|z\|^p) \tag{12}$$

*for all $x, y, z \in X$ such that $x+y+z, x+y, y+z, x+z \in X_0$, then $f$ satisfies the Fréchet equation on $X$.*

**Proof.** First observe that $\lim_{m\to\infty} m^p = 0$, so there exists an integer $m_0$ such that

$$\kappa^2\left(2(m+1)^p + 2m^p + (2m+1)^p\right) < 1 \text{ for } m \geq m_0.$$

Let us fix $m \geq \max\{n_0, m_0\}$. Replacing $(x, y, z)$ with $\big((m+1)x, mx, -mx\big)$ in (12), we have

$$\big\|2f\big((m+1)x\big) + f(mx) + f(-mx) - f\big((2m+1)x\big) - f(x)\big\| \leq c\big((m+1)^p + 2m^p\big)\|x\|^p \tag{13}$$

for all $x \in X$.

Consider the mappings $J : Y^X \to Y^X$ and $\epsilon : X \to \mathbb{R}_+$ given by

$$J\xi(x) = 2\xi\big((m+1)x\big) + \xi(mx) + \xi(-mx) - \xi\big((2m+1)x\big), \quad \xi \in Y^X,\ x \in X,$$

and

$$\epsilon(x) = c\big((m+1)^p + 2m^p\big)\|x\|^p, \quad x \in X.$$

The inequality (13) then becomes

$$\|Jf(x) - f(x)\| \leq \epsilon(x), \quad x \in X,$$

so that (6) holds true. For every $\xi, \mu \in Y^X$ and $x \in X$, we have by Lemma 1

$$\begin{aligned}
&\|J\xi(x) - J\mu(x)\| \\
&\leq \kappa^2 \left(2\left\|(\xi-\mu)\big((m+1)x\big)\right\| + \|(\xi-\mu)(mx)\| + \|(\xi-\mu)(-mx)\| + \left\|(\xi-\mu)\big((2m+1)x\big)\right\|\right) \\
&= \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i x)\|,
\end{aligned}$$

so that $J$ satisfies (5) with $f_1(x) = (m+1)x, f_2(x) = mx, f_3(x) = -mx, f_4(x) = (2m+1)x, L_1(x) = 2\kappa^2$, and $L_2(x) = L_3(x) = L_4(x) = \kappa^2$.

Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be given by

$$\Lambda\eta(x) = 2\kappa^2\eta\big((m+1)x\big) + \kappa^2\eta(mx) + \kappa^2\eta(-mx) + \kappa^2\eta\big((2m+1)x\big), \quad \eta \in \mathbb{R}_+^X,\ x \in X. \tag{14}$$

Then

$$\begin{aligned}
\Lambda\epsilon(x) &= \kappa^2\big(2\epsilon((m+1)x) + \epsilon(mx) + \epsilon(-mx) + \epsilon((2m+1)x)\big) \\
&= \kappa^2\big(2(m+1)^p + 2m^p + (2m+1)^p\big)\epsilon(x), \quad x \in X.
\end{aligned} \tag{15}$$

Since $\Lambda$ is linear, we have by induction

$$\Lambda^n\epsilon(x) = \left[\kappa^2\big(2(m+1)^p + 2m^p + (2m+1)^p\big)\right]^n \left[c\big((m+1)^p + 2m^p\big)\|x\|^p\right], \quad n \in \mathbb{N}, x \in X. \tag{16}$$

Hence, noting that $0 < \theta = \log_{2\kappa} 2 \leq 1$, it follows that

$$\begin{aligned}
\epsilon^*(x) &= \sum_{n=0}^{\infty} (\Lambda^n\epsilon)^\theta(x) \\
&= \sum_{n=0}^{\infty} \left[\kappa^2\big(2(m+1)^p + 2m^p + (2m+1)^p\big)\right]^{n\theta} \left[c\big((m+1)^p + 2m^p\big)\|x\|^p\right]^\theta \\
&= \frac{\left[c\big((m+1)^p + 2m^p\big)\|x\|^p\right]^\theta}{1 - \left[\kappa^2\left(2(m+1)^p + 2m^p + (2m+1)^p\right)\right]^\theta}, \quad x \in X.
\end{aligned} \tag{17}$$

Thus, by Theorem 2, there is a solution $F : X \to Y$ of the equation

$$2F\big((m+1)x\big) + F(mx) + F(-mx) - F\big((2m+1)x\big) = F(x)$$

such that

$$\|f(x)-F(x)\|^\theta \le \frac{4\left[c\big((m+1)^p+2m^p\big)\|x\|^p\right]^\theta}{1-\left[\kappa^2\left(2(m+1)^p+2m^p+(2m+1)^p\right)\right]^\theta}, \quad x\in X. \tag{18}$$

Moreover,

$$F(x)=\lim_{n\to\infty} J^n f(x), \quad x\in X. \tag{19}$$

To prove that $F$ satisfies the Fréchet equation on $X$, observe that

$$\begin{aligned}&\|J^n f(x+y+z)+J^n f(x)+J^n f(y)+J^n f(z)-J^n f(x+y)-J^n f(y+z)-J^n f(x+z)\|\\ &\qquad \le c\left[\kappa^2\big(2(m+1)^p+2m^p+(2m+1)^p\big)\right]^n\left(\|x\|^p+\|y\|^p+\|z\|^p\right)\end{aligned} \tag{20}$$

for all $x,y,z\in X$ such that $x+y+z, x+y, y+z, x+z\in X$. In fact, this can be obtained from (12) by induction on $n\in\mathbb{N}$.

Letting $n\to\infty$ in (20), it follows from (19) that

$$F(x+y+z)+F(x)+F(y)+F(z)-F(x+y)-F(y+z)-F(x+z)=0$$

for all $x,y,z\in X$ such that $x+y+z, x+y, y+z, x+z\in X$.

Until now, we have proved that for every integer $m\ge\max\{n_0,m_0\}$, there exists a mapping $F_m: X\to Y$ satisfying

$$F_m(x+y+z)+F_m(x)+F_m(y)+F_m(z)-F_m(x+y)-F_m(y+z)-F_m(x+z)=0,$$

for all $x,y,z\in X$ such that $x+y+z, x+y, y+z, x+z\in X$, and

$$\|f(x)-F_m(x)\|^\theta \le \frac{4\left[c\big((m+1)^p+2m^p\big)\|x\|^p\right]^\theta}{1-\left[\kappa^2\left(2(m+1)^p+2m^p+(2m+1)^p\right)\right]^\theta} \tag{21}$$

for all $x\in X$.

Now, we show that $F_m=F_k$ for all $m,k\ge\max\{m_0,n_0\}$. Fix $m,k\ge\max\{m_0,n_0\}$ and denote $\epsilon_m(x)=c\big((m+1)^p+2m^p\big)\|x\|^p$ and $\epsilon_k(x)=c\big((k+1)^p+2k^p\big)\|x\|^p$ for all $x\in X$.

By (21), we get

$$\begin{aligned}&\|F_m(x)-F_k(x)\|\\ &\le \frac{\kappa 4^{\frac{1}{\theta}}\epsilon_m(x)}{\left[1-\left[\kappa^2\left(2(m+1)^p+2m^p+(2m+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}}\\ &\quad + \frac{\kappa 4^{\frac{1}{\theta}}\epsilon_k(x)}{\left[1-\left[\kappa^2\left(2(k+1)^p+2k^p+(2k+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}}.\end{aligned} \tag{22}$$

Noting that $F_m$ and $F_k$ are fixed points of $J$, and $\Lambda$ is linear, we have by (16) and (22)

$$\begin{aligned}
&\|F_m(x) - F_k(x)\| \\
&= \|J^n F_m(x) - J^n F_k(x)\| \\
&\leq \frac{\kappa 4^{\frac{1}{\theta}} \Lambda^n \epsilon_m(x)}{\left[1 - \left[\kappa^2 \left(2(m+1)^p + 2m^p + (2m+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}} \\
&\qquad + \frac{\kappa 4^{\frac{1}{\theta}} \Lambda^n \epsilon_k(x)}{\left[1 - \left[\kappa^2 \left(2(k+1)^p + 2k^p + (2k+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}} \\
&= \frac{\kappa 4^{\frac{1}{\theta}} \left[\kappa^2 \left(2(m+1)^p + 2m^p + (2m+1)^p\right)\right]^n \epsilon_m(x)}{\left[1 - \left[\kappa^2 \left(2(m+1)^p + 2m^p + (2m+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}} \\
&\qquad + \frac{\kappa 4^{\frac{1}{\theta}} \left[\kappa^2 \left(2(k+1)^p + 2k^p + (2k+1)^p\right)\right]^n \epsilon_k(x)}{\left[1 - \left[\kappa^2 \left(2(k+1)^p + 2k^p + (2k+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $F_m = F_k$ and we denote it by $F := F_m = F_k$. Then, by (21), it follows that

$$\|f(x) - F(x)\| \leq \frac{4^{\frac{1}{\theta}} c\left((m+1)^p + 2m^p\right)\|x\|^p}{\left[1 - \left[\kappa^2 \left(2(m+1)^p + 2m^p + (2m+1)^p\right)\right]^\theta\right]^{\frac{1}{\theta}}} \tag{23}$$

for all $x \in X$. Since $p < 0$, the right hand side of (23) tends to zero as $m \to \infty$. Hence, we conclude that $f(x) = F(x)$ for all $x \in X$. Therefore, $f$ satisfies the Fréchet equation on $X$, completing the proof. □

Notice that the assumption of unboundedness of $X$ is indispensable.

**Example 2.** *Let $X_0 = [-1, 1]$, $\mathbb{R}^2$ be the quasi-Banach space in Example 1 and $f : X_0 \to \mathbb{R}^2$ be defined by $f(x) = (|x|, 0)$, $x \in X_0$. Then for all $x, y, z \in X$ such that $x + y + z, x + y, y + z, x + z \in X$,*

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(x + z)\| \leq 3(|x|^p + |y|^p + |z|^p)$$

*for $p < 0$. However $f$ does not satisfy the Fréchet equation on $X$.*

In the case of $p \geq 0$, the Fréchet equation is not hyperstable.

**Remark 1.** *Let $X = \mathbb{R} \setminus [-1, 1]$, $Y$ be a quasi-Banach space and let $f : X \to Y$ be a constant function $f(x) = c, x \in X$ for some $c \neq 0 \in Y$ and $p \geq 0$. Then $f$ satisfies*

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(x + z)\| \leq \|c\|(|x|^p + |y|^p + |z|^p)$$

*for all $x, y, z \in X$ such that $x + y + z, x + y, y + z, x + z \in X$. However $f$ does not satisfy the Fréchet equation on $X$.*

## 3. Stability of (2) on Abelian Groups

In this section, we investigate the stability of (2) and as byproducts we get stability results of (4) for various $p \geq 0$ similar to Theorem 1 (see Corollaries 2, 3 and 4 below).

**Lemma 2.** *Let $G$ be an additive abelian group and $Y$ be a linear space. If $f : G \to Y$ is a mapping satisfying (2) with $f(2x) = 2f(x)$ for all $x \in G$, then $f$ is additive.*

**Proof.** We first note that $f(0) = 0$. Replacing $(x, y, z)$ with $(x, x, -x)$ in (2), we have

$$3f(x) + f(-x) = f(2x) = 2f(x), \quad x \in G,$$

and hence,

$$f(-x) = -f(x), \quad x \in G.$$

Replacing $(x, y, z)$ with $(x, y, -y)$ in (2), we get

$$f(x+y) + f(x-y) = 2f(x), \quad x, y \in G. \tag{24}$$

Replacing $(x, y)$ with $(y, x)$ in (24), we have

$$f(x+y) + f(y-x) = 2f(y), \quad x, y \in G. \tag{25}$$

Adding (24) and (25), we obtain

$$f(x+y) = f(x) + f(y), \quad x, y \in G.$$

□

**Lemma 3.** *Let $G$ be an additive abelian group and $Y$ be a linear space. If $f : G \to Y$ is a mapping satisfying (2) with $f(2x) = 4f(x)$ for all $x \in G$, then $f$ is quadratic.*

**Proof.** Replacing $(x, y, z)$ with $(x, x, -x)$ in (2), we have

$$3f(x) + f(-x) = f(2x) = 4f(x), \quad x \in G,$$

and hence,

$$f(-x) = f(x), \quad x \in G.$$

Replacing $(x, y, z)$ with $(x, y, -y)$ in (2), we get

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) = 2f(x) + 2f(y), \quad x, y \in G.$$

Hence, by definition, $f$ is quadratic. □

**Theorem 4.** *Assume that $(X, +)$ is an abelian group, $(Y, \|\cdot\|, \kappa)$ is a quasi-Banach space and $L < 1$ is a real number such that $0 < \frac{\kappa}{3}(2L+1) < 1$. Let $\varphi : X^3 \to \mathbb{R}_+$ be a function such that*

$$\varphi(2x, 2y, 2z) \le 2L\varphi(x, y, z), \quad \varphi(x, y, z) = \varphi(-x, -y, -z), \quad x, y, z \in X.$$

*If $f : X \to Y$ is a mapping that satisfies $f(0) = 0$ and*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \le \varphi(x, y, z) \tag{26}$$

*for all $x, y, z \in X$, then there exists a unique mapping $g : X \to Y$ satisfying (2) such that*

$$\|f(x) - g(x)\| \le \frac{4^{\frac{1}{\theta}}}{\left[3^\theta - \kappa^\theta (2L+1)^\theta\right]^{\frac{1}{\theta}}} \varphi(x, x, -x), \quad x \in X. \tag{27}$$

**Proof.** Replacing $(x, y, z)$ with $(x, x, -x)$ in (26), we have

$$\|f(2x) - 3f(x) - f(-x)\| \le \varphi(x, x, -x), \quad x \in X,$$

so that

$$\left\| \frac{1}{3}f(2x) - \frac{1}{3}f(-x) - f(x) \right\| \le \frac{1}{3}\varphi(x, x, -x), \quad x \in X. \tag{28}$$

Consider the mappings $J : Y^X \to Y^X$ and $\epsilon : X \to \mathbb{R}_+$ given by

$$J\xi(x) = \frac{1}{3}\xi(2x) - \frac{1}{3}\xi(-x), \quad \xi \in Y^X,\ x \in X,$$

and

$$\epsilon(x) = \frac{1}{3}\varphi(x, x, -x), \quad x \in X.$$

The inequality (28) becomes

$$\|Jf(x) - f(x)\| \le \epsilon(x), \quad x \in X,$$

so that (6) holds true. For every $\xi, \eta \in Y^X$ and $x \in X$, we have

$$\|J\xi(x) - J\eta(x)\| \le \frac{\kappa}{3}\|\xi(2x) - \eta(2x)\| + \frac{\kappa}{3}\|\xi(-x) - \eta(-x)\|,$$

and hence, $J$ satisfies (5) with $f_1(x) = 2x$, $f_2(x) = -x$ and $L_1(x) = L_2(x) = \frac{\kappa}{3}$.

Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be given by

$$\Lambda\eta(x) = \kappa\left(\frac{1}{3}\eta(2x) + \frac{1}{3}\eta(-x)\right), \quad \eta \in \mathbb{R}_+^X,\ x \in X.$$

Then we have

$$\Lambda\epsilon(x) = \kappa\left(\frac{1}{3}\epsilon(2x) + \frac{1}{3}\epsilon(-x)\right) \le \frac{\kappa}{3}(2L+1)\epsilon(x), \quad x \in X.$$

Note that $\Lambda$ is order-preserving, that is, if $\xi(x) \ge \eta(x)$ for all $x \in X$, then

$$\Lambda\xi(x) = \Lambda\xi(x) - \Lambda\eta(x) + \Lambda\eta(x) = \Lambda(\xi - \eta)(x) + \Lambda\eta(x) \ge \Lambda\eta(x).$$

Hence, we have for all $n \in \mathbb{N}$

$$\Lambda^n\epsilon(x) \le \left(\kappa\frac{2L+1}{3}\right)^n \epsilon(x), \quad x \in X.$$

As $\frac{\kappa(2L+1)}{3} < 1$ and $0 < \theta = \log_{2\kappa} 2 \le 1$, we obtain

$$\begin{aligned}\epsilon^*(x) &= \sum_{n=0}^{\infty} (\Lambda^n\epsilon)^\theta (x) \le \sum_{n=0}^{\infty}\left(\frac{\kappa(2L+1)}{3}\right)^{n\theta}\epsilon^\theta(x) \\ &= \frac{1}{1 - \left(\frac{\kappa(2L+1)}{3}\right)^\theta}\left(\frac{1}{3}\varphi(x, x, -x)\right)^\theta = \frac{1}{3^\theta - \kappa^\theta(2L+1)^\theta}\varphi(x, x, -x)^\theta, \quad x \in X.\end{aligned}$$

Therefore, by Theorem 2, there exists a mapping $g : X \to Y$ such that

$$\begin{aligned} g(x) &= \lim_{n\to\infty} J^n f(x), \qquad x \in X, \\ g(x) &= \frac{1}{3}g(2x) - \frac{1}{3}g(-x), \qquad x \in X, \end{aligned} \tag{29}$$

and

$$\|f(x)-g(x)\|^{\theta} \leq \frac{4}{3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}}\varphi(x,x,-x)^{\theta}, \quad x \in X,$$

from which inequality (27) follows.

Now we show that $g$ satisfies (2). From (26) and the definition of $J$, we have

$$\begin{aligned}
&\|Jf(x+y+z)+Jf(x)+Jf(y)+Jf(z)-Jf(x+y)-Jf(y+z)-Jf(x+z)\| \\
&\leq \frac{\kappa}{3}\varphi(2x,2y,2z)+\frac{\kappa}{3}\varphi(-x,-y,-z) \\
&\leq \frac{\kappa(2L+1)}{3}\varphi(x,y,z), \qquad x \in X.
\end{aligned}$$

By induction, we have for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\|J^n f(x+y+z)+J^n f(x)+J^n f(y)+J^n f(z)-J^n f(x+y)-J^n f(y+z)-J^n f(x+z)\| \\
&\qquad \leq \left(\frac{\kappa(2L+1)}{3}\right)^n \varphi(x,y,z), \qquad x,y,z \in X. \qquad (30)
\end{aligned}$$

Therefore, letting $n \to \infty$ in (30), we get

$$g(x+y+z)+g(x)+g(y)+g(z)-g(x+y)-g(y+z)-g(x+z)=0, \qquad x,y,z \in X.$$

Next, we show the uniqueness of $g$. Assume that $g_1, g_2 : X \to Y$ are mappings satisfying (2) and

$$\|f(x)-g_i(x)\|^{\theta} \leq \frac{4}{3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}}\varphi(x,x,-x)^{\theta}, \quad i=1,2,\ x \in X.$$

Then, by inequality 3 in Definition 2,

$$\|g_1(x)-g_2(x)\| \leq \frac{2\cdot 4^{\frac{1}{\theta}}\kappa}{\left[3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x \in X.$$

Note by (29) that

$$g_i(x)=\frac{1}{3}g_i(2x)-\frac{1}{3}g_i(-x), \quad i=1,2,\ x \in X.$$

Then

$$\begin{aligned}
&\|g_1(x)-g_2(x)\| \\
&= \left\|\frac{1}{3}(g_1(2x)-g_2(2x))-\frac{1}{3}(g_1(-x)-g_2(-x))\right\| \\
&\leq \frac{\kappa}{3}\frac{2\cdot 4^{\frac{1}{\theta}}\kappa}{\left[3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}\right]^{\frac{1}{\theta}}}\varphi(2x,2x,-2x) \\
&\qquad + \frac{\kappa}{3}\frac{2\cdot 4^{\frac{1}{\theta}}\kappa}{\left[3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}\right]^{\frac{1}{\theta}}}\varphi(-x,-x,x) \\
&\leq \frac{\kappa(2L+1)}{3}\frac{2\cdot 4^{\frac{1}{\theta}}\kappa}{\left[3^{\theta}-\kappa^{\theta}(2L+1)^{\theta}\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x \in X.
\end{aligned}$$

Applying the same argument repeatedly, it is easy to show that for all $n \in \mathbb{N}$,

$$\|g_1(x) - g_2(x)\| \leq \left[\frac{\kappa(2L+1)}{3}\right]^n \frac{2 \cdot 4^{\frac{1}{\theta}} \kappa}{\left[3^\theta - \kappa^\theta (2L+1)^\theta\right]^{\frac{1}{\theta}}} \varphi(x, x, -x), \quad x \in X. \tag{31}$$

Letting $n \to \infty$ in (31), we obtain $g_1 = g_2$, as desired. □

Putting $\varphi(x, y, z) \equiv c$, we have the following classical Ulam stability of the functional equation under consideration.

**Corollary 1.** *Assume that $(X, +)$ is an abelian group, $(Y, \|\cdot\|, \kappa)$ is a quasi-Banach space with $\kappa < \frac{3}{2}$ and $c \geq 0$ is a constant. If $f : X \to Y$ is a mapping that satisfies $f(0) = 0$ and*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \leq c$$

*for all $x, y, z \in X$, then there exists a unique mapping $g : X \to Y$ satisfying (2) such that*

$$\|f(x) - g(x)\| \leq \frac{4^{\frac{1}{\theta}} c}{\left[3^\theta - 2^\theta \kappa^\theta\right]^{\frac{1}{\theta}}}, \quad x \in X.$$

**Proof.** We use Theorem 4 applied with $L = \frac{1}{2}$ and $\varphi(x, y, z) = c$ for all $x, y, z \in X$. □

As an example of Theorem 4, we have the following stability of (4) for $0 < p < 1$.

**Corollary 2.** *Let $(X, +)$ be an abelian subgroup of a quasi-normed space and $(Y, \|\cdot\|, \kappa)$ be a quasi-Banach space. Assume that, for some $0 < p < 1$ and some $c > 0$, the mapping $f : X \to Y$ satisfies*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \leq c(\|x\|^p + \|y\|^p + \|z\|^p),$$

*for all $x, y, z \in X$. If $1 \leq \kappa < \frac{3}{2^{p-1}+1}$, then there exists a unique mapping $g : X \to Y$ satisfying (2) such that*

$$\|f(x) - g(x)\| \leq \frac{3 \cdot 4^{\frac{1}{\theta}} c}{\left[3^\theta - \kappa^\theta (2^p + 1)^\theta\right]^{\frac{1}{\theta}}} \|x\|^p, \quad x \in X.$$

**Proof.** Taking $L = 2^{p-1}$ in Theorem 4, we obtain the result. □

Recall that an abelian group $(X, +)$ is called *uniquely* 2*-divisible* if for each $x \in X$, there exists a unique $y \in X$ such that $2y = x$. We denote $y = \frac{x}{2}$.

**Theorem 5.** *Assume that $(X, +)$ is a uniquely 2-divisible abelian group, $(Y, \|\cdot\|, \kappa)$ is a quasi-Banach space and $0 < L < \frac{1}{\kappa}$ is a real number. Let $\varphi : X^3 \to \mathbb{R}_+$ be a function such that*

$$\varphi(x, y, z) \leq \frac{L}{4} \varphi(2x, 2y, 2z), \quad \varphi(x, y, z) = \varphi(-x, -y, -z)$$

*for all $x, y, z \in X$. If $f : X \to Y$ is a mapping that satisfies*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \leq \varphi(x, y, z) \tag{32}$$

*for all $x, y, z \in X$, then there exists a unique mapping $g : X \to Y$ satisfying (2) such that*

$$\|f(x) - g(x)\| \leq \frac{L}{4} \frac{4^{\frac{1}{\theta}}}{\left[1 - (\kappa L)^\theta\right]^{\frac{1}{\theta}}} \varphi(x, x, -x), \quad x \in X. \tag{33}$$

**Proof.** We first note that $f(0) = 0$. Replacing $(x, y, z)$ with $(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2})$ in (32), we have

$$\left\| f(x) - 3f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right), \quad x \in X. \tag{34}$$

Consider the mappings $J : Y^X \to Y^X$ and $\epsilon : X \to \mathbb{R}_+$ given by

$$J\xi(x) = 3\xi\left(\frac{x}{2}\right) + \xi\left(-\frac{x}{2}\right), \quad \xi \in Y^X,\ x \in X,$$

and

$$\epsilon(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right), \quad x \in X.$$

Then inequality (34) becomes

$$\|Jf(x) - f(x)\| \le \epsilon(x), \quad x \in X,$$

so that (6) holds true. For every $\xi, \eta \in Y^X$ and $x \in X$, we have

$$\|J\xi(x) - J\eta(x)\| \le 3\kappa \left\| \xi\left(\frac{x}{2}\right) - \eta\left(\frac{x}{2}\right) \right\| + \kappa \left\| \xi\left(-\frac{x}{2}\right) - \eta\left(-\frac{x}{2}\right) \right\|,$$

so that $J$ satisfies (5) with $f_1(x) = \frac{x}{2}, f_2(x) = -\frac{x}{2}, L_1(x) = 3\kappa$ and $L_2(x) = \kappa$.

Let $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be given by

$$\Lambda\eta(x) = 3\kappa\eta\left(\frac{x}{2}\right) + \kappa\eta\left(-\frac{x}{2}\right), \quad \eta \in \mathbb{R}_+^X,\ x \in X.$$

Then we have

$$\begin{aligned} \Lambda\epsilon(x) &= 3\kappa\epsilon\left(\frac{x}{2}\right) + \kappa\epsilon\left(-\frac{x}{2}\right) = 3\kappa\varphi\left(\frac{x}{2^2}, \frac{x}{2^2}, -\frac{x}{2^2}\right) + \kappa\varphi\left(-\frac{x}{2^2}, -\frac{x}{2^2}, \frac{x}{2^2}\right) \\ &= 4\kappa\varphi\left(\frac{x}{2^2}, \frac{x}{2^2}, -\frac{x}{2^2}\right) = 4\kappa\epsilon\left(\frac{x}{2}\right), \quad x \in X. \end{aligned}$$

By induction on $n$, we get

$$\Lambda^n\epsilon(x) = 4^n\kappa^n\epsilon\left(\frac{x}{2^n}\right), \quad x \in X,$$

and hence

$$\begin{aligned} \epsilon^*(x) &= \sum_{n=0}^{\infty} (\Lambda^n\epsilon)^\theta(x) = \sum_{n=0}^{\infty} (4^n\kappa^n)^\theta \varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, -\frac{x}{2^{n+1}}\right)^\theta \\ &\le \sum_{n=0}^{\infty} 4^{n\theta}\kappa^{n\theta}\left(\frac{L}{4}\right)^{(n+1)\theta} \varphi(x, x, -x)^\theta = \left(\frac{L}{4}\right)^\theta \frac{1}{1-(\kappa L)^\theta}\varphi(x, x, -x)^\theta, \quad x \in X, \end{aligned}$$

so that (7) holds true. Therefore, by Theorem 2, there exists a mapping $g : X \to Y$ such that

$$\begin{aligned} g(x) &= \lim_{n\to\infty} J^n f(x), \quad x \in X, \\ g(x) &= 3g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right), \quad x \in X, \end{aligned} \tag{35}$$

and

$$\|f(x) - g(x)\|^\theta \le \left(\frac{L}{4}\right)^\theta \frac{4}{1-(\kappa L)^\theta}\varphi(x, x, -x)^\theta, \quad x \in X. \tag{36}$$

Inequality (33) follows from (36)

Now we show that $g$ satisfies (2). From (32) and the definition of $J$, we have

$$\begin{aligned}
&\|Jf(x+y+z)+Jf(x)+Jf(y)+Jf(z)-Jf(x+y)-Jf(y+z)-Jf(x+z)\| \\
&\leq 3\kappa\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)+\kappa\varphi\left(-\frac{x}{2},-\frac{y}{2},-\frac{z}{2}\right) \\
&= 4\kappa\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \\
&\leq \kappa L\varphi(x,y,z), \quad x\in X.
\end{aligned}$$

By induction, we have for all $n \in \mathbb{N}$ and $x, y, z \in X$,

$$\begin{aligned}
&\|J^n f(x+y+z)+J^n f(x)+J^n f(y)+J^n f(z)-J^n f(x+y)-J^n f(y+z)-J^n f(x+z)\| \\
&\leq (\kappa L)^n\varphi(x,y,z).
\end{aligned} \tag{37}$$

Therefore, letting $n \to \infty$ in (37), we obtain

$$g(x+y+z)+g(x)+g(y)+g(z)-g(x+y)-g(y+z)-g(x+z)=0, \quad x,y,z\in X.$$

Next, we show the uniqueness of $g$. Assume that $g_1, g_2 : X \to Y$ are mappings satisfying (2) and

$$\|f(x)-g_i(x)\| \leq \frac{L}{4}\frac{4^{\frac{1}{\theta}}}{\left[1-(\kappa L)^\theta\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad i=1,2,\ x\in X.$$

Then

$$\|g_1(x)-g_2(x)\| \leq \frac{\kappa L}{2}\frac{4^{\frac{1}{\theta}}}{\left[1-(\kappa L)^\theta\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x\in X.$$

By (35), we have

$$g_i(x)=3g_i\left(\frac{x}{2}\right)+g_i\left(-\frac{x}{2}\right), \quad i=1,2,\ x\in X.$$

Hence

$$\begin{aligned}
&\|g_1(x)-g_2(x)\| \\
&= 3\kappa\left\|g_1\left(\frac{x}{2}\right)-g_2\left(\frac{x}{2}\right)\right\|+\kappa\left\|g_1\left(-\frac{x}{2}\right)-g_2\left(-\frac{x}{2}\right)\right\| \\
&\leq \frac{\kappa L}{2}\frac{4^{\frac{1}{\theta}}}{\left[1-(\kappa L)^\theta\right]^{\frac{1}{\theta}}}\left(3\kappa\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right)+\kappa\varphi\left(-\frac{x}{2},-\frac{x}{2},\frac{x}{2}\right)\right) \\
&\leq \frac{(\kappa L)^2}{2}\frac{4^{\frac{1}{\theta}}}{\left[1-(\kappa L)^\theta\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x\in X.
\end{aligned}$$

In this way, it is easy to show that for all $n \in \mathbb{N}$,

$$\|g_1(x)-g_2(x)\| \leq \frac{(\kappa L)^n}{2}\frac{4^{\frac{1}{\theta}}}{\left[1-(\kappa L)^\theta\right]^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x\in X. \tag{38}$$

Letting $n \to \infty$ in (38), it follows that $g_1 = g_2$. This completes the proof. □

As an application of Theorem 5, we have the following stability of (4) for $p > 2$.

**Corollary 3.** *Let $(X,+)$ be a uniquely 2-divisible abelian subgroup of a quasi-normed space and $(Y,\|\cdot\|,\kappa)$ be a quasi-Banach space. Assume $f: X \to Y$ is a mapping that satisfies*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(x+z)\| \leq c(\|x\|^p+\|y\|^p+\|z\|^p),$$

*for $p>2$, $c>0$ and for all $x,y,z \in X$. If $1 \leq \kappa < 2^{p-2}$, then there exists a unique mapping $g: X \to Y$ satisfying (2) such that*

$$\|f(x)-g(x)\| \leq \frac{3\cdot 2^{-p}\cdot 4^{\frac{1}{\theta}}\cdot c}{\left[1-(2^{2-p}\kappa)^{\theta}\right]^{\frac{1}{\theta}}}\|x\|^p, \quad x \in X.$$

**Proof.** Taking $L=2^{2-p}$ and applying Theorem 5, we get the result. □

**Theorem 6.** *Assume that $(X,+)$ is a uniquely 2-divisible abelian group, $(Y,\|\cdot\|,\kappa)$ is a quasi-Banach space and $L<1$ is a real number. Let $\varphi: X^3 \to \mathbb{R}_+$ be a function such that*

$$\varphi(2x,2y,2z) \leq 4L\varphi(x,y,z),\ \varphi(x,y,z) \leq \frac{L}{2}\varphi(2x,2y,2z),\ \varphi(x,y,z)=\varphi(-x,-y,-z) \tag{39}$$

*for all $x,y,z \in X$. If $f: X \to Y$ is a mapping that satisfies*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(x+z)\| \leq \varphi(x,y,z)$$

*for all $x,y,z \in X$, then there exist a unique additive mapping $g_o: X \to Y$ and a unique quadratic mapping $g_e: X \to Y$ such that*

$$\|f(x)-g_o(x)-g_e(x)\| \leq \frac{4^{\frac{1}{\theta}-1}\kappa^2(1+2L)}{(1-L^{\theta})^{\frac{1}{\theta}}}\varphi(x,x,-x), \quad x \in X. \tag{40}$$

**Proof.** Note first that $f(0)=0$. Let $f_e: X \to Y$ and $f_o: X \to Y$ be the even and odd parts of $f$, respectively. That is, $f_e(x)=\frac{f(x)+f(-x)}{2}$, $f_o(x)=\frac{f(x)-f(-x)}{2}$ for $x \in X$. Then $f_e(0)=f_o(0)=0$. It is easy to show that

$$\|f_e(x+y+z)+f_e(x)+f_e(y)+f_e(z)-f_e(x+y)-f_e(y+z)-f_e(x+z)\| \leq \kappa\varphi(x,y,z) \tag{41}$$

and analogously,

$$\|f_o(x+y+z)+f_o(x)+f_o(y)+f_o(z)-f_o(x+y)-f_o(y+z)-f_o(x+z)\| \leq \kappa\varphi(x,y,z), \tag{42}$$

for all $x,y,z \in X$. Replacing $(x,y,z)$ with $(x,x,-x)$ in (41), we have

$$\|f_e(2x)-3f_e(x)-f_e(-x)\| \leq \kappa\varphi(x,x,-x),$$

so that

$$\|f_e(2x)-4f_e(x)\| \leq \kappa\varphi(x,x,-x), \quad x \in X.$$

Hence, it follows that

$$\|\frac{1}{4}f_e(2x)-f_e(x)\| \leq \frac{1}{4}\kappa\varphi(x,x,-x), \quad x \in X. \tag{43}$$

As before, define mappings $J, \Lambda$ and $\epsilon$ by

$$J\xi(x) = \frac{1}{4}\xi(2x), \quad \xi \in Y^X, \quad x \in X,$$
$$\Lambda\delta(x) = \frac{1}{4}\delta(2x), \quad \delta \in \mathbb{R}_+^X, \quad x \in X,$$
$$\epsilon(x) = \frac{1}{4}\kappa\varphi(x, x, -x), \quad x \in X.$$

Then, we have by (43)

$$\|Jf_e(x) - f_e(x)\| \le \epsilon(x),$$

so that (6) holds true.

For every $\xi, \eta \in Y^X$ and $x \in X$, we have

$$\|J\xi(x) - J\eta(x)\| \le \frac{1}{4}\|(\xi(2x) - \eta(2x))\|,$$

from which $J$ satisfies (5) with $f_1(x) = 2x$ and $L_1(x) = \frac{1}{4}$. Note that

$$\begin{aligned}\Lambda^n\epsilon(x) &= \frac{1}{4^n}\epsilon(2^n x)\\ &= \frac{1}{4^{n+1}}\kappa\varphi(2^n x, 2^n x, -2^n x)\\ &\le \frac{1}{4^{n+1}}(4L)^n\kappa\varphi(x, x, -x) = \frac{\kappa}{4}L^n\varphi(x, x, -x), \quad x \in X.\end{aligned}$$

Hence, we get

$$\epsilon^*(x) = \sum_{n=0}^{\infty}(\Lambda^n\epsilon)^\theta(x) \le \frac{(\frac{\kappa}{4})^\theta}{1 - L^\theta}\varphi(x, x, -x)^\theta, \quad x \in X,$$

so that (7) holds true. Therefore, by Theorem 2 there exists a mapping $g_e : X \to Y$ such that

$$\begin{aligned} g_e(x) &= \lim_{n\to\infty} J^n f_e(x), \quad x \in X,\\ g_e(x) &= \frac{1}{4}g_e(2x), \quad x \in X,\end{aligned} \tag{44}$$

and

$$\|g_e(x) - f_e(x)\|^\theta \le \frac{4(\frac{\kappa}{4})^\theta}{1 - L^\theta}\varphi(x, x, -x)^\theta, \quad x \in X. \tag{45}$$

Since, by (41)

$$\begin{aligned}&\|J^n f_e(x + y + z) + J^n f_e(x) + J^n f_e(y) + J^n f_e(z) - J^n f_e(x + y)\\ &\qquad - J^n f_e(y + z) - J^n f_e(x + z)\|\\ &= \frac{1}{4^n}\|f_e(2^n(x + y + z)) + f_e(2^n x) + f_e(2^n y) + f_e(e^n z)\\ &\qquad - f_e(2^n(x + y)) - f_e(2^n(y + z)) - f_e(2^n(x + z))\|\\ &\le \frac{\kappa}{4^n}\varphi(2^n x, 2^n y, 2^n z)\\ &\le \frac{\kappa}{4^n}(4L)^n\varphi(x, y, z) = \kappa L^n\varphi(x, y, z), \quad x \in X,\end{aligned}$$

it follows that $g_e$ satisfies (2). Then, on account of Lemma 3 and (44), we infer that $g_e$ is a quadratic mapping.

We apply a similar argument to the mapping $f_o$. Replacing $(x, y, z)$ with $(x, x, -x)$ in (42), we have

$$\|f_o(2x) - 2f_o(x)\| \le \kappa\varphi(x, x, -x), \quad x \in X. \tag{46}$$

Replacing $x$ with $\frac{x}{2}$ in (46), we have

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| \le \kappa\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right), \quad x \in X. \tag{47}$$

Let

$$\begin{aligned} J\xi(x) &= 2\xi\left(\frac{x}{2}\right), \quad \xi \in Y^X, \quad x \in X, \\ \Lambda\delta(x) &= 2\delta\left(\frac{x}{2}\right), \quad \delta \in \mathbb{R}_+^X, \quad x \in X, \\ \epsilon(x) &= \kappa\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right), \quad x \in X. \end{aligned}$$

Then, it follows by (47)

$$\|Jf_o(x) - f_o(x)\| \le \epsilon(x),$$

so that (6) holds true.

For every $\xi, \eta \in Y^X$ and $x \in X$, we have

$$\|J\xi(x) - J\eta(x)\| \le 2\left\|\xi\left(\frac{x}{2}\right) - \eta\left(\frac{x}{2}\right)\right\|,$$

from which $J$ satisfies (5) with $f_1(x) = \frac{x}{2}$ and $L_1(x) = 2$. Note that

$$\begin{aligned} \Lambda^n\epsilon(x) &= 2^n\epsilon\left(\frac{x}{2^n}\right) \\ &= 2^n\kappa\varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, -\frac{x}{2^{n+1}}\right) \\ &\le 2^n\kappa\left(\frac{L}{2}\right)^{n+1}\varphi(x, x, -x) = \frac{\kappa L}{2} \cdot L^n\varphi(x, x, -x), \quad x \in X. \end{aligned}$$

Hence

$$\begin{aligned} \epsilon^*(x) &= \sum_{n=0}^{\infty}(\Lambda^n\epsilon)^\theta(x) \le \sum_{n=0}^{\infty}\left(\frac{\kappa L}{2}\right)^\theta \cdot L^{n\theta}\varphi(x, x, -x)^\theta \\ &= \left(\frac{\kappa L}{2}\right)^\theta \frac{1}{1 - L^\theta}\varphi(x, x, -x)^\theta, \quad x \in X, \end{aligned}$$

so that (7) holds true. Therefore, by Theorem 2, there exists a mapping $g_o : X \to Y$ such that

$$\begin{aligned} g_o(x) &= \lim_{n\to\infty} J^n f_o(x), \quad x \in X, \\ g_o(x) &= 2g_o\left(\frac{x}{2}\right), \quad x \in X, \end{aligned} \tag{48}$$

and

$$\|g_o(x) - f_o(x)\|^\theta \le \left(\frac{\kappa L}{2}\right)^\theta \frac{4}{1 - L^\theta}\varphi(x, x, -x)^\theta, \quad x \in X. \tag{49}$$

Since, by (42)

$$\begin{aligned}
&\|J^n f_o(x+y+z) + J^n f_o(x) + J^n f_o(y) + J^n f_o(z) - J^n f_o(x+y) \\
&\qquad - J^n f_o(y+z) - J^n f_o(x+z)\| \\
&= 2^n \left\| f_o\left(\frac{x+y+z}{2^n}\right) + f_o\left(\frac{x}{2^n}\right) + f_o\left(\frac{y}{2^n}\right) + f_o\left(\frac{z}{2^n}\right) \right. \\
&\qquad \left. - f_o\left(\frac{x+y}{2^n}\right) - f_o\left(\frac{y+z}{2^n}\right) - f_o\left(\frac{x+z}{2^n}\right) \right\| \\
&\le 2^n \kappa \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \le 2^n \kappa \left(\frac{L}{2}\right)^n \varphi(x,y,z) = \kappa L^n \varphi(x,y,z)
\end{aligned}$$

for all $x, y, z \in X$, it follows that $g_o$ satisfies (2). Then by Lemma 2 and (48), we infer that $g_o$ is an additive mapping. Thus $g = g_e + g_o$ also satisfies (2).

By (45) and (49), we obtain

$$\begin{aligned}
\|f(x) - g(x)\| &\le \kappa(\|f_e(x) - g_e(x)\| + \|f_o(x) - g_o(x)\|) \\
&\le \kappa \left[ \frac{4^{\frac{1}{\theta}} \frac{\kappa}{4}}{(1-L^\theta)^{\frac{1}{\theta}}} + \left(\frac{\kappa L}{2}\right) \frac{4^{\frac{1}{\theta}}}{(1-L^\theta)^{\frac{1}{\theta}}} \right] \varphi(x,x,-x) \\
&= \frac{4^{\frac{1}{\theta}-1}\kappa^2(1+2L)}{(1-L^\theta)^{\frac{1}{\theta}}} \varphi(x,x,-x), \quad x \in X,
\end{aligned}$$

as desired. Finally, we show the uniqueness. Assume there exists another additive mapping $g_o' : X \to Y$ and a quadratic mapping $g_e' : X \to Y$ such that

$$\|f(x) - g_o'(x) - g_e'(x)\| \le \frac{4^{\frac{1}{\theta}-1}\kappa^2(1+2L)}{(1-L^\theta)^{\frac{1}{\theta}}} \varphi(x,x,-x), \quad x \in X.$$

Letting $\alpha := \frac{4^{\frac{1}{\theta}-1}\kappa^2(1+2L)}{(1-L^\theta)^{\frac{1}{\theta}}}$, and taking the even part of the mapping $f - g_o - g_e$ (resp. $f - g_o' - g_e'$), we have from (39)

$$\|f_e(x) - g_e(x)\| \le \kappa\alpha\varphi(x,x,-x) \text{ and } \|f_e(x) - g_e'(x)\| \le \kappa\alpha\varphi(x,x,-x), \quad x \in X.$$

Then

$$\begin{aligned}
\|g_e(x) - g_e'(x)\| &= \frac{1}{4}\|g_e(2x) - g_e'(2x)\| \\
&\le \frac{1}{4}\kappa^2\alpha\varphi(2x,2x,-2x) \le \frac{1}{4}\kappa^2\alpha \cdot 4L\varphi(x,x,-x) \\
&= L\kappa^2\alpha\varphi(x,x-x), \quad x \in X.
\end{aligned}$$

In this manner, we get for all $n \in \mathbb{N}$

$$\|g_e(x) - g_e'(x)\| \le L^n\kappa^2\alpha\varphi(x,x-x),$$

which goes to zero as $n \to \infty$. Hence $g_e = g_e'$. Similarly, we can show that $g_o = g_o'$. $\square$

Applying Theorem 6, we have the following stability of (4) for $1 < p < 2$.

**Corollary 4.** *Assume that* $(X,+)$ *is a uniquely* 2*-divisible abelian subgroup of a quasi-normed space and* $(Y, \|\cdot\|, \kappa)$ *is a quasi-Banach space. Let the constants* $1 < p < 2$ *and* $c > 0$ *be such that the mapping* $f : X \to Y$ *satisfies*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \leq c(\|x\|^p + \|y\|^p + \|z\|^p),$$

*for all* $x, y, z \in X$*. Then there exist a unique additive mapping* $g_o : X \to Y$ *and a unique quadratic mapping* $g_e : X \to Y$ *such that*

$$\|f(x) - g_o(x) - g_e(x)\| \leq \frac{3c \cdot 4^{\frac{1}{\theta}-1}\kappa^2(1+2L)}{(1-L^\theta)^{\frac{1}{\theta}}}\|x\|^p, \quad x \in X,$$

*where* $L = \max\{2^{p-2}, 2^{1-p}\}$.

## 4. Nonstability of the Fréchet Equation

In this part, we show that the Fréchet equation is not stable for $p \in \{1,2\}$. The following example comes from Gajda [10].

**Example 3.** *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$\phi(x) = \begin{cases} -\alpha, & x \leq -1, \\ \alpha x, & -1 < x < 1, \\ \alpha, & 1 \leq x, \end{cases}$$

*where* $\alpha > 0$*. Then the function* $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \quad x \in \mathbb{R}$$

*satisfies*

$$|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)| \leq 14\alpha(|x| + |y| + |z|),$$

*but there is no function* $g$ *satisfying* (2) *with* $c > 0$ *such that*

$$|f(x) - g(x)| \leq c|x|, \quad x \in \mathbb{R}.$$

**Proof.** Following the proof of [10] with $|x| + |y| + |z|$ instead of $|x| + |y|$, we easily get the result. □

For $p = 2$, we consider the following example coming from [36].

**Example 4.** *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$\phi(x) = \begin{cases} \alpha, & x \in (-\infty, -1] \cup [1, \infty), \\ \alpha x^2, & -1 < x < 1, \end{cases}$$

*where* $\alpha > 0$*. Then the function* $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{4^n}, \quad x \in \mathbb{R}$$

*satisfies*

$$|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(x+z)| \leq 40\alpha(|x|^2+|y|^2+|z|^2),$$

*but there is no function $g$ satisfying* (2) *with $c > 0$ such that*

$$|f(x)-g(x)| \leq c|x|^2, \quad x \in \mathbb{R}.$$

**Proof.** Following the proof of [36] with $x^2+y^2+z^2$ instead of $x^2+y^2$ and using the fact that $f$ is an even function, it is easy to get the result. $\square$

## 5. Conclusions

Using a recently developed fixed point theorem, we have proved the Hyers–Ulam stability of the Fréchet equation in quasi-Banach spaces. We also have shown the hyper-stability of the equation on a restricted domain. The method and results in this paper extend the existing ones in the literature mentioned in the Introduction.

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