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On Some New Results in Graphical Rectangular b-Metric Spaces

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Abstract: In this paper, we provide an approach to establish the Banach contraction principle (for the case $\lambda \in [0,1)$), Edelstein, Reich, and Meir–Keeler type contractions in the context of graphical rectangular b-metric space. The obtained results not only enrich and improve recent fixed point theorems of this new metric spaces but also provide positive answers to the questions raised by Mudasir Younis et al. (J. Fixed Point Theory Appl., doi:10.1007/s11784-019-0673-3, 2019).

Keywords: graphical rectangular b-metric space; Banach G-contraction; Edelstein G-contraction; Meir–Keeler G-contraction; Reich G-contraction

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1. Introduction

The Banach contraction principle [1] plays a central role in the literature ON the fixed point theory. This result has been generalized by many authors by using different types of contraction mappings in various metric spaces. In this process of generalization, Branciari [2] used quadrilateral inequality instead of the triangular inequality of metric space and gave the concept of rectangular metric space. In 2016, Shukla et al. [3] introduced a graphical version of a metric space, named as graphical metric space, in which the triangle inequality holds whenever there is a path between two elements that contain at least one intermediate element on that path. Motivated by Shukla [3], Mudasir Younis et al. [4] introduced the idea of a graphical rectangular b-metric space and proved the analogous result of Banach contraction theorem (for the case $\lambda \in [0, \frac{1}{s})$) in the aforesaid space. At the end of paper [4], the authors proposed the following questions:

- Is it possible to define a graphical (G, G')-contraction for the case $\lambda \in [\frac{1}{s}, 1)$ and prove the equivalent result of the Banach contraction principle?
- Is it possible to establish the equivalent results of Edelstein [5], Hardy–Roger [6], Kannan [7] , Meir–Keelar [8], and Reich [9] type contractions in GR_bMS?

We observe that the authors [4] used the condition that there is a path of length l ($l \in \mathbb{N}$) between the first two terms of the Picard sequence to prove their main theorem. However, due to graphical rectangular inequality, it is not possible to prove the Cauchyness of the Picard sequence having a path of even length between its first two terms.

Following this direction of research, we establish the Banach contraction principle in which we extend the range of the contraction constant λ to the case $\frac{1}{s} \leq \lambda < 1$. Moreover, we provide positive answers to the question of the existence and uniqueness of a fixed point for Edelstein, Meir–Keeler, and Reich type contraction in the aforesaid space.

2. Mathematical Preliminaries

Let X be a non-null set; a graph $G := (\mathfrak{U}(G), \mathfrak{E}(G))$ is said to be associated with X, whenever the set of vertices of the graph G is equal to set X, i.e., $\mathfrak{U}(G) = X$ and the set of edges of the graph G contains all the self loops on each vertices of the graph, i.e., $\mathfrak{E}(G) \supseteq \Delta$, where $\Delta := \{(x, y) \in X \times X : x = y\}$. In a graph G, a directed path from x to y of length n is a sequence of $n + 1$ distinct vertices $\{v_i\}_{i=0}^{n}$ such that $v_0 = x$, $v_n = y$ and $(v_{i-1}, v_i) \in \mathfrak{E}(G)$ for each $i = 1, 2, \ldots, n$. A short notation $(xPy)_G$ is used for a path from x to y in a graph G. However, for some $z \in X$, $z \in (xPy_G)$ means that z lies on the path from x to y. A sequence $\{x_n\}$ is said to be G-termwise Connected (G-TWC) if $(x_n P x_{n+1})_G$ for all $n \in \mathbb{N}$. For a mapping $A : X \to X$, a sequence $\{x_n\}$ is said to be A-Picard sequence if $Ax_n = x_{n+1}$ and $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$

$$[x]_G^n := \{y \in X : \ (xPy)_G \text{ of length n}\}$$

To avoid repetition, we assume the same terminology, notations, and basic facts as having been utilized in [4]. For more detail, one can also refer to [3,10,11]. In this paper, consider all the graphs are directed unless otherwise stated and $Fix(A)$ denotes the set of fixed points for a mapping $A : X \to X$.

Definition 1 ([4])**.** *Let X be a non-null set associated with a graph G. A graphical rectangular b-metric on X is a mapping $r_{G_b} : X \times X \to [0, \infty)$, such that, for some $s \geq 1$, it satisfies:*

- *$(GR_bM - 1)$ $r_{G_b}(p, q) = 0$ if and only if $p = q$;*
- *$(GR_bM - 2)$ $r_{G_b}(p, q) = r_{G_b}(q, p)$ for all $p, q \in X$; and*
- *$(GR_bM - 3)$ for all $p, q \in X$ and distinct points $u, v \in X \backslash \{p, q\}$, such that $(pPq)_G$ and $u, v \in (pPq)_G$, then*

$$r_{G_b}(p, q) \leq s[r_{G_b}(p, u) + r_{G_b}(u, v) + r_{G_b}(v, q)] .$$

The pair (X, r_{G_b}) is called a graphical rectangular b-metric space (GR_bMS).

Definition 2 ([4])**.** *Let (X, r_{G_b}) be a GR_bMS. A sequence $\{x_n\}$ in X is:*

(i) *Cauchy if for given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that*

$$r_{G_b}(x_k, x_l) < \epsilon \text{ for all } k, l \geq m,$$

i.e., $\lim_{k,l \to +\infty} r_{G_b}(x_k, x_l) = 0$.

(ii) *Convergent to $z \in X$, if for given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that*

$$r_{G_b}(x_k, z) < \epsilon \text{ for all } k \geq m$$

i.e., $\lim_{k \to \infty} r_{G_b}(x_k, z) = 0$.

Definition 3 ([4])**.** *Let (X, r_{G_b}) be a GR_bMS associated with a graph G. Let there be another graph G' with $\mathfrak{U}(G) = X$. If every G'-termwise connected (G'-TWC) Cauchy sequence in X converges in X, then (X, r_{G_b}) is said to be G'-complete.*

Definition 4 ([4])**.** *Let (X, r_{G_b}) be a GR_bMS associated with a subgraph G' of the graph G with $\Delta \subseteq \mathfrak{E}(G')$. A mapping $A : X \to X$ is a graphical (G, G')-contraction on X, if it satisfies:*

- *$(GC - 1)$ for each $(p, q) \in \mathfrak{E}(G')$ implies $(Ap, Aq) \in \mathfrak{E}(G')$; and*

- $(GC-2)$ *there exists* $\lambda \in [0, \frac{1}{s})$ *such that for all* $p, q \in X$ *with* $(p, q) \in \mathfrak{E}(G')$ *implies*

$$r_{G_b}(Ap, Aq) \leq \lambda r_{G_b}(p, q).$$

Definition 5 ([4])**.** *A graph* $G' = (\mathfrak{U}(G'), \mathfrak{E}(G'))$ *is said to satisfy the property* $(\mathcal{P})$, *if a* G'-*TWC* A-*Picard sequence converging in* X *ensures that there is a limit* ξ *of* $\{x_n\}$ *in* X *and* $m \in \mathbb{N}$, *such that* $(x_k, \xi) \in \mathfrak{E}(G')$ *or* $(\xi, x_k) \in \mathfrak{E}(G')$ *for all* $k > m$.

Definition 6 ([4])**.** *Let* (X, r_{G_b}) *be a* GR_bMS *and* $A : X \to X$ *is a graphical* (G, G')-*contraction mapping. The quadruple* (X, r_{G_b}, G', A) *is said to have the property* S^*, *if for each* G'-*TWC* A-*Picard sequence* $\{x_n\}$ *in* X *has the unique limit.*

The main results in [4] are the followings:

Theorem 1. *Let* (X, r_{G_b}) *be a* G'-*complete* GR_bMS *and* $A : X \to X$ *is a graphical* (G, G')-*contraction mapping, such that:*

(I) *There exists* $x_0 \in X$ *such that* $Ax_0 \in [x_0]^k_{G'}$ *for some* $k \in \mathbb{N}$.
(II) G' *has the property* $(\mathcal{P})$.

Then, the A-*Picard sequence* $\{x_n\}$ *with initial term* $x_0 \in X$ *is* G'-*TWC and converges to* z^* *and* Az^* *in* X.

Theorem 2. *Let* (X, r_{G_b}) *be a* GR_bMS *and a mapping* $A : X \to X$ *holds the conditions of Theorem 1. In addition, if the quadruple* (X, r_{G_b}, G', A) *has the property* S^*, *then* A *has a fixed point.*

Theorem 3. *Let* (X, r_{G_b}) *be a* GR_bMS *and a mapping* $A : X \to X$ *holds the conditions of Theorem 2. In addition, if* X_A *is weakly connected (as a subgraph of* G'*), then* A *has a unique fixed point, where* $X_A = \{x^* \in X : (x^*, Ax^*) \in \mathfrak{E}(G')\}$.

3. Main Results

First, we provide a definition of Banach contraction mapping in GR_bMS for $\lambda \in [0, 1)$.

Definition 7. *Let* G *be a graph associated with* GR_bMS (X, r_{G_b}). *A Banach* G-*contraction (BGC) on* X *is a mapping* $A : X \to X$ *such that:*

- $(BGC-1)$ *for each* $(p, q) \in \mathfrak{E}(G)$ *implies* $(Ap, Aq) \in \mathfrak{E}(G)$; *and*
- $(BGC-2)$ *there exists* $\lambda \in [0, 1)$ *such that for all* $p, q \in X$ *with* $(p, q) \in \mathfrak{E}(G)$ *implies*

$$r_{G_b}(Ap, Aq) \leq \lambda r_{G_b}(p, q).$$

Remark 1. *Before going to prove the next theorem, note that, for* $\lambda \in [0, 1)$ *and* $s \geq 1$, *there exists* $n_0 \in \mathbb{N}$, *such that* $0 \leq \lambda^m s^p < 1$ *for all* $m > n_0$, *where* p *is any fixed positive integer.*

Theorem 4. *Let* (X, r_{G_b}) *be a* G-*complete* GR_bMS *and an injective mapping* A *is BGC on* X, *assumes the following:*

(I) G *has the property* $(\mathcal{P})$; *and*
(II) *there exists* $x_0 \in X$ *with* $Ax_0 \in [x_0]^l_G$ *and* $A^2x_0 \in [x_0]^m_G$, *where* l, m *are odd positive integers.*

Then, the A-*Picard sequence* $\{x_n\}$ *with initial term* $x_0 \in X$ *is* G-*TWC and converges to both* z^* *and* Az^* *in* X.

Proof. Let $x_0 \in X$ be such that $Ax_0 \in [x_0]^l_G$ and $A^2x_0 \in [x_0]^m_G$, where l, m are odd positive integers. Then, there exists a path $\{y_j\}_{j=0}^l$ and $\{w_j\}_{j=0}^m$ such that

$$x_0 = y_0, \ \ Ax_0 = y_l \ \text{ and } \ (y_{j-1}, y_j) \in \mathfrak{E}(G) \text{ for each } j = 1, 2, \ldots, l$$

and

$$x_0 = w_0, \ \ A^2x_0 = w_m \ \text{ and } \ (w_{j-1}, w_j) \in \mathfrak{E}(G) \text{ for each } j = 1, 2, \ldots, m.$$

By $(BGC-1)$, we have

$$(Ay_{j-1}, Ay_j) \in \mathfrak{E}(G) \ \text{ for each } j = 1, 2, \ldots, l.$$

Therefore, $\{Ay_j\}_{j=0}^l$ is a path from $Ay_0 = Ax_0 = x_1$ to $Ay_l = A^2x_0 = x_2$ of length l. Similarly, for all $n \in \mathbb{N}$, $\{A^n y_j\}_{j=0}^l$ is a path from $A^n y_0 = A^n x_0 = x_n$ to $A^n y_l = A^n A x_0 = x_{n+1}$ of length l. Thus, $\{x_n\}$ is G-TWC A-Picard sequence. Since, for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, l$, $(A^n y_{j-1}, A^n y_j) \in \mathfrak{E}(G)$, by $(BGC-2)$, we have

$$r_{G_b}(A^n y_{j-1}, A^n y_j) \leq \lambda r_{G_b}(A^{n-1} y_{j-1}, A^{n-1} y_j) \leq \cdots \leq \lambda^n r_{G_b}(y_{j-1}, y_j) \tag{1}$$

for each $j = 1, 2, \ldots l$.

Similarly, we can show that, for all $n \in \mathbb{N}$, $\{A^n w_j\}_{j=0}^m$ is a path from $A^n w_0 = A^n x_0 = x_n$ to $A^n w_m = A^n A^2 x_0 = x_{n+2}$ of length m.

Since, for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, m$, $(A^n w_{j-1}, A^n w_j) \in \mathfrak{E}(G)$, by $(BGC-2)$, we have

$$r_{G_b}(A^n w_{j-1}, A^n w_j) \leq \lambda r_{G_b}(A^{n-1} w_{j-1}, A^{n-1} w_j) \leq \cdots \leq \lambda^n r_{G_b}(w_{j-1}, w_j) \tag{2}$$

for each $j = 1, 2, \ldots m$.

Based on (GR_bM-3), we obtain

$$\begin{aligned}
r_{G_b}(x_0, x_1) &= r_{G_b}(y_0, y_l) \\
&\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2) + r_{G_b}(y_2, y_l)] \\
&\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2)] + s^2[r_{G_b}(y_2, y_3) + r_{G_b}(y_3, y_4) + r_{G_b}(y_4, y_l)] \\
&\vdots \\
&\leq s[r_{G_b}(y_0, y_1) + r_{G_b}(y_1, y_2)] + s^2[r_{G_b}(y_2, y_3) + r_{G_b}(y_3, y_4)] + \cdots \\
&\quad + s^{\frac{l-1}{2}}[r_{G_b}(y_{l-3}, y_{l-2}) + r_{G_b}(y_{l-2}, y_{l-1}) + r_{G_b}(y_{l-1}, y_l)] \\
&= D_l
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
r_{G_b}(x_0, x_2) &= r_{G_b}(w_0, w_m) \\
&\leq s[r_{G_b}(w_0, w_1) + r_{G_b}(w_1, w_2) + r_{G_b}(w_2, w_m)] \\
&\leq s[r_{G_b}(w_0, w_1) + r_{G_b}(w_1, w_2)] + s^2[r_{G_b}(w_2, w_3) + r_{G_b}(w_3, w_4) + r_{G_b}(w_4, w_m)] \\
&\vdots \\
&\leq s[r_{G_b}(w_0, w_1) + r_{G_b}(w_1, w_2)] + s^2[r_{G_b}(w_2, w_3) + r_{G_b}(w_3, w_4)] + \cdots \\
&\quad + s^{\frac{m-1}{2}}[r_{G_b}(w_{m-3}, w_{m-2}) + r_{G_b}(w_{m-2}, w_{m-1}) + r_{G_b}(w_{m-1}, w_m)] \\
&= D_m.
\end{aligned} \tag{4}$$

In the same way, using the inequalities in Equations (1) and (3), we have

$$\begin{aligned}
r_{G_b}(x_n, x_{n+1}) = r_{G_b}(A^n x_0, A^n x_1) &= r_{G_b}(A^n y_0, A^n y_l) \\
&\leq s[r_{G_b}(A^n y_0, A^n y_1) + r_{G_b}(A^n y_1, A^n y_2)] \\
&\quad + s^2[r_{G_b}(A^n y_2, A^n y_3) + r_{G_b}(A^n y_3, A^n y_4)] + \cdots \\
&\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^n y_{l-3}, A^n y_{l-2}) + r_{G_b}(A^n y_{l-2}, A^n y_{l-1}) \\
&\qquad + r_{G_b}(A^n y_{l-1}, A^n y_l)] \\
&\leq \lambda^n D_l. \qquad (5)
\end{aligned}$$

In addition, using the inequalities in Equations (2) and (4), we get

$$\begin{aligned}
r_{G_b}(x_n, x_{n+2}) = r_{G_b}(A^n x_0, A^n x_2) &= r_{G_b}(A^n w_0, A^n w_m) \\
&\leq s[r_{G_b}(A^n w_0, A^n w_1) + r_{G_b}(A^n w_1, A^n w_2)] \\
&\quad + s^2[r_{G_b}(A^n w_2, A^n w_3) + r_{G_b}(A^n w_3, A^n w_4)] + \cdots \\
&\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^n w_{m-3}, A^n w_{m-2}) + r_{G_b}(A^n w_{m-2}, A^n w_{m-1}) \\
&\qquad + r_{G_b}(A^n w_{m-1}, A^n w_m)] \\
&= \lambda^n D_m. \qquad (6)
\end{aligned}$$

Now, we prove the Cauchyness of the sequence $\{x_n\}$, i.e., for all $p \geq 1$, $r_{G_b}(x_n, x_{n+p}) \to 0$ as $n \to +\infty$.

Case-I: If p is an odd integer, then

$$\begin{aligned}
r_{G_b}(x_n, x_{n+p}) &\leq s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\
&\quad + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\
&\quad + s^{\frac{p-1}{2}}[r_{G_b}(x_{n+p-3}, x_{n+p-2}) + r_{G_b}(x_{n+p-2}, x_{n+p-1}) \\
&\qquad + r_{G_b}(x_{n+p-1}, y_{n+p})].
\end{aligned}$$

By using the inequality in Equation (5), we have

$$\begin{aligned}
r_{G_b}(x_n, x_{n+p}) &\leq s[\lambda^n D_l + \lambda^{n+1} D_l] \\
&\quad + s^2[\lambda^{n+2} D_l + \lambda^{n+3} D_l] + \cdots \\
&\quad + s^{\frac{p-1}{2}}[\lambda^{n+p-3} D_l + \lambda^{n+p-2} D_l + \lambda^{n+p-1} D_l] \\
&\to 0 \ \ as \ \ n \to +\infty.
\end{aligned}$$

Case-II: If p is an even integer, then

$$\begin{aligned}
r_{G_b}(x_n, x_{n+p}) &\leq s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\
&\quad + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\
&\quad + s^{\frac{p-2}{2}}[r_{G_b}(x_{n+p-4}, x_{n+p-3}) + r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\
&\qquad + r_{G_b}(x_{n+p-2}, y_{n+p})].
\end{aligned}$$

By using the inequalities in Equations (5) and (6), we have

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) &\le s[\lambda^n D_l + \lambda^{n+1} D_l] \\ &+ s^2[\lambda^{n+2} D_l + \lambda^{n+3} D_l] + \cdots \\ &+ s^{\frac{p-2}{2}}[\lambda^{n+p-4} D_l + \lambda^{n+p-3} D_l + \lambda^{n+p-2} D_m] \\ &\to 0 \quad as \quad n \to +\infty. \end{aligned}$$

From Case-I and Case-II, one can say that $\{x_n\}$ is a Cauchy sequence.

Since X being G-complete implies $x_n \to z^*$ for some $z^* \in X$, according to Property ($\mathcal{P}$), there exists $k \in \mathbb{N}$ such that $(x_n, z^*) \in \mathfrak{E}(G)$ or $(z^*, x_n) \in \mathfrak{E}(G)$ for all $n > k$.

Suppose that $(x_n, z^*) \in \mathfrak{E}(G)$ for all $n > k$.

Based on $(BGC-2)$, we get

$$r_{G_b}(Ax_n, Az^*) \le \lambda r_{G_b}(x_n, z^*), \quad \text{for all } n > k.$$

This implies,

$$r_{G_b}(Ax_n, Az^*) \to 0 \quad \text{as } n \to +\infty,$$

i.e., $x_{n+1} \to Az^*$. Thus, Az^* is also a limit of $\{x_n\}$.

Similarly, we can prove this for the case $(z^*, x_n) \in \mathfrak{E}(G)$ for all $n > k$. □

Theorem 5. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ hold the conditions of Theorem 4. In addition, if the quadruple (X, r_{G_b}, G, A) has the property S^*, then A has a fixed point.*

Proof. From the proof of Theorem 4, x^* and Ax^* are the limits of a G-TWC A-Picard sequence $\{x_n\}$. By using the property S^*, we have $Ax^* = x^*$. □

Remark 2. *In GR_bMS, a sequence may converges to more than one limit. To overcome this issue, some authors have used those spaces which are Hausdorff to prove fixed point results. However, due to the property S^*, it is not further needful to take Hausdorff GR_bMS.*

Theorem 6. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ hold the conditions of Theorem 5. In addition, they satisfies that, for $z^*, w^* \in Fix(A)$, there is an odd length path between z^* and w^*. Then, A has a unique fixed point.*

Proof. From Theorem 5, we get a fixed point for A.

Suppose $z^*, w^* \in Fix(A)$. Then, by our assumption, there is an odd length (say, k) path between z^* and w^*.

Case-I: Let $k = 1$, i.e., $(z^*, w^*) \in \mathfrak{E}(G)$.

By $(BGC-1)$, $(Az^*, Aw^*) \in \mathfrak{E}(G)$.

Further, based on $(BGC-2)$, we obtain

$$r_{G_b}(Az^*, Aw^*) \le \lambda r_{G_b}(z^*, w^*)$$

implies

$$r_{G_b}(z^*, w^*) \le \lambda r_{G_b}(z^*, w^*),$$

which gives contradiction to the value of λ.

Hence, $z^* = w^*$.

Case-II: Suppose $k > 1$.

Let $\{t_i\}_{i=0}^{k}$ be the odd length path from z^* to w^*, such that $t_0 = z^*$,$t_k = w^*$ and $(t_{i-1}, t_i) \in \mathfrak{E}(G)$ for each $i = 1, 2, \ldots k$, then

$$\begin{aligned} r_{G_b}(z^*, w^*) &= r_{G_b}(A^n z^*, A^n w^*) \\ &\leq s[r_{G_b}(A^n t_0, A^n t_1) + r_{G_b}(A^n t_1, A^n t_2)] \\ &\quad + s^2[r_{G_b}(A^n t_2, A^n t_3) + r_{G_b}(A^n t_3, A^n t_4)] + \cdots \\ &\quad + s^{\frac{k-1}{2}}[r_{G_b}(A^n t_{k-3}, A^n t_{k-2}) + r_{G_b}(A^n t_{l-2}, A^n t_{l-1}) \\ &\qquad + r_{G_b}(A^n t_{k-1}, A^n t_k)] \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

This implies, $z^* = w^*$. □

Now, we give an example with contraction constant λ such that $\frac{1}{s} < \lambda < 1$, which satisfies all the conditions of Theorems 4–6 and it has unique fixed point.

Example 1. *Let $B = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and $X = \{0\} \cup B$. Let G be the graph associated with X such that $\mathfrak{E}(G) = \{(p, q) \in B \times B : p > q\} \cup \{(0, \frac{1}{2^n}) : n \in \mathbb{N}\} \cup \Delta$. A function $r_{G_b} : X \times X \to [0, \infty)$ is defined as:*

$$\begin{aligned} r_{G_b}(x, y) &= 0 \text{ if and only if } x = y, \\ r_{G_b}(0, \frac{1}{2^n}) &= r_{G_b}(\frac{1}{2^n}, 0) = \frac{1}{2^n}\,, \text{ for all } n \in \mathbb{N}, \\ r_{G_b}(\frac{1}{2^m}, \frac{1}{2^n}) &= r_{G_b}(\frac{1}{2^n}, \frac{1}{2^m}) = \frac{1}{2^{k-2}}\,, \text{ if 3 divides } |m-n|, \\ r_{G_b}(\frac{1}{2^m}, \frac{1}{2^n}) &= r_{G_b}(\frac{1}{2^n}, \frac{1}{2^m}) = \frac{1}{2^k}\,, \text{ othrewise}, \end{aligned}$$

where $k = \min\{m, n\}$. Then, (X, r_{G_b}) is GR_bMS with $s = 4$.

Now, we define a mapping $A : X \to X$ such that

$$Ax = \frac{x}{2}\,, \text{ for all } x \in X.$$

Then, A is an injective Banach G-contraction on X with $\lambda = \frac{1}{2}$.

For initial term $x_0 = \frac{1}{2}$, the sequence $\{x_n = \frac{1}{2^n}\}$ is a G-TWC A-Picard sequence. Since, for some fix $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$0 \leq (\frac{1}{2})^n (4)^p < 1\,, \text{ for all } n > n_0,$$

using this inequality, we can prove that the sequence $\{\frac{1}{2^n}\}$ is a Cauchy sequence. It satisfy all the required conditions of Theorems 4–6. Since $\{\frac{1}{2^n}\}$ converges to 0, 0 is the unique fixed point of A.

Remark 3. *The injectiveness of a mapping A on X in Theorem 4 is not superfluous because injectivity of A ensures that $\{A^n y_j\}_{j=0}^{l}$ is a path of length l and $\{A^n w_j\}_{j=0}^{m}$ is a path of length m, for all $n \in \mathbb{N}$.*

In the following example, we discuss possible difficulties in proving Theorem 4 by excluding the injective property of a mapping A on X.

Example 2. *Let* $X = \{0\} \cup \{\frac{1}{3^n} : n \in \mathbb{N}\}$ *and* $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ *be a graph associated with* X*, where* $\mathfrak{E}(G) := \Delta \cup \{(\frac{1}{3^n}, \frac{1}{3^{n+1}}) \in X \times X : n \in \mathbb{N}\}$. *Let* $r_{G_b} : X \times X \to [0, \infty)$ *be a function defined as follows:*

$$\begin{aligned} r_{G_b}(x,y) &= 0 \text{ , if and only if } x = y, \\ r_{G_b}(0, \frac{1}{3^n}) &= r_{G_b}(\frac{1}{3^n}, 0) = \frac{1}{2}, \ n \geq 1, \\ r_{G_b}(\frac{1}{3^m}, \frac{1}{3^n}) &= r_{G_b}(\frac{1}{3^n}, \frac{1}{3^m}) = 1, \text{ if 2 divides } |m-n|, \\ r_{G_b}(\frac{1}{3^m}, \frac{1}{3^n}) &= r_{G_b}(\frac{1}{3^n}, \frac{1}{3^m}) = \frac{1}{3^{n+m}}, \text{ otherwise.} \end{aligned}$$

Then, (X, r_{G_b}) *is a graphical rectangular metric space (i.e.,* GR_bMS *with* $s = 1$*).*
Define $A : X \to X$ *as*

$$Ax = \begin{cases} \frac{1}{3}, & \text{if } x = 0 \\ \frac{1}{3^4}, & \text{if } x = \frac{1}{3} \\ \frac{x}{3^2}, & \text{otherwise.} \end{cases}$$

Then, A is Banach G-contraction on X, but A is not injective. A Picard sequence $\{x_n\}$ is defined as $x_n = Ax_{n-1}$ for all $n \in \mathbb{N}$. Here, first we have to find x_0 such that $x_1 \in [x_0]_G^l$ and $x_2 \in [x_0]_G^m$, where l, m are odd integers. The only choice is $x_0 = \frac{1}{3}$, because $\{y_j = \frac{1}{3^{j+1}}\}_{j=0}^3$ is a path from x_0 to x_1 of length 3 and $\{w_j = \frac{1}{3^{j+1}}\}_{j=0}^5$ forms a path from x_0 to x_2 of length 5, i.e., $x_1 \in [x_0]_G^3$, $x_2 \in [x_0]_G^5$. However, $Ay_0 = Ay_1 = \frac{1}{3^4}$ implies $\{Ay_j\}_{j=0}^3$ is not a path of length 3; in-fact, it is not a path. In addition, $r_{G_b}(x_n, x_{n+1}) = 1$ for all $n \geq 1$, which implies $\{x_n\}$ is not a Cauchy sequence.

3.1. Edelstein G-contraction

Definition 8. *Let G be a graph associated with a* GR_bMS (X, r_{G_b}). *A path* $\{t_i\}_{i=0}^m$ *from x to y, such that* $t_0 = x$, $t_m = y$, *is said to be* ϵ*-chainable, if* $r_{G_b}(t_{i-1}, t_i) < \epsilon, \ i = 1, 2, \ldots, m.$

Let us denote

$$E_\epsilon(G) = \{(p,q) \in \mathfrak{E}(G) : r_{G_b}(p,q) < \epsilon\}.$$

Definition 9. *Let G be a graph associated with* GR_bMS (X, r_{G_b}). *An Edelstein G-contraction (EGC) on X is a mapping* $A : X \to X$ *such that:*

- $(EGC-1)$ *for each* $(p,q) \in \mathfrak{E}(G)$ *implies* $(Ap, Aq) \in \mathfrak{E}(G)$*; and*
- $(EGC-2)$ *there exists* $\lambda \in [0,1)$ *such that for all* $p, q \in X$ *with* $(p,q) \in E_\epsilon(G)$ *implies*

$$r_{G_b}(Ap, Aq) < \lambda r_{G_b}(p,q).$$

Theorem 7. *For* (X, r_{G_b}) *being a G-complete* GR_bMS *and injective Edelstein G-contraction* $A : X \to X$, *assume the following:*

(I) *if a G-TWC A-Picard sequence* $\{x_n\}$ *converges in X, then there exists* $k \in \mathbb{N}$ *and a limit* $z^* \in X$ *of* $\{x_n\}$, *such that* $(x_n, z^*) \in E_\epsilon(G)$ *or* $(z^*, x_n) \in E_\epsilon(G)$ *for all* $n > k$*; and*
(II) *there exists* $x_0 \in X$ *such that there is an* ϵ*-chainable odd length path from* x_0 *to* Ax_0 *and* x_0 *to* A^2x_0 *in the graph G.*

Then, the A-Picard sequence $\{x_n\}$ *with initial term* $x_0 \in X$ *is G-TWC and converges to* x^* *and* Ax^* *in X.*

Proof. Let $x_0 \in X$, such that there is an ϵ-chainable, odd length path $\{y_j\}_{j=0}^l$ and $\{w_j\}_{j=0}^m$ such that

$$x_0 = y_0, \ \ Ax_0 = y_l \ \text{ and } \ (y_{j-1}, y_j) \in \mathfrak{E}(G) \ \text{ for each } j = 1, 2, \ldots, l$$

and

$$x_0 = w_0,\ \ A^2x_0 = w_m \ \text{ and } \ (w_{j-1}, w_j) \in \mathfrak{E}(G) \ \text{ for each } j = 1, 2, \ldots, m.$$

By $(EGC-1)$, we have

$$(Ay_{j-1}, Ay_j) \in \mathfrak{E}(G) \ \text{ for each } j = 1, 2, \ldots, l.$$

Therefore, $\{Ay_j\}_{j=0}^{l}$ is an ϵ-chainable path from $Ay_0 = Ax_0 = x_1$ to $Ay_l = A^2x_0 = x_2$ of length l. Similarly, for all $n \in \mathbb{N}$, $\{A^n y_j\}_{j=0}^{l}$ is an ϵ-chainable path from $A^n y_0 = A^n x_0 = x_n$ to $A^n y_l = A^n A x_0 = x_{n+1}$ of length l. Thus, $\{x_n\}$ is a G-TWC A-Picard sequence.

Since $(A^n y_{j-1}, A^n y_j) \in E_\epsilon(G)$ for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, l$, by using $(EGC-2)$, for all $j = 1, 2, \ldots, l$, we get

$$r_{G_b}(A^n y_{j-1}, A^n y_j) < \lambda r_{G_b}(A^{n-1} y_{j-1}, A^{n-1} y_j) < \cdots < \lambda^n r_{G_b}(y_{j-1}, y_j) < \lambda^n \epsilon. \tag{7}$$

Further, based on (GR_bM-3) and the inequality in Equation (7), we obtain

$$\begin{aligned}
r_{G_b}(x_n, x_{n+1}) &= r_{G_b}(A^n y_0, A^n y_l) \\
&\le s[r_{G_b}(A^n y_0, A^n y_1) + r_{G_b}(A^n y_1, A^n y_2)] \\
&\quad + s^2[r_{G_b}(A^n y_2, A^n y_3) + r_{G_b}(A^n y_3, A^n y_4)] + \cdots \\
&\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^n y_{l-3}, A^n y_{l-2}) + r_{G_b}(A^n y_{l-2}, A^n y_{l-1}) \\
&\qquad + r_{G_b}(A^n y_{l-1}, A^n y_l)] \\
&< s[2\lambda^n \epsilon] + s^2[2\lambda^n \epsilon] + \cdots + s^{\frac{l-1}{2}}[3\lambda^n \epsilon] \\
&= \lambda^n \{s[2\epsilon] + s^2[2\epsilon] + \cdots + s^{\frac{l-1}{2}}[3\epsilon]\} \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{8}$$

We know that, $\{w_j\}_{j=0}^{m}$ is an ϵ-chainable path from x_0 to x_2.

By using $(EGC-1)$, we can show that $\{A^n w_j\}_{j=0}^{m}$ is an ϵ-chainable path from x_n to x_{n+2} for all $n \in \mathbb{N}$.

Using $(EGC-2)$, for all $j = 1, 2, \ldots, m$, we obtain

$$r_{G_b}(A^n w_{j-1}, A^n w_j) < \lambda r_{G_b}(A^{n-1} w_{j-1}, A^{n-1} w_j) < \cdots < \lambda^n r_{G_b}(w_{j-1}, w_j) < \lambda^n \epsilon. \tag{9}$$

Further, based on (GR_bM-3) and the inequality in Equation (9), we obtain

$$\begin{aligned}
r_{G_b}(x_n, x_{n+2}) &= r_{G_b}(A^n w_0, A^n w_m) \\
&\le s[r_{G_b}(A^n w_0, A^n w_1) + r_{G_b}(A^n w_1, A^n w_2)] \\
&\quad + s^2[r_{G_b}(A^n w_2, A^n w_3) + r_{G_b}(A^n w_3, A^n w_4)] + \cdots \\
&\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^n w_{m-3}, A^n w_{m-2}) + r_{G_b}(A^n w_{m-2}, A^n w_{m-1}) \\
&\qquad + r_{G_b}(A^n w_{m-1}, A^n w_m)] \\
&< s[2\lambda^n \epsilon] + s^2[2\lambda^n \epsilon] + \cdots + s^{\frac{m-1}{2}}[3\lambda^n \epsilon] \\
&< \lambda^n \{s[2\epsilon] + s^2[2\epsilon] + \cdots + s^{\frac{m-1}{2}}[3\epsilon]\} \\
&\to \ 0 \ \text{ as } n \to +\infty.
\end{aligned} \tag{10}$$

Now, we have to show that, the G-TWC A-Picard sequence $\{x_n\}$ is Cauchy, i.e., for $p \ge 1$, $r_{G_b}(x_n, x_{n+p}) \to 0$ as $n \to +\infty$.

Case-I: If p is an odd integer $(p > 2)$, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \le\, & s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-1}{2}}[r_{G_b}(x_{n+p-3}, x_{n+p-2}) + r_{G_b}(x_{n+p-2}, x_{n+p-1}) \\ & \quad + r_{G_b}(x_{n+p-1}, x_{n+p})]. \end{aligned}$$

From the inequality in Equation (8), we have

$$r_{G_b}(x_n, x_{n+p}) \to 0, \quad as\ n \to +\infty. \tag{11}$$

Case-II: If p is an even integer $(p > 2)$, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \le\, & s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-2}{2}}[r_{G_b}(x_{n+p-4}, x_{n+p-3}) + r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\ & \quad + r_{G_b}(x_{n+p-2}, x_{n+p})]. \end{aligned}$$

From the inequalities in Equations (8) and (10), we get

$$r_{G_b}(x_n, x_{n+p}) \to 0 \quad as\ n \to +\infty. \tag{12}$$

From Case-(I) and Case-(II), one can say that $\{x_n\}$ is a Cauchy sequence.

Since X being G-complete implies $x_n \to x^* \in X$, by our assumption, there exists $k \in \mathbb{N}$, such that $(x_n, x^*) \in E_\epsilon(G)$ or $(x^*, x_n) \in E_\epsilon(G)$, for all $n > k$. Suppose that, $(x_n, x^*) \in E_\epsilon(G)$ for all $n > k$, then

$$r_{G_b}(Ax_n, Ax^*) < \lambda r_{G_b}(x_n, x^*),$$

as $n \to +\infty$, thus we have

$$r_{G_b}(Ax_n, Ax^*) \to 0.$$

This implies $\{x_n\}$ converges both x^* and Ax^*. □

Theorem 8. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ holds the conditions of Theorem 7. In addition, if the quadruple (X, r_{G_b}, G, A) has the property S^*, then A has a fixed point.*

Proof. From the proof of Theorem 7, x^* and Ax^* are the limits of a G-TWC A-Picard sequence. By using the property S^*, we have $Ax^* = x^*$. □

Theorem 9. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ hold the conditions of Theorem 8. In addition, they satisfies that, for $z^*, w^* \in Fix(A)$ there is an ϵ-chainable, odd length (greater than one) path between z^* and w^*. Then, A has a unique fixed point.*

Proof. From Theorem 7, we get a fixed point for A.

Suppose $z^*, w^* \in Fix(A)$. Then, by our assumption, there is an ϵ-chainable, odd length (say k, $k > 1$) path between z^* and w^*.

Let $\{t_i\}_{i=0}^{k}$ be the ϵ-chainable odd length path from z^* to w^*, such that $t_0 = z^*$,$t_k = w^*$ and $(t_{i-1}, t_i) \in \mathfrak{E}(G)$ for each $i = 1, 2, \ldots k$; then,

$$\begin{aligned}
r_{G_b}(z^*, w^*) &= r_{G_b}(A^n z^*, A^n w^*) \\
&\leq s[r_{G_b}(A^n t_0, A^n t_1) + r_{G_b}(A^n t_1, A^n t_2)] \\
&\quad + s^2[r_{G_b}(A^n t_2, A^n t_3) + r_{G_b}(A^n t_3, A^n t_4)] + \cdots \\
&\quad + s^{\frac{k-1}{2}}[r_{G_b}(A^n t_{k-3}, A^n t_{k-2}) + r_{G_b}(A^n t_{l-2}, A^n t_{l-1}) \\
&\qquad + r_{G_b}(A^n t_{k-1}, A^n t_k)] \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

This implies, $z^* = w^*$. □

3.2. Meir–Keeler G-Contraction

Let (X, r_{G_b}) be a GR_bMS associated with a graph G; we define

$$E_\epsilon^\delta(G) = \{(p, q) \in \mathfrak{E}(G) : \epsilon < r_{G_b}(p, q) < \epsilon + \delta\}.$$

Definition 10. *A path in a graph G from x to y is said to be (ϵ, δ)-chainable, if there is a sequence $\{t_i\}_{i=0}^{k}$ of distinct vertices of G, such that $t_0 = x$, $t_k = y$ and $(t_{i-1}, t_i) \in E_\epsilon^\delta(G)$, for each $i = 1, 2, \ldots, k$.*

Definition 11. *Let G be a graph associated with GR_bMS (X, r_{G_b}). A mapping $A : X \to X$ is a Meir–Keeler G-contraction (MKGC), if it satisfies:*

- $(MKGC-1)$ *for each $(p, q) \in \mathfrak{E}(G)$ implies $(Ap, Aq) \in \mathfrak{E}(G)$;*
- $(MKGC-2)$ *for given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$r_{G_b}(Ap, Aq) < \epsilon, \quad \textit{whenever } (p, q) \in E_\epsilon^\delta(G).$$

Lemma 1. *Let (X, r_{G_b}) be a GR_bMS and an injective mapping $A : X \to X$ is a MKGC. If $(x, y) \in E_\epsilon^\delta(G)$, then $\lim_{n\to+\infty} r_{G_b}(A^n x, A^n y) = 0$.*

Proof. Let $(x, y) \in E_\epsilon^\delta(G)$. By $(MKGC-2)$, we have

$$r_{G_b}(Ax, Ay) < \epsilon.$$

If $r_{G_b}(Ax, Ay) = 0$, then $Ax = Ay$. Since A is injective, which implies $x = y$, it gives a contradiction to our assumption, thus $r_{G_b}(Ax, Ay) \neq 0$. Now, choose ϵ_1 and δ_1 such that $\epsilon = \epsilon_1 + \delta_1$ and $\epsilon_1 < r_{G_b}(Ax, Ay)$. This implies $(Ax, Ay) \in E_{\epsilon_1}^{\delta_1}(G)$; again, by $(MKGC-2)$, we have

$$r_{G_b}(A^2 x, A^2 y) < \epsilon_1.$$

This implies

$$r_{G_b}(A^2 x, A^2 y) < \epsilon_1 < \epsilon < r_{G_b}(Ax, Ay).$$

Continuing this way, one can say that, $\{r_{G_b}(A^n x, A^n y)\}$ is a positive term decreasing sequence which converges to 0, i.e., $\lim_{n\to+\infty} r_{G_b}(A^n x, A^n y) = 0$. □

Theorem 10. *Let (X, r_{G_b}) be a G-complete GR_bMS and an injective mapping $A : X \to X$ is a MKGC; assume the following:*

(I) *there exists $x_0 \in X$, such that there is an (ϵ, δ)-chainable odd length path from x_0 to Ax_0 and x_0 to $A^2 x_0$ in G; and*

(II) if a G-TWC A-Picard sequence $\{x_n\}$ converges in X, then there exists $k \in \mathbb{N}$ and a limit $z^* \in X$ of $\{x_n\}$, such that $(x_n, z^*) \in E_{\epsilon_n}^{\delta_n}(G)$ or $(z^*, x_n) \in E_{\epsilon_n}^{\delta_n}(G)$ for all $n > k$.

Then, the A-Picard sequence $\{x_n\}$ with initial term $x_0 \in X$ is G-TWC and converges to x^* and Ax^* in X.

Proof. Let $x_0 \in X$, such that there is an (ϵ, δ)-chainable odd length paths $\{y_i\}_{i=0}^{l}$ and $\{w_i\}_{i=0}^{m}$ in G from x_0 to Ax_0 and x_0 to A^2x_0, respectively, such that

$$x_0 = y_0, \quad Ax_0 = y_l \text{ and } (y_{j-1}, y_j) \in E_\epsilon^\delta(G) \text{ for each } j = 1, 2, \ldots, l$$

and

$$x_0 = w_0, \quad A^2x_0 = w_m \text{ and } (w_{j-1}, w_j) \in E_\epsilon^\delta(G) \text{ for each } j = 1, 2, \ldots, m.$$

By using (MKGC-1), we can easily prove that the sequence $\{x_n\}$ is a G-TWC A-Picard sequence. Based on (GR_bM-3) and Lemma 1, we have

$$\begin{aligned} r_{G_b}(x_n, x_{n+1}) &= r_{G_b}(A^n y_0, A^n y_l) \\ &\le s[r_{G_b}(A^n y_0, A^n y_1) + r_{G_b}(A^n y_1, A^n y_2)] \\ &\quad + s^2[r_{G_b}(A^n y_2, A^n y_3) + r_{G_b}(A^n y_3, A^n y_4)] + \cdots \\ &\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^n y_{l-3}, A^n y_{l-2}) + r_{G_b}(A^n y_{l-2}, A^n y_{l-1}) \\ &\qquad + r_{G_b}(A^n y_{l-1}, A^n y_l)] \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned} r_{G_b}(x_n, x_{n+2}) &= r_{G_b}(A^n w_0, A^n w_m) \\ &\le s[r_{G_b}(A^n w_0, A^n w_1) + r_{G_b}(A^n w_1, A^n w_2)] \\ &\quad + s^2[r_{G_b}(A^n w_2, A^n w_3) + r_{G_b}(A^n w_3, A^n w_4)] + \cdots \\ &\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^n w_{m-3}, A^n w_{m-2}) + r_{G_b}(A^n w_{m-2}, A^n w_{m-1}) \\ &\qquad + r_{G_b}(A^n w_{m-1}, A^n w_m)] \\ &\to \ 0 \text{ as } n \to +\infty. \end{aligned} \tag{14}$$

Now, we have to show that, the G-TWC A-Picard sequence $\{x_n\}$ is Cauchy, i.e., for $p \ge 1$, $r_{G_b}(x_n, x_{n+p}) \to 0$ as $n \to +\infty$.

Case-I: If p is an odd integer, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) &\le s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\ &\quad + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ &\quad + s^{\frac{p-1}{2}}[r_{G_b}(x_{n+p-3}, x_{n+p-2}) + r_{G_b}(x_{n+p-2}, x_{n+p-1}) \\ &\qquad + r_{G_b}(x_{n+p-1}, x_{n+p})]. \end{aligned}$$

From the inequality in Equation (13), we can say that

$$r_{G_b}(x_n, x_{n+p}) \to 0 \quad \textit{as } n \to +\infty. \tag{15}$$

Case-II: If p is an even integer, then

$$\begin{aligned} r_{G_b}(x_n, x_{n+p}) \leq & s[r_{G_b}(x_n, x_{n+1}) + r_{G_b}(x_{n+1}, x_{n+2})] \\ & + s^2[r_{G_b}(x_{n+2}, x_{n+3}) + r_{G_b}(x_{n+3}, x_{n+4})] + \cdots \\ & + s^{\frac{p-2}{2}}[r_{G_b}(x_{n+p-4}, x_{n+p-3}) + r_{G_b}(x_{n+p-3}, x_{n+p-2}) \\ & \quad + r_{G_b}(x_{n+p-2}, x_{n+p})]. \end{aligned}$$

From the inequalities in Equations (13) and (14),

$$r_{G_b}(x_n, x_{n+p}) \to 0 \quad as\ n \to +\infty. \tag{16}$$

From Case-(I) and Case-(II), one can say that $\{x_n\}$ is a Cauchy sequence.

Since X is a G-complete, implies $x_n \to x^*$ for some $x^* \in X$.

However, from Condition (II), there exists $k \in \mathbb{N}$ such that $(x_n, x^*) \in E^{\delta_n}_{\epsilon_n}(G)$ or $(x^*, x_n) \in E^{\delta_n}_{\epsilon_n}(G)$ for all $n > k$.

Let us take $(x_n, x^*) \in E^{\delta_n}_{\epsilon_n}(G)$; by Meir–Keeler G-contraction, we have

$$r_{G_b}(Ax_n, Ax^*) < r_{G_b}(x_n, x^*).$$

Obviously, $r_{G_b}(Ax_n, Ax^*) \to 0$ as $n \to +\infty$ i.e., $x_{n+1} \to Ax^*$. □

Theorem 11. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ hold all the conditions of Theorem 10 along with that the quadruple (X, r_{G_b}, G, A) satisfies the property S^*. Then, A has a fixed point.*

Proof. From the proof of Theorem 10, x^* and Ax^* are limits of a G-TWC A-Picard sequence $\{x_n\}$. By using the property S^*, we have $Ax^* = x^*$. □

Theorem 12. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ hold all the conditions of the Theorem 11. In addition, if there exists an (ϵ, δ)-chainable odd length path (which is greater than one) between any two fixed point of A, then A has unique fixed point.*

Proof. From Theorem 11, we get a fixed point for A.

Suppose $z^*, w^* \in Fix(A)$. Then, by our assumption, there is an odd length (say, k, $k > 1$) path $\{t_i\}_{i=0}^k$ from z^* to w^*, such that $t_0 = z^*, t_k = w^*$ and $(t_{i-1}, t_i) \in E^{\delta}_{\epsilon}(G)$ for all $i = 1, 2, \ldots k$.

By using $(GR_bM - 3)$ and Lemma 1, we have

$$\begin{aligned} r_{G_b}(z^*, w^*) = & r_{G_b}(A^n z^*, A^n w^*) \\ \leq & s[r_{G_b}(A^n t_0, A^n t_1) + r_{G_b}(A^n t_1, A^n t_2)] \\ & + s^2[r_{G_b}(A^n t_2, A^n t_3) + r_{G_b}(A^n t_3, A^n t_4)] + \cdots \\ & + s^{\frac{k-1}{2}}[r_{G_b}(A^n t_{k-3}, A^n t_{k-2}) + r_{G_b}(A^n t_{k-2}, A^n t_{k-1}) \\ & \quad + r_{G_b}(A^n t_{k-1}, A^n t_k)] \\ \to & 0 \quad \text{as } n \to +\infty. \end{aligned}$$

This implies, $z^* = w^*$. □

3.3. Reich G-Contraction

Definition 12. *Let G be a graph associated with GR_bMS (X, r_{G_b}). A mapping $A : X \to X$ is a Reich G-contraction (RGC) if it satisfies:*

- *(RGC-1) for each $(p,q) \in \mathfrak{E}(G)$ implies $(Ap,Aq),(p,Ap),(q,Aq) \in \mathfrak{E}(G)$; and*
- *(RGC-2) for each $(p,q) \in \mathfrak{E}(G)$, there exists non-negative numbers a,b,c such that $a+b+c<1$ and*

$$r_{G_b}(Ap,Aq) \leq ar_{G_b}(p,Ap) + br_{G_b}(q,Aq) + cr_{G_b}(p,q).$$

Lemma 2. *Let (X, r_{G_b}) be a GR_bMS associated with a graph G and a mapping $A : X \to X$ is a RGC. If $(x, Ax) \in \mathfrak{E}(G)$, then*

$$r_{G_b}(A^n x, A^{n+1}x) \leq \beta^n r_{G_b}(x, Ax),$$

where $\beta < 1$.

Proof. Let $(x, Ax) \in \mathfrak{E}(G)$; then, $(Ax, A^2x) \in \mathfrak{E}(G)$. Using $(RGC-2)$, we have

$$\begin{aligned} r_{G_b}(Ax, A^2x) &\leq ar_{G_b}(x, Ax) + br_{G_b}(Ax, A^2x) + cr_{G_b}(x, Ax) \\ &= (a+c) r_{G_b}(x, Ax) + br_{G_b}(Ax, A^2x) \\ &\leq \left(\frac{a+c}{1-b}\right) r_{G_b}(x, Ax) = \beta r_{G_b}(x, Ax), \end{aligned} \tag{17}$$

where $\beta = \frac{a+c}{1-b}$. Since $a+b+c<1$ implies $\beta < 1$, similarly, we have

$$\begin{aligned} r_{G_b}(A^n x, A^{n+1}x) &\leq \beta r_{G_b}(A^{n-1}x, A^n x) \\ &\leq \beta \left(\beta r_{G_b}(A^{n-2}x, A^{n-1}x)\right) \\ &\vdots \\ &\leq \beta^n r_{G_b}(x, Ax). \end{aligned}$$

□

Lemma 3. *Let (X, r_{G_b}) be a GR_bMS associated with a graph G and a mapping $A : X \to X$ is a RGC. If $(x, y) \in \mathfrak{E}(G)$, then $r_{G_b}(A^n x, A^n y) \to 0$ as $n \to +\infty$.*

Proof. Let $(x, y) \in \mathfrak{E}(G)$, then $(A^n x, A^n y) \in \mathfrak{E}(G)$ for all $n \in \mathbb{N}$. By using $(RGC-2)$, we have

$$\begin{aligned} r_{G_b}(A^n x, A^n y) &\leq ar_{G_b}(A^{n-1}x, A^n x) + br_{G_b}(A^{n-1}y, A^n y) + cr_{G_b}(A^{n-1}x, A^{n-1}y) \\ &\leq a\beta^{n-1} r_{G_b}(x, Ax) + b\beta^{n-1} r_{G_b}(y, Ay) + cr_{G_b}(A^{n-1}x, A^{n-1}y), \qquad (18) \\ &\leq a\beta^{n-1} r_{G_b}(x, Ax) + b\beta^{n-1} r_{G_b}(y, Ay) + r_{G_b}(A^{n-1}x, A^{n-1}y) \\ &\leq a\beta^{n-1} r_{G_b}(x, Ax) + b\beta^{n-1} r_{G_b}(y, Ay) \\ &\quad + a\beta^{n-2} r_{G_b}(x, Ax) + b\beta^{n-2} r_{G_b}(y, Ay) + cr_{G_b}(A^{n-1}x, A^{n-1}y) \\ &\vdots \\ &\leq [\beta^{n-1} + \beta^{n-2} + \cdots + 1][ar_{G_b}(x, Ax) + br_{G_b}(y, Ay)] + r_{G_b}(x, y) \\ &\leq \frac{1}{1-\beta}[ar_{G_b}(x, Ax) + br_{G_b}(y, Ay)] + r_{G_b}(x, y). \qquad (19) \end{aligned}$$

This implies that, the sequence $\{r_{G_b}(A^n x, A^n y)\}$ is an infinite bounded sequence. Then, the sequence $\{r_{G_b}(A^n x, A^n y)\}$ has a limit point. Now, as $n \to +\infty$, the inequality in Equation (18) becomes

$$S_\infty \leq cS_\infty,$$

where $S_\infty = \lim_{n \to +\infty} r_{G_b}(A^n x, A^n y)$.

This implies $S_\infty = 0$. □

Since the graphical rectangular metric is not necessarily continuous [12–14], to prove the fixed point theorem in GR_bMS by using the Reich G-contraction, we assume the continuity of the metric function.

Theorem 13. *Let (X, r_{G_b}) be a G-complete GR_bMS and an injective mapping $A : X \to X$ is a RGC, if it satisfies:*

(I) G have the property $(\mathcal{P})$;
(II) There exists $x_0 \in X$, such that $Ax_0 \in [x_0]^l_G$ and $A^2x_0 \in [x_0]^m_G$, where l, m are odd integers; and
(III) The metric r_{G_b} is continuous.

Then, the A-Picard sequence $\{x_n\}$ with initial term $x_0 \in X$, is G-TWC and converges to x^*, which is a fixed point of A.

Proof. Let $x_0 \in X$, such that there is a odd length path $\{y_j\}_{j=0}^l$ and $\{w_j\}_{j=0}^m$ such that

$$x_0 = y_0, \;\; Ax_0 = y_l \;\text{ and }\; (y_{j-1}, y_j) \in \mathfrak{E}(G) \text{ for each } j = 1, 2, \ldots, l$$

and

$$x_0 = w_0, \;\; A^2x_0 = w_m \;\text{ and }\; (w_{j-1}, w_j) \in \mathfrak{E}(G) \;\text{ for each } j = 1, 2, \ldots, m.$$

By $(RGC - 1)$, we have

$$(Ay_{j-1}, Ay_j) \in \mathfrak{E}(G) \;\text{ for each } j = 1, 2, \ldots, l.$$

Therefore, $\{Ay_j\}_{j=0}^l$ is a path from $Ay_0 = Ax_0 = x_1$ to $Ay_l = A^2x_0 = x_2$ of length l. Continuing this process, for all $n \in \mathbb{N}$, we obtain $\{A^ny_j\}_{j=0}^l$ a path from $A^ny_0 = A^nx_0 = x_n$ to $A^ny_l = A^nAx_0 = x_{n+1}$ of length l. Thus, $\{x_n\}$ is G-TWC sequence.

Now, using $(GR_bM - 3)$ and Lemma 3, we have

$$\begin{aligned} r_{G_b}(x_n, x_{n+1}) &= r_{G_b}(A^ny_0, A^ny_l) \\ &\leq s[r_{G_b}(A^ny_0, A^ny_1) + r_{G_b}(A^ny_1, A^ny_2)] \\ &\quad + s^2[r_{G_b}(A^ny_2, A^ny_3) + r_{G_b}(A^ny_3, A^ny_4)] + \cdots \\ &\quad + s^{\frac{l-1}{2}}[r_{G_b}(A^ny_{l-3}, A^ny_{l-2}) + r_{G_b}(A^ny_{l-2}, A^ny_{l-1}) \\ &\qquad + r_{G_b}(A^ny_{l-1}, A^ny_l)] \\ &\to 0 \;\text{ as } n \to +\infty \end{aligned} \tag{20}$$

and

$$\begin{aligned} r_{G_b}(x_n, x_{n+2}) &= r_{G_b}(A^nw_0, A^nw_m) \\ &\leq s[r_{G_b}(A^nw_0, A^nw_1) + r_{G_b}(A^nw_1, A^nw_2)] \\ &\quad + s^2[r_{G_b}(A^nw_2, A^nw_3) + r_{G_b}(A^nw_3, A^nw_4)] + \cdots \\ &\quad + s^{\frac{m-1}{2}}[r_{G_b}(A^nw_{m-3}, A^nw_{m-2}) + r_{G_b}(A^nw_{m-2}, A^nw_{m-1}) \\ &\qquad + r_{G_b}(A^nw_{m-1}, A^nw_m)] \\ &\to \;0 \;\text{ as } n \to +\infty. \end{aligned} \tag{21}$$

Based on the inequalities in Equations (20) and (21), we can easily prove $\{x_n\}$ is a Cauchy sequence (by using similar method as in Theorem 7).

Since (X, r_{G_b}) is G-complete, this implies that $x_n \to x^* \in X$. From Condition (I), there exists $k \in \mathbb{N}$ such that $(x_n, z^*) \in \mathfrak{E}(G)$ or $(x^*, x_n) \in \mathfrak{E}(G)$ for all $n > k$.

Let us take $(x_n, x^*) \in \mathfrak{E}(G)$, by $(RGC-1)$, $(Ax_n, Ax^*) \in \mathfrak{E}(G)$.
By $(RGC-2)$, we have

$$r_{G_b}(Ax_n, Ax^*) \leq ar_{G_b}(x_n, Ax_n) + br_{G_b}(x^*, Ax^*) + cr_{G_b}(x_n, x^*).$$

As $n \to +\infty$, and using continuity of the metric r_{G_b}, we have

$$r_{G_b}(L, Ax^*) \leq br_{G_b}(x^*, Ax^*),$$

where L is any limit of $\{x_n\}$, thus the above inequality also holds for x^*. Therefore,

$$r_{G_b}(x^*, Ax^*) \leq br_{G_b}(x^*, Ax^*)$$

implies

$$r_{G_b}(x^*, Ax^*) \leq br_{G_b}(x^*, Ax^*),$$

which implies $Ax^* = x^*$. $\square$

Theorem 14. *Let (X, r_{G_b}) and an injective mapping $A : X \to X$ assume all the conditions of Theorem 13. In addition, they satisfies that, for $z^*, w^* \in Fix(A)$, there is an odd length path between z^* and w^*. Then, A has a unique fixed point.*

Proof. From Theorem 13, we get a fixed point for A.

Suppose $z^*, w^* \in Fix(A)$. Then, by our assumption, there is an odd length (say, k) path between z^* and w^*.

Case-I: Let $k = 1$, i.e., $(z^*, w^*) \in \mathfrak{E}(G)$.

By $(RGC-1)$, $(Az^*, Aw^*) \in \mathfrak{E}(G)$.

Now, based on $(RGC-2)$, we have

$$r_{G_b}(Az^*, Aw^*) \leq ar_{G_b}(z^*, Az^*) + br_{G_b}(w^*, Aw^*) + cr_{G_b}(z^*, w^*)$$
$$\Rightarrow \quad r_{G_b}(z^*, w^*) \leq cr_{G_b}(z^*, w^*),$$

which gives contradiction to the value of c.

Hence, $z^* = w^*$.

Case-II: Suppose $k > 1$.

Let $\{t_i\}_{i=0}^{k}$ be the odd length path from z^* to w^*, such that $t_0 = z^*$ and $t_k = w^*$; then,

$$\begin{aligned} r_{G_b}(z^*, w^*) &= r_{G_b}(A^n z^*, A^n w^*) \\ &\leq s[r_{G_b}(A^n t_0, A^n t_1) + r_{G_b}(A^n t_1, A^n t_2)] \\ &\quad + s^2[r_{G_b}(A^n t_2, A^n t_3) + r_{G_b}(A^n t_3, A^n t_4)] + \cdots \\ &\quad + s^{\frac{k-1}{2}}[r_{G_b}(A^n t_{k-3}, A^n t_{k-2}) + r_{G_b}(A^n t_{l-2}, A^n t_{l-1}) \\ &\qquad + r_{G_b}(A^n t_{k-1}, A^n t_k)] \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

This implies $z^* = w^*$. $\square$

4. Conclusions

We extend the range of λ for the case $\lambda \in [0, 1)$ and establish the Banach contraction theorem in the context of GR_bMS that provides a positive answer to the question proposed in [4]. Moreover, we define Edelstein G-contraction, Meir–Keeler G-contraction, and Reich G-contraction in the aforesaid space and prove the fixed point results that generalize some well known results in the literature. Our results give a precise technique and directions for further study in this new space.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Branciari, A. A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. *Publ. Math. Debr.* **2000**, *57*, 31–37.
3. Shukla, S.; Radenović, S.; Vetro, C. Graphical metric space: A generalized setting in fixed point theory. *Rev. Real Acad. Cienc. Ser. A. Mat.* **2017**, *111*, 641–655. [CrossRef]
4. Younis, M.; Singh, D.; Goyal, A. A novel approach of graphical rectangular *b*-metric spaces with an application to the vibrations of a vertical heavy hanging cable. *J. Fixed Point Theory Appl.* **2019**, *21*, 33. [CrossRef]
5. Edelstein, M. An extension of Banach contraction principle. *Proc. Am. Math. Soc.* **1961**, *12*, 7–10.
6. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math Bull.* **1973**, *16*, 201–206. [CrossRef]
7. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
8. Meir, A.; Keeler, E. A theorem on contraction mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329. [CrossRef]
9. Reich, S. Some remarks concerning contraction mappings. *Can. Math. Bull.* **1971**, *14*, 121–124. [CrossRef]
10. Chuensupantharat, N.; Kumam, P.; Chauhan, V.; Singh, D.; Menon, R. Graphic contraction mappings via graphical *b*-metric spaces with applications. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 3149–3165. [CrossRef]
11. Baradol, P.; Gopal, D.; Radenović, S. Computational fixed points in graphical rectangular metric spaces with application. *J. Comput. Appl. Math.* **2020**, *375*, 112805. [CrossRef]
12. Kadelburg, Z.; Radenović, S. On generalized metric spaces: A survey. *TWMS J. Pure Appl. Math.* **2014**, *5*, 3–13.
13. Samet, B. Discussion on: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari. *Publ. Math. Debr.* **2010**, *76*, 493–494.
14. Sarma, I.R.; Rao, J.M.; Rao, S.S. Contractions over generalized metric spaces. *J. Nonlinear Sci. Appl.* **2009**, *2*, 180–182. [CrossRef]