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A Study on Cubic H-Relations in a Topological Universe Viewpoint

Jeong-Gon Lee [1,*], Kul Hur [2] and Xueyou Chen [3]

[1] Division of Applied Mathematics, Nanoscale Science and Technology Institute, Wonkwang University, Iksan 54538, Korea
[2] Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea; kulhur@wku.ac.kr
[3] School of Mathematics, Shandong University of Technology, Zibo 255049, China; xueyou-chen@163.com
[*] Correspondence: jukolee@wku.ac.kr

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Abstract: We introduce the concrete category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] of cubic H-relational spaces and P-preserving [resp. R-preserving] mappings between them and study it in a topological universe viewpoint. In addition, we prove that it is Cartesian closed over **Set**. Next, we introduce the subcategory $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] of $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] and investigate it in the sense of a topological universe. In particular, we obtain exponential objects in $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] quite different from those in $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$].

Keywords: cubic H-relational space; cubic H-reflexive relation; topological category; cartesian closed category; topological universe

1. Introduction

In 1984, Nel [1] introduced the concept of a topological universe which implies quasitopos [2]. Its notion has already been put to effective use several areas of mathematics in [3–5]. After then, Kim et al. [6] and Lee et al. [7] constructed the category $\mathbf{NSet}(H)$ of neutrosophic H-sets and morphisms between them and the category $\mathbf{NCSet}(H)$ of neutrosophic crisp sets and morphisms between them, and they studied each category in the sense of a topological universe. On the other hand, Cerruti [8] constructed the category of L-fuzzy relations and obtained some of its properties. Hur [9,10] [resp. Hur et al. [11] and Lim et al [12] formed the category $\mathbf{Rel}(H)$ of H-fuzzy relational spaces [resp. $\mathbf{IRel}(H)$ of H-intuitionistic fuzzy relational spaces and $\mathbf{VRel}(H)$ of vague relational spaces] and each category was investigated in topological universe viewpoint.

In 2012, Jun et al. [13] introduced the notion of a cubic set and investigated some of its properties. After that time, Ahn and Ko [14] studied cubic subalgebras and filters of CI-algebras. Akram et al. [15] applied the concept of cubic sets to KU-algebras. Jun et al. [16] dealt with cubic structures of ideals of BCI-algebras. Jun and Khan [17] found some properties of cubic ideals in semigroups. Jun et al. [18] studied cubic subgroups. Zeb et al. [19] defined the notion of a cubic topology and investigated some of its properties. Recently, Mahmood et al. [20] dealt with multicriteria decision making based on cubic sets. Rashed et al. [21] applied the concept of cubic sets to graph theory. Yaqoob et al. [22] introduced the notion of a cubic finite switchboard state machine and studied its various properties. Ma et al. [23] define a cubic relation on H_v-LA-semigroup and investigated some of its properties. Kim et al. [24] defined cubic relations and obtained some their properties.

In this paper, we study the category of cubic relations and morphisms between them in the sense of a topological universe proposed by Nel. First, we define the concept of a cubic H-relational space for a Heyting algebra H and introduce the concrete category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] of cubic H-relational spaces and P-preserving [resp. R-preserving] mappings between them, and obtained some categorical structures and give examples. In particular, we prove that the category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] is Cartesian closed over **Set**, where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them. Next, we introduce the subcategory $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] of $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] and investigate it in the sense of a topological universe. In particular, we obtain exponential objects in $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] quite different from those in $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$].

2. Preliminaries

In this section, we list some basic definitions for category theory which are needed in the next sections. Let us recall that a concrete category is a category of sets which are endowed with an unspecified structure. Refer to [25] for the notions of a topological category and a cotopological category.

Definition 1 ([25])**.** *Let* **A** *be a concrete category.*

(i) *The* **A***-fiber of a set* X *is the class of all* **A***-structures on* X.
(ii) **A** *is said to be properly fibered over* **Set***, if it satisfies the following:*

  (a) *(Fiber-smallness) for each set* X*, the* **A***-fiber of* X *is a set,*
  (b) *(Terminal separator property) for each singleton set* X*, the* **A***-fiber of* X *has precisely one element,*
  (c) *if* ξ *and* η *are* **A***-structures on a set* X *such that* $id : (X, \xi) \to (X, \eta)$ *and* $id : (X, \eta) \to (X, \xi)$ *are* **A***-morphisms, then* $\xi = \eta$.

Definition 2 ([26])**.** *A category* **A** *is said to be Cartesian closed, if it satisfies the following conditions:*

(i) *for each* **A***-object* A *and* B*, there exists a product* $A \times B$ *in* **A***,*
(ii) *exponential objects exist in* A*, i.e., for each* **A***-object* A*, the functor* $A \times - : A \to A$ *has a right adjoint, i.e., for any* **A***-object* B*, there exist an* **A***-object* B^A *and a* **A***-morphism* $e_{A,B} : A \times B^A \to B$ *(called the evaluation) such that for any* **A***-object* C *and any* **A***-morphism* $f : A \times C \to B$*, there exists a unique* **A***-morphism* $\bar{f} : C \to B^A$ *such that* $e_{A,B} \circ (1_A \times \bar{f}) = f$*, i.e., the diagram commutes:*

$$\begin{tikzcd}
A \times B^A \arrow[rr, "e_{A,B}"] & & B \\
& A \times C \arrow[ul, "\exists 1_A \times \bar{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Definition 3 ([1])**.** *A category* **A** *is called a topological universe over* **Set** *if it satisfies the following conditions:*

(i) **A** *is well-structured, i.e., (a)* **A** *is concrete category; (b) fiber-smallness condition; (c)* **A** *has the terminal separator property,*
(ii) **A** *is cotopological over* **Set***,*
(iii) *final episinks in* A *are preserved by pullbacks, i.e., for any episink* $(g_j : X_j \to Y)_J$ *and any* **A***-morphism* $f : W \to Y$*, the family* $(e_j : U_j \to W)_J$*, obtained by taking the pullback* f *and* g_j*, for each* $j \in J$*, is again a final episink.*

Now refer to [13,27–34] for the concepts of fuzzy sets, fuzzy relations, interval-valued fuzzy sets and interval-valued fuzzy relations, neutrosophic crisp sets, neutrosophic sets and operation between them, respectively.

3. Properties of the Categories $\mathbf{HRel}_P(H)$ and $\mathbf{HRel}_R(H)$

In this section, first, we write the concept of a cubic set introduced by Jun et al. [13] (Also, see [13] for the equality $\mathcal{A} = \mathcal{B}$ and orders $\mathcal{A} \sqsubset \mathcal{B}$, $\mathcal{A} \Subset \mathcal{B}$ for any cubic sets $\mathcal{A}$, $\mathcal{B}$, the complement $\mathcal{A}^c$ of a cubic set $\mathcal{A}$, and the unions $\mathcal{A} \sqcup \mathcal{B}$, $\mathcal{A} \Cup \mathcal{B}$ and intersections $\mathcal{A} \sqcap \mathcal{B}$, $\mathcal{A} \Cup \mathcal{B}$ of two cubic sets $\mathcal{A}$, $\mathcal{B}$). Next, we introduce the category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] consisting of all cubic H-relational spaces and all P-preserving [resp. R-preserving] mappings between any two cubic H-relational spaces and it has the similar structures as those of $\mathbf{CSet}_P(H)$ [resp. $\mathbf{CSet}_R(H)$] (See [35]).

Throughout this section and next section, H denotes a complete Heyting algebra (Refer to [36,37] for its definition) and $[H]$ denotes the set of all closed subintervals of H.

Definition 4 ([13])**.** *Let X be a nonempty set. Then a complex mapping $\mathcal{A} =< A, \lambda >: X \to [I] \times I$ is called a cubic set in X, where $I = [0, 1]$ and $[I]$ be the set of all closed subintervals of I.*

A cubic set $\mathcal{A} =< A, \lambda >$ in which $A(x) = \mathbf{0}$ and $\lambda(x) = 1$ (resp. $A(x) = \mathbf{1}$ and $\lambda(x) = 0$) for each $x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

A cubic set $\mathcal{B} =< B, \mu >$ in which $B(x) = \mathbf{0}$ and $\mu(x) = 0$ (resp. $B(x) = \mathbf{1}$ and $\mu(x) = 1$) for each $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a cubic empty (resp. whole) set in X.

We denote the set of all cubic sets in X by $([I] \times I)^X$.

Definition 5. *Let X be a nonempty set. Then a complex mapping $\mathcal{R} =< R, \lambda >: X \times X \to [H] \times H$ is called a cubic H-relation in X. The pair $(X, \mathcal{R})$ is called a cubic H-relational space. In particular, a cubic H-relation from X to X is called a H-relation in or on X. We will denote the set of all cubic H-relations in X as resp. $([H] \times H)^{X \times X}$. In fact, each member $\mathcal{R} =< R, \lambda >\in ([H] \times H)^{X \times Y}$ is a cubic H-set in $X \times X$ (See [35]).*

Definition 6. *Let $(X, \mathcal{R}_X) = (X, < R_X, \lambda_X >)$ and $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$ be two cubic H-relational spaces. Then a mapping $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is called:*

(i) a P-order preserving mapping, if it satisfies the following condition:

$$\mathcal{R}_X \sqsubset \mathcal{R}_Y \circ f^2 =< R_Y \circ f^2, \lambda_Y \circ f^2 >, \text{ i.e., for each } (x, y) \in X \times X,$$

$$< [R_X^-(x, y), R_X^+(x, y)], \lambda(x, y) >$$

$$\leq_P < [R_Y^-(f(x), f(y)), R_Y^+(f(x), f(y))], \lambda_Y(f(x), f(y)) >, \text{ i.e.,}$$

$$R_X^-(x, y) \leq (R_Y^- \circ f^2)(x, y),\ R_X^+(x, y) \leq (R_Y^+ \circ f^2)(x, y),\ \lambda_X(x, y) \leq (\lambda_Y \circ f^2)(x, y),$$

(ii) a R-order preserving mapping, if it satisfies the following condition:

$$\mathcal{R}_X \Subset \mathcal{R}_Y \circ f^2 =< R_Y \circ f^2, \lambda_Y \circ f^2 >, \text{ i.e., for each } (x, y) \in X \times X,$$

$$< [R_X^-(x, y), R_X^+(x, y)], \lambda(x, y) >$$

$$\leq_R < [R_Y^-(f(x), f(y)), R_Y^+(f(x), f(y))], \lambda_Y(f(x), f(y)) >, \text{ i.e.,}$$

$$R_X^-(x, y) \leq (R_Y^- \circ f^2)(x, y),\ R_X^+(x, y) \leq (R_Y^+ \circ f^2)(x, y),\ \lambda_X(x, y) \geq (\lambda_Y \circ f^2)(x, y),$$

where $f^2 = f \times f$.

Proposition 1. *Let $(X, \mathcal{R}_X) = (X, < R_X, \lambda_X >)$, $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$ and $(Z, \mathcal{R}_Z) = (Z, < R_Z, \lambda_Z >)$ be three cubic H- relational spaces.*

(1) The identity mapping $1_X : (X, \mathcal{R}_X) \to (X, \mathcal{R}_X)$ is a P-order [resp. R-oder] preserving mapping.
(2) If $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ and $g : (Y, \mathcal{R}_Y) \to (Z, \mathcal{R}_Z)$ are P-preserving [resp. R-preserving] mappings, then $g \circ f : (XX, \mathcal{R}_X) \to (Z, \mathcal{R}_Z)$ is a P-preserving [resp. R-preserving] mapping.

Proof. (1) The proof follows from the definitions of P-orders and R-orders, and identity mappings.

(2) Suppose $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ and $g : (Y, \mathcal{R}_Y) \to (Z, \mathcal{R}_Z)$ are P-preserving mappings and let $(x, y) \in X \times X$. Then

$$\mathcal{R}_X(x,y) =< [R^-_X(x,y), R^+_X(x,y)], \lambda_X(x,y) >$$
$$\leq_P < [(R^-_Y \circ f^2)(x,y), R^+_Y \circ f^2)(x,y)], \lambda_Y \circ f^2)(x,y) >$$

$$\begin{aligned}
&\text{[Since } f \text{ is a P-preserving mapping]}\\
&=< [R^-_Y(f(x), f(y)), R^+_Y(f(x), f(y))], \lambda_Y(f(x), f(y)) >\\
&\leq_P [R^-_Z(g(f(x)), g(f(y))), R^+_Z(g(f(x)), g(f(y)))], \lambda_Z(g(f(x)), g(f(y))) >\\
&\text{[Since } g \text{ is a P-preserving mapping]}\\
&= [R^-_Z \circ (g \circ f)^2(x,y), R^+_Z \circ (g \circ f)^2(x,y)], \lambda_Z \circ (g \circ f)^2(x,y) >.
\end{aligned}$$

Thus, $\mathcal{R}_X \sqsubset \mathcal{R}_Z \circ (g \circ f)^2$. So $g \circ f$ is a P-preserving mapping. $\square$

We will denote the collection consisting of all cubic H-relational spaces and all P-preserving [resp. R-preserving] mappings between any two cubic H-relational spaces as $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$]. Then from Proposition 1, we can easily see that $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] forms a concrete category. In the sequel, a P-preserving [resp. R-preserving] mapping between any two cubic H-spaces will be called a $\mathbf{CRel}_P(H)$-mapping [resp. $\mathbf{CRel}_R(H)$-mapping].

Lemma 1. *The category* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] is topological over* **Set**.

Proof. Let X be a set and let $(X_j, \mathcal{R}_j)_{j \in J} = (X_j, < R_j, \lambda_j >)$ be any family of cubic H-relational spaces indexed by a class J. Suppose $(f_j : X \to X_j)_J$ be a source of mappings. We define a mapping $\mathcal{R}_{X,P} =< R_{X,P}, \lambda_{X,P} >: X \times X \to [H] \times H$ as follows: for each $(x, y) \in X \times X$,

$$\mathcal{R}_{X,P}(x) = [\sqcap_{j \in J} (\mathcal{R}_j \circ f_j^2)](x,y), \text{ i.e.,}$$

$$\mathcal{R}_{X,P}(x,y) =< [\bigwedge_{j \in J} R^-_j(f_j(x), f_j(y)), \bigwedge_{j \in J} R^+_j(f_j(x), f_j(y)), \bigwedge_{j \in J} \lambda_j(f_j(x), f_j(y)) > .$$

Then clearly, for each $j \in J$ and $(x, y) \in X \times X$,

$$< [R^-_{X,P}(x,y), R^+_{X,P}(x,y)], \lambda_{X,P}(x,y) >$$
$$\leq_P < [R^-_j(f_j(x), f_j(y)), R^+_j(f_j(x), f_j(y)), \lambda_j(f_j(x), f_j(y)) > .$$

Thus, $\mathcal{R}_{X,P} \sqsubset \mathcal{R}_j \circ f_j^2$, for each $j \in J$. So $f_j : (X, \mathcal{R}_{X,P}) \to (X_j, \mathcal{R}_j)$ is a $\mathbf{CRel}_P(H)$-mapping, for each $j \in J$.

For any object $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y)$, let $g : Y \to X$ be any mapping for which $f_j \circ g : (Y, \mathcal{R}_Y) \to (X_j, \mathcal{R}_j)$ is a $\mathbf{CRel}_P(H)$-mapping, for each $j \in J$ and let $(y, y') \in Y \times Y$. Then for each $j \in J$,

$$\mathcal{R}_Y(y, y') \leq_P [\mathcal{R}_j \circ (f_j \circ g)^2](y, y') = [(\mathcal{R}_j \circ f_j^2) \circ g^2](y, y'), \text{ i.e.,}$$

$$< [R^-_Y(y, y'), R^+_Y(y, y')], \lambda_Y(y, y') >$$
$$\leq_P < [(R^-_j \circ f_j^2)(g(y), g(y')), (R^+_j \circ f_j^2)(g(y), g(y')], (\lambda_j \circ f_j^2)(g(y), g(y') > .$$

Thus,

$$\begin{aligned}
&< [R^-_Y(y, y'), R^+_Y(y, y')], \lambda_Y(y, y') >\\
&\leq_P < [\textstyle\bigwedge_{j \in J} (R^-_j \circ f_j^2)(g(y), g(y'), \bigwedge_{j \in J} (R^-_j \circ f_j^2)(g(y), g(y')],\\
&\textstyle\bigwedge_{j \in J} (\lambda_j \circ f_j^2)(g(y), g(y') >
\end{aligned}$$

$$= [\sqcap_{j\in J}(\mathcal{R}_j \circ f_j)](g(y), g(y^{'})$$
$$= (\mathcal{R}_{X,P} \circ g^2)(y, y^{'}). \text{ [By the definition of } \mathcal{R}_{X,P}]$$

So $\mathcal{R}_Y \sqsubset \mathcal{R}_{X,P} \circ g^2$. Hence $g : (Y, \mathcal{R}_Y) \to (X, \mathcal{R}_{X,P})$ is a $\mathbf{CRel}_P(H)$-mapping. Therefore $(f_j : (X, \mathcal{R}_{X,P}) \to (X_j, \mathcal{R}_j))_J$ is an initial source in $\mathbf{CRel}_P(H)$.

Now define a mapping $\mathcal{R}_{X,R} =< R_{X,R}, \lambda_{X,R} >: X \times X \to [H] \times H$ as below: for each $(x, y) \in X \times X$,

$$\mathcal{R}_{X,R}(x) = [\Cap_{j\in J}(\mathcal{R}_j \circ f_j^2)](x, y), \text{ i.e.,}$$

$$\mathcal{R}_{X,R}(x, y) =< [\bigwedge_{j\in J} R_j^-(f_j(x), f_j(y)), \bigwedge_{j\in J} R_j^+(f_j(x), f_j(y)), \bigvee_{j\in J} \lambda_j(f_j(x), f_j(y)) > .$$

Then clearly, for each $j \in J$ and $(x, y) \in X \times X$,

$$< [R_{X,R}^-(x, y), R_{X,R}^+(x, y)], \lambda_{X,R}(x, y) >$$
$$\leq_R < [R_j^-(f_j(x), f_j(y)), R_j^+(f_j(x), f_j(y))], \lambda_j(f_j(x), f_j(y)) > .$$

Thus, $\mathcal{R}_{X,R} \Subset \mathcal{R}_j \circ f_j^2$, for each $j \in J$. So $f_j : (X, \mathcal{R}_{X,R}) \to (X_j, \mathcal{R}_j)$ is a $\mathbf{CRel}_R(H)$-mapping, for each $j \in J$.

For any object $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y)$, let $g : Y \to X$ be any mapping for which $f_j \circ g : (Y, \mathcal{R}_Y) \to (X_j, \mathcal{R}_j)$ is a $\mathbf{CRel}_R(H)$-mapping, for each $j \in J$ and let $(y, y^{'}) \in Y \times Y$. Then for each $j \in J$,

$$\mathcal{R}_Y(y, y^{'}) \leq_R [\mathcal{R}_j \circ (f_j \circ g)^2](y, y^{'}) = [(\mathcal{R}_j \circ f_j^2) \circ g^2](y, y^{'}), \text{ i.e.,}$$

$$< [R_Y^-(y, y^{'}), R_Y^+(y, y^{'})], \lambda_Y(y, y^{'}) >$$
$$\leq_R < [(R_j^- \circ f_j^2)(g(y), g(y^{'})), (R_j^+ \circ f_j^2)(g(y), g(y^{'})], (\lambda_j \circ f_j^2)(g(y), g(y^{'}) > .$$

Thus,

$$< [R_Y^-(y, y^{'}), R_Y^+(y, y^{'})], \lambda_Y(y, y^{'}) >$$
$$\leq_R < [\bigwedge_{j\in J}(R_j^- \circ f_j^2)(g(y), g(y^{'}), \bigwedge_{j\in J}(R_j^- \circ f_j^2)(g(y), g(y^{'})],$$
$$\bigvee_{j\in J}(\lambda_j \circ f_j^2)(g(y), g(y^{'}) >$$
$$= [\Cap_{j\in J}(\mathcal{R}_j \circ f_j)](g(y), g(y^{'})$$
$$= (\mathcal{R}_{X,R} \circ g^2)(y, y^{'}). \text{ [By the definition of } \mathcal{R}_{X,R}]$$

So $\mathcal{R}_Y \sqsubset \mathcal{R}_{X,R} \circ g^2$. Hence $g : (Y, \mathcal{R}_Y) \to (X, \mathcal{R}_{X,R})$ is a $\mathbf{CRel}_R(H)$-mapping. Therefore $(f_j : (X, \mathcal{R}_{X,R}) \to (X_j, \mathcal{R}_j))_J$ is an initial source in $\mathbf{CRel}_R(H)$. This completes the proof. □

Example 1. *(1) (***Inverse image of a cubic** H**-relation***) Let X be a set, let $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$ be a cubic H-relational space and let $f : X \to Y$ be a mapping. Then there exists a unique initial cubic H-relation of P-order type $\mathcal{R}_{X,P}$ [resp. R-order type $\mathcal{R}_{X,R}$] in X for which $f : (X, \mathcal{R}_{X,P}) \to (Y, \mathcal{R}_Y)$ is a* $\mathbf{CRel}_P(H)$*-mapping [resp. $f : (X, \mathcal{R}_{X,R}) \to (Y, \mathcal{R}_Y)$ is a* $\mathbf{CRel}_R(H)$*-mapping]. In fact,*

$$\mathcal{R}_{X,P} = \mathcal{R}_Y \circ f^2 =< R_Y \circ f^2, \lambda_Y \circ f^2 >= \mathcal{R}_{X,R}.$$

In this case, $\mathcal{R}_{X,P}$ [resp. $\mathcal{R}_{X,R}$] is called the inverse image under f of the cubic H-relation $\mathcal{R}_Y$ in Y.

In particular, if $X \subset Y$ and $f : X \to Y$ is the inclusion mapping, then the inverse image $\mathcal{R}_{X,P}$ [resp. $\mathcal{R}_{X,R}$] of $\mathcal{R}_Y$ under f is called a cubic H-subrelation of $(Y, \mathcal{R}_Y)$. In fact,

$$\mathcal{R}_{X,P}(x, y) = \mathcal{R}_Y(x, y) = \mathcal{R}_{X,R}(x, y), \text{ for each } (x, y) \in X \times X.$$

*(2) (**Cubic H-product relation**) Let $((X_j,\mathcal{R}_j))_{j\in J}=((X_j,<R_j,\lambda_j>))_{j\in J}$ be any family of cubic H-relational spaces and let $X=\Pi_{j\in J}X_j$. For each $j\in J$, let $pr_j: X\to X_j$ be the ordinary projection. Then there exists a unique cubic H-relation of P-order type, $\mathcal{R}_{X,P}$ in X for which $pr_j:(X,\mathcal{R}_{X,P})\to(X_j,\mathcal{R}_j)$ is a $\mathbf{CRel}_P(H)$-mapping, for each $j\in J$. In this case, $\mathcal{R}_{X,P}$ is called the cubic H-product relation of $(\mathcal{R}_j)_{j\in J}$ and $(X,\mathcal{R}_{X,P})$ is called the cubic H-product relational space of $((X_j,\mathcal{R}_j))_{j\in J}$, and denoted as the following, respectively:*

$$\mathcal{R}_{X,P}=\Pi_{j\in J}\mathcal{R}_j$$

and

$$(X,\mathcal{R}_{X,P})=(\Pi_{j\in J}X_j,\Pi_{j\in J}\mathcal{R}_j)=(\Pi_{j\in J}X_j,<\Pi_{j\in J}R_j,\Pi_{j\in J}\lambda_j>).$$

In fact, $\mathcal{R}_{X,P}(x)=[\sqcap_{j\in J}(\mathcal{R}_j\circ pr_j)](x,y)$, for each $(x,y)\in X\times X$.

Similarly, there exists a unique cubic H-relation of R-order type, $\mathcal{R}_{X,R}$ in X for which $pr_j:(X,\mathcal{R}_{X,R})\to(X_j,\mathcal{R}_j)$ is a $\mathbf{CRel}_R(H)$-mapping, for each $j\in J$. In this case, $\mathcal{R}_{X,R}$ is called the cubic H-product* relation of $(\mathcal{R}_j)_{j\in J}$ and $(X,\mathcal{R}_{X,R})$ is called the cubic H-product* relational space of $((X_j,\mathcal{R}_j))_{j\in J}$, and denoted as the following, respectively:

$$\mathcal{R}_{X,R}=\Pi^*_{j\in J}\mathcal{R}_j$$

and

$$(X,\mathcal{R}_{X,R})=(\Pi_{j\in J}X_j,\Pi^*_{j\in J}\mathcal{R}_j)=(\Pi_{j\in J}X_j,<\Pi_{j\in J}R_j,\Pi^*_{j\in J}\lambda_j>).$$

In fact, $\mathcal{R}_{X,R}(x,y)=[\Cap_{j\in J}(\mathcal{R}_j\circ pr_j)](x,y)$, for each $(x,y)\in X\times X$.

In particular, if $J=\{1,2\}$, then for each $(x_1,y_1),\ (x_2,y_2)\in X_1\times X_2$,

$$(\mathcal{R}_1\times\mathcal{R}_2)((x_1,y_1),(x_2,y_2))$$
$$=<[R_1^-(x_1,x_2)\wedge R_2^-(y_1,y_2),R_1^+(x_1,x_2)\wedge R_2^+(y_1,y_2)],\lambda_1(x_1,x_2)\wedge\lambda_2(y_1,y_2)>$$

and

$$(\mathcal{R}_1\times^*\mathcal{R}_2)((x_1,y_1),(x_2,y_2))$$
$$=<[R_1^-(x_1,x_2)\wedge R_2^-(y_1,y_2),R_1^+(x_1,x_2)\wedge R_2^+(y_1,y_2)],\lambda_1(x_1,x_2)\vee\lambda_2(y_1,y_2)>.$$

The following is obvious from Lemma 3.9 and Theorem 1.6 in [25] or Proposition in Section 1 in [38].

Corollary 1. *The category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] is complete and cocomplete over* **Set**.

Furthermore, we can easily see that $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] is well-powered and cowell-powered. It is well-known that a concrete category is topological if and only if it is cotopological (See Theorem 1.5 in [25]). However, we prove directly that $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] is cotopological.

Lemma 2. *The category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] is cotopological over* **Set**.

Proof. Let X be any set and let $(X_j,\mathcal{R}_j)_{j\in J}=(X_j,<R_j,\lambda_j>)$ be any family of cubic H-relational spaces indexed by a class J. Suppose $(f_j: X_j\to X)_{j\in J}$ is a sink of mappings. We define a mapping $\mathcal{R}_{X,P}=<R_{X,P},\lambda_{X,P}>: X\times X\to[H]\times H$ as follows: for each $(x,y)\in X\times X$,

$$\mathcal{R}_{X,P}(x,y)=(\sqcup_{j\in J}\sqcup_{(x_j,y_j)\in f^{-2}(x,y)}\mathcal{R}_j)(x_j,y_j)=\bigvee_{j\in J}\bigvee_{(x_j,y_j)\in f^{-2}(x,y)}\mathcal{R}_j(x_j,y_j).$$

Then we can easily see that

$$f_j:(X_j,\mathcal{R}_j)\to(X,\mathcal{R}_{X,P})\text{ is a }\mathbf{CRel}_P(H)-\text{mapping, for each }j\in J.$$

For any cubic H-relational space $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$, let $g : X \to Y$ be any mapping such that $g \circ f_j : (X_j, \mathcal{R}_j) \to (Y, \mathcal{R}_Y)$ is a $\mathbf{CRel}_P(H)$-mapping, for each $j \in J$ and let $(x, y) \in X \times X$. Then for each $j \in J$ and each $(x_j, y_j) \in f_j^{-2}(x, y)$,

$$\begin{aligned}
&\mathcal{R}_j(x_j, y_j)\\
&=< [R_j^-(x_j, y_j), [R_j^+(x_j, y_j)], \lambda_j(x_j, y_j) >\\
&\leq_P < [(R_Y^- \circ (g \circ f_j)^2)(x_j, y_j), (R_Y^+ \circ (g \circ f_j)^2)(x_j, y_j)], (\lambda_Y \circ (g \circ f_j)^2)(x_j, y_j) >\\
&=< [(R_Y^- \circ g^2)(f_j(x_j), f_j(y_j)), (R_Y^+ \circ g^2)(f_j(x_j), f_j(y_j))], (\lambda_Y \circ g^2)(f_j(x_j), f_j(y_j)) >\\
&=< [(R_Y^- \circ g^2)(x, y), (R_Y^+ \circ g^2)(x, y), (\lambda_Y \circ g^2)(x, y) >\\
&= (\mathcal{R}_Y \circ g^2)(x, y).
\end{aligned}$$

Thus, by the definition of $\mathcal{R}_{X,P}$, $\mathcal{R}_{X,P}(x, y) \leq_P (\mathcal{R}_Y \circ g^2)(x, y)$. So $\mathcal{R}_{X,P} \sqsubset \mathcal{R}_Y \circ g^2$. Hence $g : (X, \mathcal{R}_{X,P}) \to (Y, \mathcal{R}_Y)$ is a $\mathbf{CRel}_P(H)$-mapping. Therefore $\mathbf{CRel}_P(H)$ is cotopological over **Set**.

Now we define a mapping $\mathcal{R}_{X,R} =< R_{X,R}, \lambda_{X,R} >: X \times X \to [H] \times H$ as follows: for each $(x, y) \in X \times X$,

$$\begin{aligned}
&\mathcal{R}_{X,R}(x, y)\\
&= (\Cup_{j \in J} \Cup_{(x_j, y_j) \in f^{-2}(x,y)} \mathcal{R}_j)(x_j, y_j)\\
&=< [\bigvee_{j \in J} \bigvee_{(x_j, y_j) \in f^{-2}(x)} R_j^-(x_j, y_j), \bigvee_{j \in J} \bigvee_{(x_j, y_j) \in f^{-2}(x,y)} R_j^+(x_j, y_j)],\\
&\qquad \bigwedge_{j \in J} \bigwedge_{x_j \in f^{-2}(x,y)} \lambda_j(x_j, y_j) > .
\end{aligned}$$

Then we can easily see that

$$f_j : (X_j, \mathcal{R}_j) \to (X, \mathcal{R}_{X,R}) \text{ is a } \mathbf{CRel}_R(H) - \text{mapping, for each } j \in J.$$

For any cubic H-relational space $(Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$, let $g : X \to Y$ be any mapping such that $g \circ f_j : (X_j, \mathcal{R}_j) \to (Y, \mathcal{R}_Y)$ is a $\mathbf{CRel}_R(H)$-mapping, for each $j \in J$ and let $(x, y) \in X \times X$. Then for each $j \in J$ and each $(x_j, y_j) \in f_j^{-2}(x, y)$,

$$\begin{aligned}
&\mathcal{R}_j(x_j, y_j)\\
&=< [R_j^-(x_j, y_j), [R_j^+(x_j, y_j)], \lambda_j(x_j, y_j) >\\
&\leq_R < [(R_Y^- \circ (g \circ f_j)^2)(x_j, y_j), (R_Y^+ \circ (g \circ f_j)^2)(x_j, y_j)], (\lambda_Y \circ (g \circ f_j)^2)(x_j, y_j) >\\
&=< [(R_Y^- \circ g^2)(f_j(x_j), f_j(y_j)), (R_Y^+ \circ g^2)(f_j(x_j), f_j(y_j))], (\lambda_Y \circ g^2)(f_j(x_j), f_j(y_j)) >\\
&=< [(R_Y^- \circ g^2)(x, y), (R_Y^+ \circ g^2)(x, y)], (\lambda_Y \circ g^2)(x, y) >\\
&= (\mathcal{R}_Y \circ g^2)(x, y).
\end{aligned}$$

Thus, by the definition of $\mathcal{R}_{X,R}$, $\mathcal{R}_{X,R}(x, y) \leq_R (\mathcal{R}_Y \circ g^2)(x, y)$. So $\mathcal{R}_{X,R} \Subset \mathcal{R}_Y \circ g^2$. Hence $g : (X, \mathcal{R}_{X,R}) \to (Y, \mathcal{R}_Y)$ is a $\mathbf{CRel}_R(H)$-mapping. Therefore $\mathbf{CRel}_R(H)$ is cotopological over **Set**. This completes the proof. □

Example 2. (**Cubic H-quotient relation**) *Let $(X, \mathcal{R}) = (X, < R, \lambda >)$ be a cubic H-relational space, let $\sim$ be an equivalence relation on X and let $\pi : X \to X/\sim$ be the canonical mapping. We define a mapping $\mathcal{R}_{X/\sim,P} : X/\sim \times X/\sim \to [H] \times H$ as below: for each $([x], [y]) \in X/\sim \times X/\sim$,*

$$\begin{aligned}
&\mathcal{R}_{X/\sim,P}([x], [y])\\
&= [\sqcup_{(x',y') \in \pi^{-2}([x],[y])} \mathcal{R}](x', y')\\
&=< [\bigvee_{(x',y') \in \pi^{21}([x],[y])} R^-(x', y'), \bigvee_{(x',y') \in \pi^{-2}([x],[y])} R^+(x', y')],\\
&\qquad \bigvee_{(x',y') \in \pi^{-2}([x],[y])} \lambda(x', y') > .
\end{aligned}$$

Then we can easily see that $\mathcal{R}_{X/\sim,P}$ is a cubic H-relation in $X/\sim$. Furthermore, $\pi : (X, \mathcal{R}) \to (X/\sim, \mathcal{R}_{X/\sim,P})$ is a $\mathbf{CRel}_P(H)$-mapping. Thus, $\mathcal{R}_{X/\sim,P}$ is the final cubic H-relation in $X/\sim$.

Now we define a mapping $\mathcal{R}_{X/\sim,R} : X/\sim \times X/\sim \to [H] \times H$ as follows: for each $([x], [y]) \in X/\sim \times X/\sim$,

$$\mathcal{R}_{X/\sim,R}([x]) = [\Cup_{(x',y') \in \pi^{-2}([x],[y])} \mathcal{R}](x', y')$$

$$=< [\bigvee_{(x',y')\in\pi^{-2}([x],[y])} R^{-}(x',y'), \bigvee_{(x',y')\in\pi^{-2}([x],[y])} R^{+}(x',y')], \bigwedge_{(x',y')\in\pi^{-2}([x],[y])} \lambda(x',y') > .$$

Then we can easily see that $\mathcal{R}_{X/\sim,R}$ is a cubic H-relation in $X/\sim$. Furthermore, $\pi : (X,\mathcal{R}) \to (X/\sim, \mathcal{A}_{X/\sim,R})$ is a **Crel**$_R(H)$*-mapping. Thus, $\mathcal{R}_{X/\sim,R}$ is the final cubic H-relation in $X/\sim$.*

In this case, $\mathcal{R}_{X/\sim,P}$ [resp. $\mathcal{A}_{X/\sim,R}$] is called the cubic H-quotient [resp. H-quotient] relation in X induced by $\sim$.*

Definition 7 ([38]). *Let* **A** *be a concrete category and let $f, g : A \to B$ be two* **A***-morphisms. Then a pair (E, e) is called an equalizer in* **A** *of f and g, if the following conditions hold:*

(i) *$e : E \to A$ is an* **A***-morphism,*
(ii) *$f \circ e = g \circ e$,*
(iii) *for any* **A***-morphism $e' : E' \to A$ such that $f \circ e' = g \circ e'$, there exists a unique* **A***-morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$.*

In this case, we say that **A** *has equalizers.*

Dual notion: Coequalizer.

Proposition 2. *The category* **CRel**$_P(H)$ *[resp.* **CRel**$_R(H)$*] has equalizers.*

Proof. Let $f, g : (X,\mathcal{R}_X) \to (Y,\mathcal{R}_Y)$ be two **CRel**$_P(H)$-mappings, where $\mathcal{R}_X =< R_X, \lambda_X >$ and $\mathcal{R}_Y =< R_Y, \lambda_Y >$. Let $E = \{a \in X : f(a) = g(a)\}$ and define a mapping $\mathcal{R}_{E,P} : E \times E \to [H] \times H$ as follows: for each $(a,b) \in E \times E$,

$$\mathcal{R}_{E,P}(a,b) = \mathcal{R}_X(a,b)) =< [R^-_X(a,b), R^+_X(a,b)], \lambda_X(a,b) > .$$

Then clearly, $\mathcal{R}_{E,P}$ is a cubic H-relation in E and $\mathcal{R}_{E,P} \sqsubset \mathcal{R}_X$. Consider the inclusion mapping $i : E \to X$. Then clearly, $i : (E, \mathcal{A}_{P,E}) \to (X, \mathcal{A})$ is a **CSet**$_P(H)$-mapping and $f \circ i = g \circ i$.

Let $k : (E', \mathcal{R}_{E'}) \to (X, \mathcal{A}_X)$ be a **CRel**$_P(H)$-mapping such that $f \circ k = g \circ k$. We define a mapping $\bar{k} : E' \to E$ as follows: for each $e' \in E'$,

$$\bar{k}(e') = i^{-1} \circ k(e').$$

Then clearly, $k = i \circ \bar{k}$.

Let $(e', f') \in E' \times E'$. Since $k : (E', \mathcal{R}_{E'}) \to (X, \mathcal{R}_{E,P})$ is a **CRel**$_P(H)$-mapping,

$$\begin{aligned}\mathcal{R}_{E,P} \circ (\bar{k})^2(e',f') &= \mathcal{R}_{E,P} \circ (\bar{k})^2(e',f') \\ &= \mathcal{R}_{E,P} \circ (i^{-2} \circ k^2(e',f')) \\ &= \mathcal{R}_{E,P} \circ k^2(e',f') \\ &\geq_P \mathcal{R}_{E'}(e',f').\end{aligned}$$

Thus, $\mathcal{R}_{E'} \sqsubset \mathcal{R}_{E,P} \circ (\bar{k})^2$. So $\bar{k} : (E', \mathcal{R}_{E'}) \to (E, \mathcal{R}_{E,P})$ is a **CRel**$_P(H)$-mapping.

Now in order to prove the uniqueness of $\bar{k}$, let $\bar{r} : E' \longrightarrow E$ such that $i \circ \bar{r} = k$. Then $\bar{r} = i^{-1} \circ k = \bar{k}$. Thus, $\bar{k}$ is unique. Hence **CRel**$_P(H)$ has equalizers.

Similarly, we can prove that **CRel**$_R(H)$ has the equalizer $\mathcal{R}_{E,P}$. □

For two cubic H-relations $\mathcal{R}_X =< R_X, \lambda_X >$ in X and $\mathcal{R}_Y =< R_Y, \lambda_Y >$ in Y, the product of P-order type [resp. R-order type], denoted by $\mathcal{R}_X \times_P \mathcal{Y}_Y$ [resp. $\mathcal{R}_X \times_R \mathcal{R}_Y$], is a cubic H-relation in $X \times Y$ defined by: for any (x,y), $(x',y') \in X \times Y$,

$$(\mathcal{R}_X \times_P \mathcal{R}_Y)((x,y),(x',y')) =< R_X(x,x') \wedge R_Y(y,y'), \lambda_X(x,x') \wedge \lambda_X(y,y') >$$

[resp. $(\mathcal{R}_X \times_R \mathcal{R})_Y((x,y),(x^{'},y^{'})) =< R_X(x,x^{'}) \wedge R_Y(y,y^{'}), \lambda_X(x,x^{'}) \vee \lambda_X(y,y^{'}) >$].

Lemma 3. *Final episinks in* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] are preserved by pullbacks.*

Proof. Let $(g_j : (X_j, \mathcal{R}_j) \to (Y, \mathcal{R}_Y))_{j \in J}$ be any final episink in $\mathbf{CRel}_P(H)$ and let $f : (W, \mathcal{R}_W) \to (Y, \mathcal{R}_Y)$ be any $\mathbf{CRel}_P(H)$-mapping,, where $\mathcal{R}_j =< R_j, \lambda_j >$, $\mathcal{R}_Y =< R_Y, \lambda_Y >$ and $\mathcal{R}_W =< R_W, \lambda_W >$. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}$$

and let us define a mapping $\mathcal{R}_{U_j,P} =< R_{U_j,P}, \lambda_{U_j,P} >: U_j \times U_j \to [H] \times H$ as follows: for each $((w,x_j),(w^{'},x_j^{'})) \in U_j \times U_j$,

$$\begin{aligned}
&\mathcal{R}_{U_j,P}((w,x_j),(w^{'},x_j^{'})) \\
&= (\mathcal{R}_W \times_P \mathcal{R}_j)\mid_{U_j \times U_j}((w,x_j),(w^{'},x_j^{'})) \\
&= (\mathcal{R}_W \times_P \mathcal{R}_j)((w,x_j),(w^{'},x_j^{'})) \\
&=< R_W(w,w^{'}) \wedge R_j(x_j,x_j^{'}), \lambda_W(w,w^{'}) \wedge \lambda_j(x_j,x_j^{'}) > \\
&=< (R_W \times R_j)((w,x_j),(w^{'},x_j^{'})), (\lambda_W \times \lambda_j)((w,x_j),(w^{'},x_j^{'})) >, \text{ i.e.,}
\end{aligned}$$

$$\mathcal{R}_{U_j,P} =< R_W \times R_j \mid_{U_j \times U_j}, \lambda_W \times \lambda_j \mid_{U_j \times U_j} > .$$

For each $j \in J$, let $e_j : U_j \to W$ and $p_j : U_j \to X_j$ be the usual projections. Then clearly, $e_j : (U_j, \mathcal{R}_{U_j,P}) \to (W, \mathcal{R}_W)$ and $p_j : (U_j, \mathcal{R}_{U_j,P}) \to (X_j, \mathcal{R}_j)$ are $\mathbf{CRel}_P(H)$-mappings and $g_j \circ p_j = f \circ e_j$, for each $j \in J$. Thus, we have the following pullback square in $\mathbf{CRel}_P(H)$:

$$\begin{tikzcd}
(U_j, \mathcal{R}_{U_j,P}) \arrow[r, "p_j"] \arrow[d, "e_j"'] & (X_j, \mathcal{R}_j) \arrow[d, "g_j"] \\
(W, \mathcal{R}_W) \arrow[r, "f"'] & (Y, \mathcal{R}_Y).
\end{tikzcd}$$

We will prove that $(e_j : (U_j, \mathcal{R}_{U_j,P}) \to (W, \mathcal{R}_W))_{j \in J}$ is a final episink in $\mathbf{CRel}_P(H)$. Let $w \in W$. Since $(g_j)_{j \in J}$ is an episink in $\mathbf{CSet}_P(H)$, there is $j \in J$ such that $g_j(x_j) = f(w)$, for some $x_j \in X_j$. Thus, $(w, x_j) \in U_j$ and $e_j(w, x_j) = w$. So $(e_j)_{j \in J}$ is an episink in $\mathbf{CRel}_P(H)$.

Finally, let us show that $(e_j)_J$ is final in $\mathbf{CRel}_P(H)$. Let $\mathcal{R}_W^*$ be the final structure in W regarding $(e_j)_{j \in J}$ and let $(w, w^{'}) \in W \times W$. Then

$$\begin{aligned}
\mathcal{R}_W(w,w^{'}) &=< R_W(w,w^{'}), \lambda_W(w,w^{'}) > \\
&=< R_W(w,w^{'}) \wedge R_W(w,w^{'}), \lambda_W(w,w^{'}) \wedge \lambda_W(w,w^{'}) > \\
&\leq_P < R_W(w,w^{'}) \wedge R_Y \circ f^2(w,w^{'}), \lambda_W(w,w^{'}) \wedge \lambda_Y \circ f^2(w,w^{'}) > \\
&\quad [\text{Since } f : (W, \mathcal{R}_W) \to (Y, \mathcal{R}_Y) \text{ is a } \mathbf{CRel}_P(H)\text{-mapping}] \\
&=< R_W(w,w^{'}) \wedge [\textstyle\bigvee_{j \in J} \bigvee_{(x_j,x_j^{'}) \in g_j^{-2}(f(w),f(w^{'}))} R_j(x_j,x_j^{'})], \\
&\quad \lambda_W(w,w^{'}) \wedge [\textstyle\bigvee_{j \in J} \bigvee_{(x_j,x_j^{'}) \in g_j^{-2}(f(w),f(w^{'}))} \lambda_j(x_j,x_j^{'})] > \\
&\quad [\text{Since } (g_j : (R_j, \mathcal{R}_j) \to (Y, \mathcal{R}_Y))_{j \in J} \text{ is a final episink in } \mathbf{CRel}_P(H)] \\
&=< \textstyle\bigvee_{j \in J} \bigvee_{(x_j,x_j^{'}) \in g_j^{-2}(f(w),f(w^{'}))} [R_W(w,w^{'}) \wedge R_j(x_j,x_j^{'})]], \\
&\quad \textstyle\bigvee_{j \in J} \bigvee_{(x_j,x_j^{'}) \in g_j^{-2}(f(w),f(w^{'}))} [\lambda_W(w,w^{'}) \wedge \lambda_j(x_j,x_j^{'})] > \\
&=< \textstyle\bigvee_{j \in J} \bigvee_{((w,x_j),(w^{'},x_j^{'})) \in e_j^{-2}(w,w^{'})} [R_W(w,w^{'}) \wedge R_j(x_j,x_j^{'})]], \\
&\quad \textstyle\bigvee_{j \in J} \bigvee_{((w,x_j),(w^{'},x_j^{'})) \in e_j^{-2}(w,w^{'})} [\lambda_W(w,w^{'}) \wedge \lambda_j(x_j,x_j^{'})] >
\end{aligned}$$

$$=< \bigvee_{j\in J} \bigvee_{((w,x_j),(w',x_j'))\in e_j^{-2}(w,w')}[R_{U_j,P}((w,x_j,(w',x_j')],$$
$$\bigvee_{j\in J} \bigvee_{((w,x_j),(w',x_j'))\in e_j^{-2}(w,w')}[\lambda_{U_j,P}((w,x_j,(w',x_j')] >$$
$$= \mathcal{R}^*_W(w,w').$$

Thus, $\mathcal{R}_W \sqsubset \mathcal{R}^*_W$. Since $(e_j : (U_j, \mathcal{R}_{U_j}) \to (W, \mathcal{R}_W))_{j\in J}$ is final, $1_W : (W, \mathcal{R}^*_W) \to (W, \mathcal{R}_W)$ is a $\mathbf{CRel}_P(H)$-mapping. So $\mathcal{R}^*_W \sqsubset \mathcal{R}_W$. Hence $\mathcal{R}^*_W = \mathcal{R}_W$. Therefore $(e_j)_{j\in J}$ is final.

Now we define a mapping $\mathcal{R}_{U_j,R} =< R_{U_j,R}, \lambda_{U_j,R} >: U_j \to [H] \times H$ as follows: for each $((w,x_j),(w',x_j')) \in U_j \times U_j$,

$$\begin{aligned}&\mathcal{R}_{U_j,R}((w,x_j),(w',x_j'))\\ &= (\mathcal{R}_W \times_R \mathcal{R}_j) \mid_{U_j\times U_j} ((w,x_j),(w',x_j'))\\ &= (\mathcal{R}_W \times_R \mathcal{R}_j)((w,x_j),(w',x_j'))\\ &=< R_W(w,w') \wedge R_j(x_j,x_j'), \lambda_W(w,w') \vee \lambda_j(x_j,x_j') > .\end{aligned}$$

For each $j \in J$, let $e_j : U_j \to W$ and $p_j : U_j \to X_j$ be the usual projections. Then we can similarly prove that final episinks in $\mathbf{Rel}_R(H)$ are preserved by pullbacks. This completes the proof. □

For any singleton set $\{a\}$, since the cubic set $\mathcal{R}_{\{a\}}$ in $\{a\}$ is not unique, the category $\mathbf{CRel}(H)$ is not properly fibered over **Set**. Then from Definitions 1 and 3, Lemmas 2 and 3, we have the following result.

Theorem 1. *The category* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

Theorem 2. *The category* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] is Cartesian closed over* **Set**.

Proof. From Lemma 1, it is clear that $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] has products. Then it is sufficient to prove that $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] has exponential objects.

For any cubic H-relational spaces $\mathbf{X} = (X, \mathcal{R}_X) = (X, < R_X, \lambda_X >)$ and $\mathbf{Y} = (Y, \mathcal{R}_Y) = (Y, < R_Y, \lambda_Y >)$, let Y^X be the set of all ordinary mappings from X to Y. We define two mappings $R_{Y^X} : Y^X \times Y^X \to [H]$ and $\lambda_{Y^X} : Y^X \times Y^X \to H$ as follows: for each $(f,g) \in Y^X \times Y^X$,

$R_{Y^X}(f,g) = \bigvee\{h \in H : R_X(x,y) \wedge h \leq R_Y(f(x), f(y)),$ for each $(x,y) \in X \times X\}$

and

$\lambda_{Y^X}(f,g) = \bigvee\{h \in H : \lambda_X(x,y) \wedge h \leq \lambda_Y(f(x), f(y)),$ for each $(x,y) \in X \times X\}$. Then clearly, $\mathcal{A}_{Y^X} =< A_{Y^X}, \lambda_{Y^X} >$ is a cubic H-relation in Y^X. Moreover, by the definitions of R_{Y^X} and λ_{Y^X},

$$R^-_X(x,y) \wedge R^-_{Y^X}(f,g) \leq R^-_Y(f(x), f(y)),\ R^+_X(x,y) \wedge R^+_{Y^X}(f,g) \leq R^-_Y(f(x), f(y))$$

and

$$\lambda_X(x,y) \wedge \lambda_{Y^X}(f,g) \leq \lambda_Y(f(x), f(y)),$$

for each $(x,y) \in X \times X$.

Let $\mathbf{Y^X} = (Y^X, \mathcal{R}_{Y^X})$ and let us define a mapping $e_{X,Y} : X \times Y^X \to Y$ as follows: for each $(x,f) \in X \times Y^X$,

$$e_{X,Y}(x,f) = f(x).$$

Let $(x,f),\ (y,g) \in X \times Y^X$. Then

$$\begin{aligned}(R^-_X \times_P R^-_{Y^X})((x,f),(y,g)) &= R^-_X(x,y) \wedge R^-_{Y^X}(f,g)\\ &\leq R^-_Y(f(x), f(y))\\ &= R^-_Y \circ e^2_{X,Y}((x,f),(y,g)),\\ &\text{[By the definition of } e_{X,Y}]\end{aligned}$$

$$\begin{aligned}(R^+_X \times_P R^+_{Y^X})((x,f),(y,g)) &= R^+_X(x,y) \wedge R^+_{Y^X}(f,g)\\ &\le A^+_Y(f(x),f(y))\\ &= A^+_Y \circ e^2_{X,Y}((x,f),(y,g)) \text{ and}\\ (\lambda_X \times_P \lambda_{Y^X})((x,f),(y,g)) &= \lambda_X(x,y) \wedge \lambda_{Y^X}(f,g)\\ &\le \lambda_Y(f(x),f(y))\\ &= \lambda_Y \circ e^2_{X,Y}((x,f),(y,g)).\end{aligned}$$

Thus, $e_{X,Y} : \mathbf{X} \times_P \mathbf{Y^X} \to \mathbf{Y}$ is a $\mathbf{CRel}_P(H)$-mapping, where $\mathbf{X} \times_P \mathbf{Y^X} = (X \times Y^X, < A_X \times_P A_{Y^X}, \lambda_X \times_P \lambda_{Y^X} >)$.

For any cubic H-relational space $\mathbf{Z} = (Z, < A_Z, \lambda_Z >)$, let $k : \mathbf{X} \times_P \mathbf{Z} \to \mathbf{Y}$ be a $\mathbf{CRel}_P(H)$-mapping. We define a mapping $\bar{k} : Z \to Y^X$ as follows: for each $z \in Z$ and each $x \in X$,

$$[\bar{k}(z)](x) = k(x,z).$$

Then we can prove that $\bar{k}$ is a unique $\mathbf{CRel}_P(H)$-mapping such that $e_{X,Y} \circ (1_X \times \bar{k}) = k$.

Now we define two mappings $R_{Y^X,R} : Y^X \times Y^X \to [H]$ and $\lambda_{Y^X,R} : Y^X \times Y^X \to H$ as follows: for each $(f,g) \in Y^X \times Y^X$ and each $x \in X$,

$$R_{Y^X,R}(f,g) = R_{Y^X,P}(f,g)$$

and

$$\lambda_{Y^X,R}(f,g) = \bigwedge\{h \in H : \lambda_X(x,y) \vee h \ge \lambda_Y(f(x),f(y)), \text{ for each } (x,y) \in X \times X\}.$$

Then clearly, $\mathcal{R}_{Y^X,R} =< R_{Y^X,R}, \lambda_{Y^X,R} >$ is a cubic H-relation in Y^X. Moreover, by the definitions of $R_{Y^X,R}$ and $\lambda_{Y^X,R}$,

$$R_X(x,y) \wedge R_{Y^X,R}(f,g) \le R_Y(f(x),f(y))$$

and

$$\lambda_X(x,y) \vee \lambda_{Y^X,R}(f,g) \ge \lambda_Y(f(x),f(y)),$$

for each $x \in X$. Let $\mathbf{Y^X} = (Y^X, \mathcal{R}_{Y^X,R})$ and let us define a mapping $e_{X,Y} : X \times Y^X \to Y$ as follows: for each $(x,f) \in X \times Y^X$,

$$e_{X,Y}(x,f) = f(x).$$

Let $(x,f),\ (y,g) \in X \times Y^X$. Then by the definitions of $R_{Y^X,R}$ and $\lambda_{Y^X,R}$, we have the followings:

$$(R_X \times_R A_{Y^X,R})((x,f),(y,g)) \le R_Y \circ e^2_{X,Y}((x,f),(y,g))$$

and

$$(\lambda_X \times_R \lambda_{P,Y^X})((x,f),(y,g)) \ge \lambda_Y \circ e^2_{X,Y}((x,f),(y,g)).$$

Thus, $\mathcal{R}_X \times_R \mathcal{R}_{Y^X,R} \Subset \mathcal{R}_Y \circ e^2_{X,Y}$. So $e_{X,Y} : \mathbf{X} \times_R \mathbf{Y^X} \to \mathbf{Y}$ is a $\mathbf{CRel}_R(H)$-mapping, where $\mathbf{X} \times_R \mathbf{Y^X} = (X \times Y^X, < R_X \times_R R_{Y^X,R}, \lambda_X \times_R \lambda_{Y^X,R} >)$.

For any cubic H-relational space $\mathbf{Z} = (Z, < R_Z, \lambda_Z >)$, let $k : \mathbf{X} \times_R \mathbf{Z} \to \mathbf{Y}$ be a $\mathbf{CRel}_R(H)$-mapping. We define a mapping $\bar{k} : Z \to Y^X$ as follows: for each $z \in Z$ and each $x \in X$,

$$[\bar{k}(z)](x) = k(x,z).$$

Then we can prove that $\bar{k}$ is a unique $\mathbf{CRel}_R(H)$-mapping such that

$$e_{X,Y} \circ (1_X \times \bar{k}) = k.$$

This completes the proof. □

Remark 1. *The category* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] is not a topos (See [39] for its definition), since it has no subobject classifier.*

Example 3. *Let* $I = \{0, 1\}$ *be two points chain, respectively and let* $X = \{a\}$. *Let* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be the cubic* H*-relations in* X *defined by:*

$$\mathcal{R}_1(a) =< \mathbf{0}, 0 > \text{ and } \mathcal{R}_2(a) =< \mathbf{1}, 1 > .$$

Let $1_X : (X, \mathcal{R}_1) \to (X, \mathcal{R}_2)$ *be the identity mapping. Then clearly,* 1_X *is both monomorphism and epimorphism in* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*]. However,* 1_X *is not an isomorphism in* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*]. Thus,* $\mathbf{CRel}(H)$ *has no subobject classifier.*

4. The Categories $\mathbf{CRel}_{P,R}(H)$ and $\mathbf{CRel}_{R,R}(H)$

In this section, we obtain two subcategories $\mathbf{CRel}_{P,R}(H)$ and $\mathbf{CRel}_{R,R}(H)$ of $\mathbf{CRel}_P(H)$ and $\mathbf{CRel}_R(H)$, respectively which are topological universes over **Set**.

It is interesting that final structures and exponential objects in $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] are shown to be quite different from those in $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$].

First of all, we list two well-known results.

Result 1 (Theorem 2.5 [25]). Let **A** be a well-powered and co(well-powered) topological category. Then the followings are equivalent:

(1) **B** is bireflective in **A**,
(2) **B** is closed under the formation of initial sources, i.e., for any initial source $(f_j : A \to A^j)_{j\in J}$ in **A** with $A_j \in \mathbf{B}$ for each $j \in J$, then $A \in \mathbf{B}$.

Result 2 (Theorem 2.6 [25]). If **A** is a topological category and **B** is a bireflective subcategory of **A**, then **B** is also a topological category. Moreover, every source in **B** which is initial in **A** is initial in **B**

Definition 8. *Let* X *be a nonempty set and let* $\mathcal{R} =< R, \lambda >$ *be a cubic* H*-relation in* X. *Then* $\mathcal{R}$ *is said to be reflexive, if* R *and* λ *are reflexive, i.e.,* $R(x, x) = \mathbf{1}$ *and* $\lambda(x, x) = 1$*, for each* $x \in X$.

The class of all cubic H-reflexive relational spaces and $\mathbf{CRel}_P(H)$-mappings [resp. $\mathbf{CRel}_R(H)$-mappings between them forms a subcategory of $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] denoted by $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$].

The following is the immediate result of Definitions 1 and 8.

Lemma 4. *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is properly fibered over* **Set**.

Lemma 5. *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is closed under the formation of initial sources in The category* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*]*

Proof. Let $f_j : (X, \mathcal{R}_{X,P}) \to (X_j, \mathcal{R}_j))_{j\in J}$ be an initial source in $\mathbf{CRel}_P(H)$ such that each $(X_j, \mathcal{R}_j)$ belongs to $\mathbf{CRel}_{P,R}(H)$, where $(X, \mathcal{R}_{X,P}) = (X, < R_{X,P}, \lambda_{X,P} >)$ and $(X_j, \mathcal{R}_j) = (X_j, < R_j, \lambda_j >)$. Let $x \in X$ and let $j \in J$. Since R_j and λ_j are reflexive, $R_j \circ f_j^2(x, x) = \mathbf{1}$ and $\lambda_j \circ f_j^2(x, x) = 1$. Then

$$R_{X,P}(x, x) = \bigwedge_{j\in J} R_j \circ f_j^2(x, x) = \mathbf{1} \text{ and } \lambda_{X,P}(x, x) = \bigwedge_{j\in J} \lambda_j \circ f_j^2(x, x) = 1.$$

Thus, $\mathcal{R}_{X,P}(x, x) =< \mathbf{1}, 1 >$. So $\mathcal{R}_{X,P}$ is reflexive.

Now let $f_j : (X, \mathcal{R}_{X,R}) \to (X_j, \mathcal{R}_j))_{j\in J}$ be an initial source in $\mathbf{CRel}_R(H)$ such that each $(X_j, \mathcal{R}_j)$ belongs to $\mathbf{CRel}_{R,R}(H)$. Then clearly, for each $x \in X$,

$$R_{X,R}(x,x) = R_{X,P}(x,x) = \mathbf{1} \text{ and } \lambda_{X,R}(x,x) = \bigvee_{j\in J} \lambda_j \circ f_j^2(x,x) = 1.$$

Thus, $\mathcal{R}_{X,R}(x,x) =< \mathbf{1}, 1 >$. So $\mathcal{R}_{X,R}$ is reflexive. This completes the proof. □

From Results 1, 2 and Lemma 5, we have the followings.

Proposition 3. (1) *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is a bireflective subcategory of* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*].*

(2) *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is topological over* **Set**.

It is well-known that a category **A** is topological if and only if it is cotopological. Then by (2) of the above Proposition, the category $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] is cotopological over **Set**. However, we will prove that $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] is cotopological over **Set**, directly.

Lemma 6. *the category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] has final structure over* **Set**.

Proof. Let X be a nonempty set and let $((X_j, \mathcal{R}_j)) = ((X_j, < R_j, \lambda_j >)_{j\in J}$ be any family of cubic H-relational spaces indexed by a class J. We define two mappings $R_{X,P} : X \to [H]$ and $\lambda^{X,P} : X \to H$, respectively as below: for each $(x,y) \in X \times X$,

$$R_{X,P}(x,y) = \begin{cases} \bigvee_{j\in J} \bigvee_{(x_j,y_j)\in f_j^{-2}(x,y)} R_j(x_j,y_j) & \text{if } (x,y) \in (X \times X - \triangle_X) \\ \mathbf{1} & \text{if } (x,y) \in \triangle_X \end{cases}$$

and

$$\lambda_{X,P}(x,y) = \begin{cases} \bigvee_{j\in J} \bigvee_{(x_j,y_j)\in f_j^{-2}(x,y)} \lambda_j(x_j,y_j) & \text{if } (x,y) \in (X \times X - \triangle_X) \\ 1 & \text{if } (x,y) \in \triangle_X, \end{cases}$$

where $\triangle_X = \{(x,x) : x \in X\}$. Then clearly, $\mathcal{R}_{X,P}$ is the cubic H-reflexive relation in X given by: for each $(x,y) \in X \times X$,

$$\mathcal{R}_{X,P}(x,y) = \begin{cases} \sqcup_{j\in J} \sqcup_{(x_j,y_j)\in f_j^{-2}(x,y)} \mathcal{R}_j(x_j,y_j) & \text{if } (x,y) \in (X \times X - \triangle_X) \\ < \mathbf{1}, 1 > & \text{if } (x,y) \in \triangle_X. \end{cases}$$

Moreover, we can easily check that $(X, \mathcal{R}_{X,P}) = (X, < R_{X,P}, \lambda_{X,P} >)$ is a final structure in $\mathbf{CRel}_{P,R}(H)$. Thus, $(f_j : (X_j, \mathcal{R}_j) \to (X, \mathcal{R}_{X,P}))_{j\in J}$ is a final sink in $\mathbf{CRel}_{P,R}(H)$.

Now we define two mappings $R_{X,R} : X \to [H]$ and $\lambda^{X,R} : X \to H$, respectively as follows: for each $(x,y) \in X \times X$,

$$R_{X,R}(x,y) = R_{X,P}(x,y)$$

and

$$\lambda_{X,R}(x,y) = \begin{cases} \bigwedge_{j\in J} \bigwedge_{(x_j,y_j)\in f_j^{-2}(x,y)} \lambda_j(x_j,y_j) & \text{if } (x,y) \in (X \times X - \triangle_X) \\ 1 & \text{if } (x,y) \in \triangle_X. \end{cases}$$

Then clearly, $\mathcal{R}_{X,R}$ is the cubic H-reflexive relation in X given by: for each $(x,y) \in X \times X$,

$$\mathcal{R}_{X,R}(x,y) = \begin{cases} \Cup_{j\in J} \Cup_{(x_j,y_j)\in f_j^{-2}(x,y)} \mathcal{R}_j(x_j,y_j) & \text{if } (x,y) \in (X \times X - \triangle_X) \\ < \mathbf{1}, 1 > & \text{if } (x,y) \in \triangle_X. \end{cases}$$

Moreover, we can easily show that $(f_j : (X_j, \mathcal{R}_j) \to (X, \mathcal{R}_{X,R}))_{j\in J}$ is a final sink in $\mathbf{CRel}_{R,R}(H)$. □

Lemma 7. *Final episinks in* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] are preserved by pullbacks.*

Proof. Let $(g_j : (X_j, \mathcal{R}_j) \to (Y, \mathcal{R}_{Y,P}))_{j\in J}$ be any final episink in $\mathbf{CRel}_{P,R}(H)$ and let $f : (W, \mathcal{R}_W) \to (Y, \mathcal{R}_{Y,P})$ be any $\mathbf{CRel}_P(H)$-mapping, where $(W, \mathcal{R}_W)$ is a cubic H-reflexive relational space. For each $j \in J$, let us take U_j, $\mathcal{R}_{U_j,P}$, e_j and p_j as in the first proof of Lemma 3. Then we can easily check that $\mathbf{CRel}_{P,R}(H)$ is closed under the formation of pullbacks in $\mathbf{CRel}_P(H)$. Thus, it is enough to prove that $(e_j)_{j\in J}$ is final.

Suppose $\mathcal{R}^*_W$ is the final cubic H-relation in W regarding $(e_j)_{j\in J}$ and let $(w, w') \in (W \times W - \triangle_X)$. Then

$$\begin{aligned}
\mathcal{R}_W(w,w') &= < R_W(w,w'), \lambda_W(w,w') > \\
&= < R_W(w,w') \wedge R_W(w,w'), \lambda_W(w,w') \wedge \lambda_W(w,w') > \\
&\leq_P < R_W(w,w') \wedge R_Y \circ f^2(w,w'), \lambda_W(w,w') \wedge \lambda_Y \circ f^2(w,w') > \\
&\quad \text{[Since } f : (W, \mathcal{R}_W) \to (Y, \mathcal{R}_Y) \text{ is a } \mathbf{CRel}_P(H)\text{-mapping]} \\
&= < R_W(w,w') \wedge [\textstyle\bigvee_{j\in J} \bigvee_{(x_j,x'_j)\in g_j^{-2}(f(w),f(w'))} R_j(x_j,x'_j)], \\
&\quad \lambda_W(w,w') \wedge [\textstyle\bigvee_{j\in J} \bigvee_{(x_j,x'_j)\in g_j^{-2}(f(w),f(w'))} \lambda_j(x_j,x'_j)] > \\
&\quad \text{[Since } (g_j : (R_j, \mathcal{R}_j) \to (Y, \mathcal{R}_Y))_{j\in J} \text{ is a final episink in } \mathbf{CRel}_P(H)] \\
&= < \textstyle\bigvee_{j\in J} \bigvee_{(x_j,x'_j)\in g_j^{-2}(f(w),f(w'))} [R_W(w,w') \wedge R_j(x_j,x'_j)]], \\
&\quad \textstyle\bigvee_{j\in J} \bigvee_{(x_j,x'_j)\in g_j^{-2}(f(w),f(w'))} [\lambda_W(w,w') \wedge \lambda_j(x_j,x'_j)] > \\
&= < \textstyle\bigvee_{j\in J} \bigvee_{((w,x_j),(w',x'_j))\in e_j^{-2}(w,w')} [R_W(w,w') \wedge R_j(x_j,x'_j)]], \\
&\quad \textstyle\bigvee_{j\in J} \bigvee_{((w,x_j),(w',x'_j))\in e_j^{-2}(w,w')} [\lambda_W(w,w') \wedge \lambda_j(x_j,x'_j)] > \\
&= < \textstyle\bigvee_{j\in J} \bigvee_{((w,x_j),(w',x'_j))\in e_j^{-2}(w,w')} [R_{U_j,P}((w,x_j,(w',x'_j)], \\
&\quad \textstyle\bigvee_{j\in J} \bigvee_{((w,x_j),(w',x'_j))\in e_j^{-2}(w,w')} [\lambda_{U_j,P}((w,x_j,(w',x'_j)] > \\
&= \mathcal{R}^*_W(w,w').
\end{aligned}$$

Thus, $\mathcal{R}_W \sqsubset \mathcal{R}^*_W$. On the other hand, by a similar argument in the first proof of Lemma 3, $\mathcal{R}^*_W \sqsubset \mathcal{R}_W$ on $W \times W - \triangle_W$. So $\mathcal{R}^*_W = \mathcal{R}_W$ on $W \times W - \triangle_W$. Now let $(w, w) \in \triangle_W$. Then clearly, $\mathcal{R}^*_W(w,w) =< \mathbf{1}, 1 >= \mathcal{R}_W(w,w)$. Thus, $\mathcal{R}^*_W = \mathcal{R}_W$ on $\triangle_W$. Hence $\mathcal{R}^*_W = \mathcal{R}_W$ on W.

Now for each $j \in J$, let us $\mathcal{R}_{U_j,R} =< R_{U_j,R}, \lambda_{U_j,R} >: U_j \to [H] \times H$ be the mapping as in the second proof of Lemma 3. Then we can similarly prove that final episinks in $\mathbf{Rel}_{R,R}(H)$ are preserved by pullbacks. This completes the proof. □

The following is the immediate result of Lemma 4, Proposition 3 (2) and Lemma 7.

Theorem 3. *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is a topological universe over* **Set**. *In particular,* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is Cartesian closed over* **Set** (See [1]) *and a concrete quasitopos* (See [40]).

In [41], Noh obtained exponential objects in $\mathbf{Rel}(I)$, where $\mathbf{Rel}(I)$ denotes the category of fuzzy relations. By applying his construction of an exponential object in $\mathbf{Rel}(I)$ to the category $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$], we have the following.

Proposition 4. *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] has an exponential object.*

Proof. For any $\mathbf{X} = (X, \mathcal{R}_X) = (X, < R_X, \lambda_X >$, $\mathbf{Y} = (Y, \mathcal{R}_Y) = (X, < R_Y, \lambda_Y >) \in \mathrm{Ob}(\mathbf{CRel}_{P,R}(H))$ and let $Y^X = \hom(\mathbf{X}, \mathbf{Y})$. For any $(f, g) \in Y^X \times Y^X$, let

$$\mathrm{D}(f,g) = \{(x,y) \in X \times X : R_X(x,y) > R_Y(f(x),g(y)),\ \lambda_X(x,y) > \lambda_Y(f(x),g(y))\}.$$

We define a mapping $\mathcal{R}_{Y^X,P} =< R_{Y^X,P}, \lambda_{Y^X,P} >: Y^X \times Y^X \to [H] \times H$ as follows: for each $(f, g) \in Y^X \times Y^X$,

$$\mathcal{R}_{Y^X,P}(f,g)$$

$$= \begin{cases} < \bigwedge_{(x,y)\in \mathrm{D}(f,g)} R_Y(f(x), f(y)), \bigwedge_{(x,y)\in \mathrm{D}(f,g)} \lambda_Y(f(x), f(y)) > & \text{if } \mathrm{D}(f,g) \neq \phi \\ < \mathbf{1}, 1 > & \text{if } \mathrm{D}(f,g) = \phi. \end{cases}$$

Then by the definition of $\mathrm{D}(f,g)$, $\mathrm{D}(f,f) = \phi$, for each $f \in Y^X$. Thus, $\mathcal{R}_{Y^X,P}(f,f) =< \mathbf{1}, 1 >$, for each $f \in Y^X$. So $\mathcal{R}_{Y^X,P}$ is a cubic H-reflexive relation in Y^X.

Let $\mathbf{Y^X} = (Y^X, \mathcal{R}_{Y^X,P})$ and we define the mapping $e_{X,Y} : \mathbf{X} \times_P \mathbf{Y^X} \to \mathbf{Y}$ as follows: for each $(a, f) \in X \times Y^X$,

$$e_{X,Y}(a, f) = f(a).$$

Let (a, f), $(b, g) \in X \times Y^X$.

Case 1: Suppose $\mathrm{D}(f,g) = \phi$. Then

$$\begin{aligned}
&(\mathcal{R}_X \times_P \mathcal{R}_{Y^X,P})((a, f),(b, g)) \\
&=< R_X(a,b) \wedge R_{Y^X,P}(f,g), \lambda_X(a,b) \wedge \lambda_{Y^X,P}(f,g) > \\
&=< R_X(a,b), \lambda_X(a,b) > \\
&\;\;[\text{By the definition of } R_{Y^X,P},\ R_{Y^X,P}(f,g) = \mathbf{1},\ \lambda_{Y^X,P}(f,g) = 1] \\
&\leq_P < R_Y(f(x), g(y)), \lambda_Y(f(x), g(y)) > [\text{Since } \mathrm{D}(f,g) = \phi] \\
&=< R_Y \circ e^2_{X,Y}((a, f),(b, g)).
\end{aligned}$$

Case 2: Suppose $\mathrm{D}(f,g) \neq \phi$. Then

$$\begin{aligned}
&(\mathcal{R}_X \times_P \mathcal{R}_{Y^X,P})((a, f),(b, g)) \\
&=< R_X(a,b) \wedge [\textstyle\bigwedge_{(x,y)\in \mathrm{D}(f,g)} R_Y(f(x), f(y))], \\
&\quad \lambda_X(a,b) \wedge [\textstyle\bigwedge_{(x,y)\in \mathrm{D}(f,g)} \lambda_Y(f(x), f(y))] > \\
&\leq_P < R_Y(f(x), g(y)), \lambda_Y(f(x), g(y)) > \\
&=< R_Y \circ e^2_{X,Y}((a, f),(b, g)).
\end{aligned}$$

Thus, in either case, $\mathcal{R}_X \times \mathcal{R}_{Y^X,P} \sqsubset R_Y \circ e^2_{X,Y}$. So $e_{X,Y}$ is a $\mathbf{CRel}_P(H)$-mapping.

Let $\mathbf{Z} = (Z, \mathcal{R}_Z) = (Z, < R_Z, \lambda_Z >)$ be any cubic H-reflexive relational space and let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be any $\mathbf{CRel}_P(H)$-mapping. We define the mapping $\bar{h} : Z \to Y^X$ as follows: for each $c \in Z$ and each $a \in X$,

$$[\bar{h}(c)](a) = h(a, c).$$

Let $c \in Z$ and let a, $b \in X$. Then

$$\begin{aligned}
&\mathcal{R}_Y \circ [\bar{h}(c)]^2(a,b) \\
&= \mathcal{R}_Y([\bar{h}(c)](a), [\bar{h}(c)](b)) \\
&=< R_Y([\bar{h}(c)](a), [\bar{h}(c)](b)), \lambda_Y([\bar{h}(c)](a), [\bar{h}(c)](b)) > \\
&=< R_Y(h(a,c), h(b,c)), \lambda_Y(h(a,c), h(b,c)) > \\
&=< R_Y \circ h^2(h(a,c), h(b,c)), \lambda_Y \circ h^2(h(a,c), h(b,c)) > \\
&= \mathcal{R}_Y \circ h^2((a,c),(b,c)) \\
&\geq_P (\mathcal{R}_X \times_P \mathcal{R}_Z)((a,c),(b,c)) \\
&=< (R_X \times_P R_Z)((a,c),(b,c)), (\lambda_X times_P \lambda_Z)((a,c),(b,c)) > \\
&=< R_X(a,b) \wedge R_Z(c,c), \lambda_X(a,b) \wedge \lambda_Z(c,c) > \\
&=< R_X(a,b), \lambda_X(a,b) > [\text{Since } \mathcal{R}_Z \text{ is reflexive}] \\
&= \mathcal{R}_X(a,b).
\end{aligned}$$

Thus, $\mathcal{R}_X \sqsubset \mathcal{R}_Y \circ [\bar{h}(c)]^2$. So $\bar{h}(c) : \mathbf{X} \to \mathbf{Y}$ is a $\mathbf{CRel}_P(H)$-mapping. Hence $\bar{h}$ is well-defined. Let c, $c' \in Z$.

Case 1: Suppose $\mathrm{D}(\bar{h}(c), \bar{h}(c')) = \phi$. Then

$$\begin{aligned}
\mathcal{R}_{Y^X,P} \circ \bar{h}^2(c, c') &= \mathcal{R}_{Y^X,P}(\bar{h}(c), \bar{h}(c')) \\
&=< \mathbf{1}, 1 > [\text{By the definition of } \mathcal{R}_{Y^X,P}] \\
&\geq_P \mathcal{R}_Z(c, c').
\end{aligned}$$

Case 2: Suppose $\mathrm{D}(\bar{h}(c), \bar{h}(c')) \neq \phi$. Then

$$\mathcal{R}_{Y^X,P}(\bar{h}(c), \bar{h}(c')) =< R_{Y^X,P}(\bar{h}(c), \bar{h}(c')), \lambda_{Y^X,P}(\bar{h}(c), \bar{h}(c')) >$$

$$
\begin{aligned}
&=< \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} R_Y([\bar{h}(c)](a),[\bar{h}(c^{'})](b)),\\
&\quad \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} \lambda_Y([\bar{h}(c)](a),[\bar{h}(c^{'})](b)) >\\
&=< \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} R_Y(h(a,c),h(b,c^{'})),\\
&\quad \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} \lambda_Y(h(a,c),h(b,c^{'})) >\\
&\geq_P < \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} [R_X(a,b)\wedge R_Z(c,c^{'})],\\
&\quad \bigwedge_{(a,b)\in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))} [\lambda_X(a,b)\wedge \lambda_Z(c,c^{'})] >.
\end{aligned}
$$

On one hand, for any $(a,b) \in \mathrm{D}(\bar{h}(c),\bar{h}(c^{'}))$,

$$
\begin{aligned}
R_X(a,b) &> R_Y([\bar{h}(c)](a),[\bar{h}(c^{'})](b))\\
&= R_Y(h(a,c),h(b,c^{'}))\\
&\geq R_X(a,b)\wedge R_Z(c,c^{'}).
\end{aligned}
$$

Thus, $R_X(a,b) > R_Z(c,c^{'})$. Similarly, we have $\lambda_X(a,b) > \lambda_Z(c,c^{'})$. So

$$\mathcal{R}_{Y^X,P}(\bar{h}(c),\bar{h}(c^{'})) \geq_P \mathcal{R}_Z(c,c^{'}).$$

Hence in either cases, $\mathcal{R}_Z \sqsubset \mathcal{R}_{Y^X,P} \circ \bar{h}^2$. Therefore $\bar{h}$ is a $\mathbf{CRel}_P(H)$-mapping. Furthermore, $\bar{h}$ is unique and $e_{X,Y} \circ (1_X \times \bar{h}) = h$.

Now for any $\mathbf{X} = (X,\mathcal{R}_X) = (X,< R_X,\lambda_X >$, $\mathbf{Y} = (Y,\mathcal{R}_Y) = (X,< R_Y,\lambda_Y >) \in \mathrm{Ob}(\mathbf{CRel}_{R,R}(H))$ and let $Y^X = \mathrm{hom}(\mathbf{X},\mathbf{Y})$. For any $(f,g) \in Y^X \times Y^X$, let

$$\mathrm{D}^{'}(f,g) = \{(x,y) \in X\times X : R_X(x,y) > R_Y(f(x),g(y)),\ \lambda_X(x,y) < \lambda_Y(f(x),g(y))\}.$$

We define a mapping $\mathcal{R}_{Y^X,R} =< R_{Y^X,R},\lambda_{Y^X,R} >$: $Y^X \times Y^X \to [H]\times H$ as follows: for each $(f,g) \in Y^X \times Y^X$,

$$
\begin{aligned}
&\mathcal{R}_{Y^X,R}(f,g)\\
&= \begin{cases} < \bigwedge_{(x,y)\in \mathrm{D}^{'}(f,g)} R_Y(f(x),f(y)), \bigvee_{(x,y)\in \mathrm{D}^{'}(f,g)} \lambda_Y(f(x),f(y)) > & \text{if } \mathrm{D}^{'}(f,g) \neq \phi\\ < \mathbf{1},1 > & \text{if } \mathrm{D}^{'}(f,g) = \phi. \end{cases}
\end{aligned}
$$

Then we can easily check that $\mathcal{R}_{Y^X,R}$ is a cubic H-reflexive relation in Y^X. Moreover, by the similar argument of the above proof, we can show that $\mathcal{R}_{Y^X,R}$ is an exponential object in Y^X. This completes the proof. □

Remark 2. (1) *We can see that exponential objects in* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] is quite different from those in* $\mathbf{CRel}_P(H)$ *[resp.* $\mathbf{CRel}_R(H)$*] constructed in Theorem 1.*

(2) *The category* $\mathbf{CRel}_{P,R}(H)$ *[resp.* $\mathbf{CRel}_{R,R}(H)$*] has no subject classifier.*

Example 4. *Let* $H = \{0,1\}$ *be the two points chain and let* $X = \{a,b\}$. *Let* $\mathcal{R}_{1,P} =< R_{1,P},\lambda_{1,P} >$ *and* $\mathcal{R}_{2,P} =< R_{2,P},\lambda_{2,P} >$ *be cubic H-reflexive relations in X given by:*

$$\mathcal{R}_{1,P}(a,a) = \mathcal{R}_{1,P}(b,b) =< \mathbf{1},1 >,\ \mathcal{R}_{1,P}(a,b) = \mathcal{R}_{1,P}(b,a) =< \mathbf{0},0 >$$

and

$$\mathcal{R}_{2,P}(a,a) = \mathcal{R}_{2,P}(b,b) =< \mathbf{1},1 >,\ \mathcal{R}_{2,P}(a,b) = \mathcal{R}_{2,P}(b,a) =< \mathbf{1},1 > .$$

Let $1_X : (X,\mathcal{R}_{1,P}) \to (X,\mathcal{R}_{2,P})$ *be the identity mapping. Then clearly,* 1_X *is both monomorphism and epimorphism in* $\mathbf{CRel}_P(H)$. *However,* 1_X *is not an isomorphism in* $\mathbf{CRel}_P(H)$.

5. Conclusions

We constructed the concrete category $\mathbf{CRel}_P(H)$ [resp. $\mathbf{CRel}_R(H)$] of cubic H-relational spaces and P-preserving [resp. R-preserving] mappings between them and studied it in the sense of a topological universe. In particular, we proved that it is Cartesian closed over **Set**. Next, We introduced the category $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] of cubic H-reflexive relational spaces and P-preserving [resp. R-preserving] mappings between them and investigated it in a viewpoint of a topological universe. In particular, we obtained exponential objects in $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$] quite different from those in $\mathbf{CRel}_{P,R}(H)$ [resp. $\mathbf{CRel}_{R,R}(H)$]. Also we proved that $\mathbf{CRel}_{P}(H)$ [resp. $\mathbf{CRel}_{R}(H)$] is a topological universe but $\mathbf{CRel}_{P}(H)$ [resp. $\mathbf{CRel}_{R}(H)$] not a topological universe. In the future, we will expect one to study some full subcategories of the category $\mathbf{CRel}_{P}(H)$ [resp. $\mathbf{CRel}_{R}(H)$].

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