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On the Nonlocal Fractional Delta-Nabla Sum Boundary Value Problem for Sequential Fractional Delta-Nabla Sum-Difference Equations

Jiraporn Reunsumrit [1,†] and Thanin Sitthiwirattham [2,*,†]

1 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jiraporn.r@sci.kmutnb.ac.th
2 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
* Correspondence: thanin_sit@dusit.ac.th
† These authors contributed equally to this work.

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Abstract: In this paper, we propose sequential fractional delta-nabla sum-difference equations with nonlocal fractional delta-nabla sum boundary conditions. The Banach contraction principle and the Schauder's fixed point theorem are used to prove the existence and uniqueness results of the problem. The different orders in one fractional delta differences, one fractional nabla differences, two fractional delta sum, and two fractional nabla sum are considered. Finally, we present an illustrative example.

Keywords: sequential fractional delta-nabla sum-difference equations; nonlocal fractional delta-nabla sum boundary value problem; existence; uniqueness

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1. Introduction

Nowaday, fractional calculus is attractive knowledge for many reseachers in many fields. In particular, the fractional calculus has been used in many research works related to biological, biomechanics, magnetic fields, echanics of micro/nano structures, and physical problems (see [1–7]). We can find fractional delta difference calculus and fractional nabla difference calculus in [8–24] and [25–36], respectively. Definitions and properties of fractional difference calculus are presented in the book [37].

We note that there are a few papers using the delta-nabla calculus as a tool. For example, Malinowska and Torres [38] presented the delta-nabla calculus of variations. Dryl and Torres [39,40] studied the delta-nabla calculus of variations for composition functionals on time scales, and a general delta-nabla calculus of variations on time scales with application to economics. Ghorbanian and Rezapour [41] proposed a two-dimensional system of delta-nabla fractional difference inclusions. Liu, Jin and Hou [42] investigated existence of positive solutions for discrete delta-nabla fractional boundary value problems with p-Laplacian.

In this paper, we aim to extend the study of delta-nabla calculus that has appeared in discrete fractional boundary value problems. We have found that the research works related to delta-nabla calculus were presented as above. However, the boundary value problem for sequential fractional delta-nabla difference equation has not been studied before. Our problem is sequential fractional delta-nabla sum-difference equations with nonlocal fractional delta-nabla sum boundary conditions as given by

$$\begin{aligned} \Delta^{\alpha}\nabla^{\beta}u(t) &= F\left[t+\alpha-1, u(t+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(t+\alpha-1), \left(\mathcal{T}^{\phi}u\right)(t+\alpha-1)\right] \\ \Delta^{-\omega}u(\alpha+\omega-2) &= \kappa\nabla^{-\gamma}u(\alpha-2) \\ u(T+\alpha) &= \lambda u(\eta), \;\; \eta \in \mathbb{N}_{\alpha-1,T+\alpha-1} \end{aligned} \tag{1}$$

where $t \in \mathbb{N}_{0,T} := \{0,1,\ldots,T\}$; $\alpha \in (1,2]$; $\beta,\theta,\phi,\omega,\gamma \in (0,1]$; $\alpha+\beta \in (2,3]$; $T > 0$; κ,λ are given constants; $F \in C(\mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{R}^3, \mathbb{R})$; and for $\varphi,\psi \in C(\mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}, [0,\infty))$, we define

$$\begin{aligned} \left(\mathcal{S}^{\theta}u\right)(t) &:= \left[\nabla^{-\theta}\varphi u\right](t) = \frac{1}{\Gamma(\theta)}\sum_{s=\alpha-2}^{t}(t-\rho(s))^{\overline{\theta-1}}\,\varphi(t,s)\,u(s), \\ \left(\mathcal{T}^{\phi}u\right)(t) &:= \left[\Delta^{-\phi}\psi u\right](t+\phi) = \frac{1}{\Gamma(\phi)}\sum_{s=\alpha-\phi-2}^{t-\phi}(t-\sigma(s))^{\underline{\phi-1}}\,\psi(t,s+\phi)\,u(s+\phi). \end{aligned}$$

The objective of this research is to investigate the solution of the boundary value problem (1). The basic knowledge is disscussed in Section 2, the existence results are presented in Section 3, and an example is provided in Section 4.

2. Preliminaries

We give the notations, definitions, and lemmas as follows. The forward operator and the backward operator are defined as $\sigma(t) := t+1$, and $\rho(t) := t-1$, respectively.

For $t,\alpha \in \mathbb{R}$, we define the generalized falling and rising functions as follows:

- The generalized falling function
$$t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$$
for any $t+1-\alpha$ is not a pole of the Gamma function. If $t+1-\alpha$ is a pole and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.
- The generalized rising function
$$t^{\overline{\alpha}} := \frac{\Gamma(t+\alpha)}{\Gamma(t)}$$
for any t is not a pole of the Gamma function. If t is a pole and $t+\alpha$ is not a pole, then $t^{\overline{\alpha}} = 0$.

Definition 1. *For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the α-order fractional delta sum of f is defined by*

$$\Delta^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s), \;\; t \in \mathbb{N}_{a+\alpha}.$$

The α-order fractional nabla sum of f is defined by

$$\nabla^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s), \;\; t \in \mathbb{N}_{a}.$$

Definition 2. *For $\alpha > 0$, $N \in \mathbb{N}$ where $0 \le N-1 < \alpha < N$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional delta difference of f is defined by*

$$\Delta^{\alpha}f(t) := \Delta^{N}\Delta^{-(N-\alpha)}f(t) = \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{t+\alpha}(t-\sigma(s))^{\underline{-\alpha-1}}f(s), \;\; t \in \mathbb{N}_{a+N-\alpha}.$$

The α-order Riemann-Liouville fractional nabla difference of f is defined by

$$\nabla^{\alpha} f(t) := \nabla^{N} \nabla^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{-\alpha-1}} f(s), \ \ t \in \mathbb{N}_{a+N}.$$

Lemma 1 ([15]). *Let $0 \leq N-1 < \alpha \leq N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_a \to \mathbb{R}$. Then,*

$$\Delta^{-\alpha} \Delta^{\alpha} y(t) \ = \ y(t) + C_1 (t-a)^{\underline{\alpha-1}} + C_2 (t-a)^{\underline{\alpha-2}} + \ldots + C_N (t-a)^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

Lemma 2 ([28]). *Let $0 \leq N-1 < \alpha \leq N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_{a+1} \to \mathbb{R}$. Then,*

$$\nabla^{-\alpha} \nabla^{\alpha} y(t) = \begin{cases} y(t), & \alpha \notin \mathbb{N} \\ y(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^{k} f(a), & \alpha = N, \end{cases}$$

for some $t \in \mathbb{N}_{a+N}$.

We next provide a linear variant of our problem (1).

Lemma 3. *Let $\Lambda \neq 0$; $\alpha \in (1,2]$; $\beta, \omega, \gamma \in (0,1)$; $\alpha + \beta \in (2,3]$; $T > 0$; κ, λ are given constants; and $h \in C\big(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}\big)$. Then the problem*

$$\Delta^{\alpha} \nabla^{\beta} u(t) \ = \ h(t+\alpha-1), \ \ t \in \mathbb{N}_{0,T} \tag{2}$$

$$\Delta^{-\omega} u(\alpha+\omega-2) \ = \ \kappa \nabla^{-\gamma} u(\alpha-2) \tag{3}$$

$$u(T+\alpha) \ = \ \lambda u(\eta), \ \ \eta \in \mathbb{N}_{\alpha-1,T+\alpha-1} \tag{4}$$

has the unique solution

$$\begin{aligned} u(t) \ = \ & \frac{\mathcal{O}[h]}{\Lambda \Gamma(\beta)} \sum_{s=\alpha-1}^{t} (t-\rho(s))^{\overline{\beta-1}} s^{\underline{\alpha-1}} \\ & + \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{t} \sum_{r=0}^{s-\alpha} (t-\rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}} h(r+\alpha-1) \end{aligned} \tag{5}$$

where the functional $\mathcal{O}[h]$ and the constant Λ are defined by

$$\begin{aligned} \mathcal{O}[h] \ = \ & \frac{\lambda}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{\eta} \sum_{r=0}^{s-\alpha} (\eta-\rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}} h(r+\alpha-1) \\ & - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} (T+\alpha-\rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}} h(r+\alpha-1), \end{aligned} \tag{6}$$

$$\Lambda \ = \ \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-1}^{T+\alpha} (T+\alpha-\rho(s))^{\overline{\beta-1}} s^{\underline{\alpha-1}} - \frac{\lambda}{\Gamma(\beta)} \sum_{s=\alpha-1}^{\eta} (\eta-\rho(s))^{\overline{\beta-1}} s^{\underline{\alpha-1}}. \tag{7}$$

Proof. Using the fractional delta sum of order α for (2), we obtain

$$\nabla^{\beta} u(t) \ = \ C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1), \tag{8}$$

for $t \in \mathbb{N}_{\alpha-2,T+\alpha}$.

Taking the fractional nabla sum of order β for (8), we get

$$u(t) = \frac{1}{\Gamma(\beta)}\sum_{s=\alpha-2}^{t}(t-\rho(s))^{\overline{\beta-1}}\left[C_1 s^{\underline{\alpha-1}} + C_2 s^{\underline{\alpha-2}}\right] + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t}\sum_{r=0}^{s-\alpha}(t-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}h(r+\alpha-1), \tag{9}$$

for $t \in \mathbb{N}_{\alpha-2,T+\alpha}$.

Using the fractional delta sum of order ω for (9), we have

$$\Delta^{-\omega}u(t) = \sum_{s=\alpha}^{t-\omega}\sum_{r=\alpha}^{s}\frac{(t-\sigma(s))^{\underline{\omega-1}}(s-\rho(r))^{\overline{\beta-1}}}{\Gamma(\omega)\Gamma(\beta)}\left[C_1 r^{\underline{\alpha-1}} + C_2 r^{\underline{\alpha-2}}\right] + \sum_{s=\alpha}^{t-\omega}\sum_{r=\alpha}^{s}\sum_{\xi=0}^{r-\alpha}\frac{(t-\sigma(s))^{\underline{\omega-1}}(s-\rho(r))^{\overline{\beta-1}}(r-\sigma(\xi))^{\underline{\alpha-1}}}{\Gamma(\omega)\Gamma(\beta)\Gamma(\alpha)}h(\xi+\alpha-1), \tag{10}$$

for $t \in \mathbb{N}_{\alpha+\omega-1,T+\alpha+\omega}$.

Taking the fractional nabla sum of order γ for (9), we have

$$\nabla^{-\gamma}u(t) = \sum_{s=\alpha}^{t}\sum_{r=\alpha-2}^{s}\frac{(t-\rho(s))^{\overline{\gamma-1}}(s-\rho(r))^{\overline{\beta-1}}}{\Gamma(\gamma)\Gamma(\beta)}\left[C_1 r^{\underline{\alpha-1}} + C_2 r^{\underline{\alpha-2}}\right] + \sum_{s=\alpha}^{t}\sum_{r=\alpha}^{s}\sum_{\xi=0}^{r-\alpha}\frac{(t-\rho(s))^{\overline{\gamma-1}}(s-\rho(r))^{\overline{\beta-1}}(r-\sigma(\xi))^{\underline{\alpha-1}}}{\Gamma(\omega)\Gamma(\beta)\Gamma(\alpha)}h(\xi+\alpha-1), \tag{11}$$

for $t \in \mathbb{N}_{\alpha-2,T+\alpha}$.

By substituting $t = \alpha+\omega-2$ and $t = \alpha-2$ into (10) and (11), respectively; and using the condition (3), we obtain

$$C_2 = 0$$

Substitute $C_2 = 0$ and apply the condition (4). Then, we obtain

$$C_1 = \frac{\mathcal{O}[h]}{\Lambda}$$

where $\mathcal{O}[h]$ and Λ are defined by (6)–(7), respectively. Substituting the constants C_1 and C_2 into (9), we obtain (5). □

3. Existence and Uniqueness Result

Define $\mathcal{C} = C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R})$ is the Banach space of all function u and define the norm as

$$\|u\|_{\mathcal{C}} = \|u\| + \|\nabla^{-\theta}u\| + \|\Delta^{-\phi}u\|$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|u(t)|$, $\|\nabla^{-\theta}u\| = \max_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|\nabla^{-\theta}u(t)|$ and $\|\Delta^{-\phi}u\| = \max_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|\Delta^{-\phi}u(t+\phi)|$.

In addition, we define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{F}u)(t) = \frac{\mathcal{O}[h]}{\Lambda\Gamma(\beta)}\sum_{s=\alpha-1}^{t}(t-\rho(s))^{\overline{\beta-1}}s^{\underline{\alpha-1}} + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t}\sum_{r=0}^{s-\alpha}(t-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}} \times F\left[r+\alpha-1, u(r+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(r+\alpha-1), \left(\mathcal{T}^{\phi}u\right)(r+\alpha-1)\right], \tag{12}$$

where $\Lambda \neq 0$ is defined by (7) and the functional $\mathcal{O}[F(u)]$ is defined by

$$\begin{aligned}
\mathcal{O}[F(u)] \;=\;& \frac{\lambda}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{\eta} \sum_{r=0}^{s-\alpha} (\eta - \rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}} \times \\
& F\left[r+\alpha-1, u(r+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(r+\alpha-1), \left(\mathcal{T}^{\phi}u\right)(r+\alpha-1)\right] \\
& - \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} (T+\alpha - \rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}} \times \\
& F\left[r+\alpha-1, u(r+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(r+\alpha-1), \left(\mathcal{T}^{\phi}u\right)(r+\alpha-1)\right]. \qquad (13)
\end{aligned}$$

Obviously, the operator $\mathcal{F}$ has the fixed points if and only if the boundary value problem (1) has solutions. Firstly, we show the existence and uniqueness result of the boundary value problem (1) by using the Banach contraction principle.

Theorem 1. *Assume that* $F : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous,* $\varphi, \psi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0,\infty)$ *with* $\varphi_0 = \max\{\varphi(t-1,s) : (t,s) \in \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}\}$ *and* $\psi_0 = \max\{\psi(t-1,s) : (t,s) \in \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}\}$. *Suppose that the following conditions hold:*

(H_1) *there exist constants* $L_1, L_2, L_3 > 0$ *such that for each* $t \in \mathbb{N}_{\alpha-2,T+\alpha}$ *and* $u, v \in \mathbb{R}$

$$\begin{aligned}
& \left| F\left(t, u, \left(\mathcal{S}^{\theta}u\right), \left(\mathcal{T}^{\phi}u\right)\right) - F\left(t, v, \left(\mathcal{S}^{\theta}v\right), \left(\mathcal{T}^{\phi}v\right)\right) \right| \\
\leq\; & L_1|u-v| + L_2\left|\left(\mathcal{S}^{\theta}u\right) - \left(\mathcal{S}^{\theta}v\right)\right| + L_3\left|\left(\mathcal{T}^{\phi}u\right) - \left(\mathcal{T}^{\phi}v\right)\right|.
\end{aligned}$$

Then the problem (1) *has a unique solution on* $\mathbb{N}_{\alpha-2,T+\alpha}$ *provided that*

$$\chi := \left\{ L_1 + L_2\varphi_0 \frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0 \frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)} \right\} [\Omega_1 + \Omega_2 + \Omega_3] \;<\; 1 \qquad (14)$$

where

$$\Theta = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[\lambda \eta^{\underline{\alpha}} (\eta - \alpha + 1)^{\overline{\beta}} + (T+\alpha)^{\underline{\alpha}} (T+1)^{\overline{\beta}} \right], \qquad (15)$$

$$\Omega_1 = \frac{1}{\Gamma(\beta+1)} \left[\frac{\Theta}{|\Lambda|} (T+\alpha)^{\underline{\alpha-1}} (T+2)^{\overline{\beta-1}} + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} (T+1)^{\overline{\beta}} \right], \qquad (16)$$

$$\Omega_2 = \frac{1}{\Gamma(\theta+1)\Gamma(\beta+1)} \left[\frac{\Theta}{|\Lambda|} (T+\alpha)^{\underline{\alpha-1}} (T+2)^{\overline{\beta-1}} (T+2)^{\overline{\theta}} + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} (T+1)^{\overline{\beta}} (T+1)^{\overline{\theta}} \right], \qquad (17)$$

$$\Omega_3 = \frac{1}{\Gamma(\phi+1)\Gamma(\beta+1)} \left[\frac{\Theta}{|\Lambda|} (T+\alpha)^{\underline{\alpha-1}} (T+2)^{\overline{\beta-1}} (T+1)^{\underline{\phi}} + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} (T+1)^{\overline{\beta}} T^{\underline{\phi}} \right]. \qquad (18)$$

Proof. We shall show that $\mathcal{F}$ is a contraction. For any $u, v \in \mathcal{C}$ and for each $t \in \mathbb{N}_{\alpha-2,T+\alpha}$, we have

$$\begin{aligned}
\Big|\mathcal{O}[F(u)] - \mathcal{O}[F(v)]\Big| \leq \; & \Bigg| \tfrac{\lambda}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{\eta} \sum_{r=0}^{s-\alpha} (\eta - \rho(s))^{\overline{\beta-1}} (s - \sigma(r))^{\underline{\alpha-1}} \times \\
& \Big[L_1|u-v| + L_2\Big| (\mathcal{S}^{\theta}u) - (\mathcal{S}^{\theta}v) \Big| + L_3\Big| (\mathcal{T}^{\phi}u) - (\mathcal{T}^{\phi}v) \Big| \Big] \\
& - \tfrac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} (T+\alpha - \rho(s))^{\overline{\beta-1}} (s - \sigma(r))^{\underline{\alpha-1}} \times \\
& \Big[L_1|u-v| + L_2\Big| (\mathcal{S}^{\theta}u) - (\mathcal{S}^{\theta}v) \Big| + L_3\Big| (\mathcal{T}^{\phi}u) - (\mathcal{T}^{\phi}v) \Big| \Big] \Bigg| \\
\leq \; & \|u-v\|_{\mathcal{C}} \left\{ L_1 + L_2\varphi_0 \tfrac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0 \tfrac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)} \right\} \Bigg| \lambda \sum_{s=\alpha}^{\eta} \sum_{r=0}^{s-\alpha} \tfrac{(\eta-\rho(s))^{\overline{\beta-1}}}{\Gamma(\beta)} \times \\
& \tfrac{(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\alpha)} - \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \tfrac{(T+\alpha-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)} \Bigg| \\
\leq \; & \|u-v\|_{\mathcal{C}} \left\{ L_1 + L_2\varphi_0 \tfrac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0 \tfrac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)} \right\} \Theta,
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
& \Big|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\Big| \\
\leq \; & \left| \frac{\mathcal{O}[F(u)] - \mathcal{O}[F(v)]}{\Lambda} \right| \sum_{s=\alpha-1}^{T+\alpha} (T+\alpha-\rho(s))^{\overline{\beta-1}} s^{\underline{\alpha-1}} + \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \frac{(T+\alpha-\rho(s))^{\overline{\beta-1}}}{\Gamma(\beta)} \times \\
& \frac{(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\alpha)} \Big[L_1|u-v| + L_2\Big| \Big(\mathcal{S}^{\theta}u\Big) - \Big(\mathcal{S}^{\theta}v\Big) \Big| + L_3\Big| (\mathcal{T}^{\phi}u) - (\mathcal{T}^{\phi}v) \Big| \Big] \\
\leq \; & \|u-v\|_{\mathcal{C}} \left\{ L_1 + L_2\varphi_0 \frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0 \frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)} \right\} \left\{ \frac{\Theta(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|} \times \right. \\
& \left. \sum_{s=\alpha-1}^{T+\alpha} (T+\alpha-\rho(s))^{\overline{\beta-1}} + \sum_{s=\alpha}^{T+\alpha} \sum_{r=0}^{s-\alpha} \frac{(T+\alpha-\rho(s))^{\overline{\beta-1}} (s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)} \right\} \\
\leq \; & \|u-v\|_{\mathcal{C}} \left\{ L_1 + L_2\varphi_0 \frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0 \frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)} \right\} \Omega_1.
\end{aligned} \tag{20}$$

Next, we consider the following $(\nabla^{-\theta}\mathcal{F}u)$ and $(\Delta^{-\phi}\mathcal{F}u)$ as

$$\begin{aligned}
(\nabla^{-\theta}\mathcal{F}u)(t) \;=\; & \frac{\mathcal{O}[F(u)]}{\Lambda} \sum_{s=\alpha}^{t} \sum_{r=\alpha-1}^{s} \frac{(t-\rho(s))^{\overline{\theta-1}} (s-\rho(r))^{\overline{\beta-1}}}{\Gamma(\theta)\Gamma(\beta)} r^{\underline{\alpha-1}} \\
& + \sum_{s=\alpha}^{t} \sum_{r=\alpha}^{s} \sum_{\xi=0}^{r-\alpha} \frac{(t-\rho(s))^{\overline{\theta-1}} (s-\rho(r))^{\overline{\beta-1}} (r-\sigma(\xi))^{\underline{\alpha-1}}}{\Gamma(\theta)\Gamma(\beta)\Gamma(\alpha)} \times \\
& F\Big[\xi+\alpha-1, u(\xi+\alpha-1), \Delta^{\theta}u(\xi+\alpha-\theta+1), \nabla^{\gamma}u(\xi+\alpha+1)\Big]\Big],
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
(\Delta^{-\phi}\mathcal{F}u)(t+\phi) \;=\; & \frac{\mathcal{O}[F(u)]}{\Lambda} \sum_{s=\alpha}^{t-\phi} \sum_{r=\alpha-1}^{s} \frac{(t+\phi-\sigma(s))^{\underline{\phi-1}} (s-\rho(r))^{\overline{\beta-1}}}{\Gamma(\phi)\Gamma(\beta)} r^{\underline{\alpha-1}} \\
& + \sum_{s=\alpha}^{t-\phi} \sum_{r=\alpha}^{s} \sum_{\xi=0}^{r-\alpha} \frac{(t+\phi-\sigma(s))^{\underline{\phi-1}} (s-\rho(r))^{\overline{\beta-1}} (r-\sigma(\xi))^{\underline{\alpha-1}}}{\Gamma(\phi)\Gamma(\beta)\Gamma(\alpha)} \times \\
& F\Big[\xi+\alpha-1, u(\xi+\alpha-1), \Delta^{\theta}u(\xi+\alpha-\theta+1), \nabla^{\gamma}u(\xi+\alpha+1)\Big]\Big].
\end{aligned} \tag{22}$$

Similarly to the above, we have

$$\left|(\nabla^{-\theta}\mathcal{F}u)(t)-(\nabla^{-\theta}\mathcal{F}v)(t)\right| \le \|u-v\|_{\mathcal{C}}\left\{L_1+L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)}+L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right\}\Omega_2, \tag{23}$$

$$\left|\Delta^{-\phi}\mathcal{F}u)(t+\phi)-\Delta^{-\phi}\mathcal{F}v)(t+\phi)\right| \le \|u-v\|_{\mathcal{C}}\left\{L_1+L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)}+L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right\}\Omega_3. \tag{24}$$

From (20), (23) and (24), we get

$$\begin{aligned}\left\|(\mathcal{F}u)-(\mathcal{F}v)\right\|_{\mathcal{C}} \le\ & \|u-v\|_{\mathcal{C}}\left\{L_1+L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)}+L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right\}[\Omega_1+\Omega_2+\Omega_3] \\ =\ & \chi\|u-v\|_{\mathcal{C}}.\end{aligned} \tag{25}$$

By (14), we have $\left\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right\|_{\mathcal{C}} < \|u-v\|_{\mathcal{C}}$.

Consequently, $\mathcal{F}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is a unique solution of the problem (1) on $t\in\mathbb{N}_{\alpha-2,T+\alpha}$. □

4. Existence of at Least One Solution

Next, we provide Arzelá-Ascoli theorem and Schauder's fixed point theorem that will be used to prove the existence of at least one solution of (1).

Lemma 4 ([43])**.** *(Arzelá-Ascoli theorem) A set of function in $C[a,b]$ with the sup norm is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 5 ([43])**.** *A set is compact if it is closed and relatively compact.*

Lemma 6 ([44])**.** *(Schauder's fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T: D\to D$ be the map such that the set $Tu: u\in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^*\in U$: $Tu^*=u^*$.*

Theorem 2. *Assuming that (H_1) holds, problem (1) has at least one solution on $\mathbb{N}_{\alpha-3,T+\alpha}$.*

Proof. The proof is organized as follows.
Step I. Verify $\mathcal{F}$ map bounded sets into bounded sets in $B_R=\{u\in C(\mathbb{N}_{\alpha-2,T+\alpha}):\|u\|_{\mathcal{C}}\le R\}$.

Let $\max_{t\in\mathbb{N}_{\alpha-2,T+\alpha}}|F(t,0,0,0)|=M$ and choose a constant

$$R \ge \frac{M\,[\Omega_1+\Omega_2+\Omega_3]}{1-[\Omega_1+\Omega_2+\Omega_3]\left\{L_1+L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)}+L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right\}}. \tag{26}$$

For each $u \in B_R$, we obtain

$$
\begin{aligned}
& \left|\mathcal{O}[F(u)]\right| \\
\leq \; & \left|\lambda \sum_{s=\alpha}^{\eta}\sum_{r=0}^{s-\alpha} \frac{(\eta-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)} - \sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha} \frac{(T+\alpha-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}\right| \\
& \Big[\Big|F\big[r+\alpha-1, u(r+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(r+\alpha-\theta+1), \left(\mathcal{T}^{\phi}v\right)(r+\alpha+1)\big] \\
& -F(r+\alpha-1,0,0,0)\Big| + \left|F(r+\alpha-1,0,0,0)\right|\Big] \\
\leq \; & \left\{\left[L_1 + L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right]\|u\|_{\mathcal{C}} + M\right\}\Theta
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
& \left|(\mathcal{F}u)(t)\right| \\
\leq \; & \left|\frac{\mathcal{O}[F(u)]}{\Lambda}\right| \sum_{s=\alpha-1}^{T+\alpha}(T+\alpha-\rho(s))^{\overline{\beta-1}}s^{\underline{\alpha-1}} + \sum_{s=\alpha}^{T+\alpha}\sum_{r=0}^{s-\alpha}\frac{(T+\alpha-\rho(s))^{\overline{\beta-1}}}{\Gamma(\beta)} \times \\
& \frac{(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Big[\Big|F\big[r+\alpha-1, u(r+\alpha-1), \left(\mathcal{S}^{\theta}u\right)(r+\alpha-\theta+1), \\
& \left(\mathcal{T}^{\phi}v\right)(r+\alpha+1)\big] - F(r+\alpha-1,0,0,0)\Big| + \left|F(r+\alpha-1,0,0,0)\right|\Big] \\
\leq \; & \left\{\left[L_1 + L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right]\|u\|_{\mathcal{C}} + M\right\}\Omega_1.
\end{aligned}
\tag{28}
$$

In addition, we have

$$
\left|(\nabla^{-\theta}\mathcal{F}u)(t)\right| \leq \left\{\left[L_1 + L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right]\|u\|_{\mathcal{C}} + M\right\}\Omega_2, \tag{29}
$$

$$
\left|(\Delta^{-\phi}\mathcal{F}u)(t+\phi)\right| \leq \left\{\left[L_1 + L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right]\|u\|_{\mathcal{C}} + M\right\}\Omega_3. \tag{30}
$$

From (28), (29) and (30), we have

$$
\begin{aligned}
\left\|(\mathcal{F}u)(t)\right\|_{\mathcal{C}} \leq & \left\{\left[L_1 + L_2\varphi_0\frac{(T+3)^{\overline{\theta}}}{\Gamma(\theta+1)} + L_3\psi_0\frac{(T+2)^{\underline{\phi}}}{\Gamma(\phi+1)}\right]\|u\|_{\mathcal{C}} + M\right\}[\Omega_1+\Omega_2+\Omega_3] \\
\leq & \; R.
\end{aligned}
\tag{31}
$$

So, $\|\mathcal{F}u\|_{\mathcal{C}} \leq R$. Therefore $\mathcal{F}$ is uniformly bounded.

Step II. Since F and H are continuous, the operator $\mathcal{F}$ is continuous on B_R.

Step III. Prove that $\mathcal{F}$ is equicontinuous on B_R. For any $\epsilon > 0$, there exists a positive constant $\rho^* = \max\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, , \delta_6\}$ such that for $t_1, t_2 \in \mathbb{N}_{\alpha-2,T+\alpha}$
$\left|(t_2-\alpha+2)^{\overline{\beta}} - (t_1-\alpha+2)^{\overline{\beta}}\right| < \frac{\epsilon\Gamma(\beta+1)|\Lambda|}{6(T+\alpha)^{\underline{\alpha-1}}\Theta\|F\|}$, where $|t_2-t_1| < \delta_1$,
$\left|(t_2-\alpha+1)^{\overline{\beta}} - (t_1-\alpha+1)^{\overline{\beta}}\right| < \frac{\epsilon\Gamma(\alpha+1)\Gamma(\beta+1)}{6(T+\alpha)^{\underline{\alpha}}\|F\|}$, where $|t_2-t_1| < \delta_2$, $\Big|(t_2-\alpha+2)^{\overline{\theta}} - (t_1-\alpha+2)^{\overline{\theta}}\Big| < \frac{\epsilon\Gamma(\beta+1)\Gamma(\theta+1)|\Lambda|}{6(T+2)^{\overline{\beta}}(T+\alpha)^{\underline{\alpha-1}}\Theta\|F\|}$, where $|t_2-t_1| < \delta_3$,
$\left|(t_2-\alpha+1)^{\overline{\theta}} - (t_1-\alpha+1)^{\overline{\theta}}\right| < \frac{\epsilon\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\theta+1)}{6(T+1)^{\overline{\beta}}(T+\alpha)^{\underline{\alpha}}\Theta\|F\|}$, where $|t_2-t_1| < \delta_4$,

$$\begin{aligned}
\left|(t_2-\alpha+1)^{\underline{\phi}}-(t_1-\alpha+1)^{\underline{\phi}}\right| &< \tfrac{\epsilon\Gamma(\beta+1)\Gamma(\phi+1)|\Lambda|}{6(T+2)^{\overline{\beta}}(T+\alpha)^{\underline{\alpha-1}}\Theta\|F\|}, &&\text{where } |t_2-t_1|<\delta_5,\\
\left|(t_2-\alpha)^{\underline{\phi}}-(t_1-\alpha)^{\underline{\phi}}\right| &< \tfrac{\epsilon\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\phi+1)}{6(T+1)^{\overline{\beta}}(T+\alpha)^{\underline{\alpha}}\|F\|}, &&\text{where } |t_2-t_1|<\delta_6.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\right|\\
\leq\;& \left|\frac{\mathcal{O}[F(u)]}{\Lambda}\right|\left|\frac{1}{\Gamma(\beta)}\sum_{s=\alpha-1}^{t_2}(t_2-\rho(s))^{\overline{\beta-1}}s^{\underline{\alpha-1}}-\frac{1}{\Gamma(\beta)}\sum_{s=\alpha-1}^{t_1}(t_1-\rho(s))^{\overline{\beta-1}}s^{\underline{\alpha-1}}\right|\\
&\left|\sum_{s=\alpha}^{t_2}\sum_{r=0}^{s-\alpha}\frac{(t_2-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}\times\right.\\
&F\left[r+\alpha-1,u(r+\alpha-1),\left(\mathcal{S}^{\theta}u\right)(r+\alpha-1),\left(\mathcal{T}^{\phi}u\right)(r+\alpha-1)\right]\\
&-\sum_{s=\alpha}^{t_1}\sum_{r=0}^{s-\alpha}\frac{(t_1-\rho(s))^{\overline{\beta-1}}(s-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}\times\\
&\left.F\left[r+\alpha-1,u(r+\alpha-1),\left(\mathcal{S}^{\theta}u\right)(r+\alpha-1),\left(\mathcal{T}^{\phi}u\right)(r+\alpha-1)\right]\right|\\
<\;& \frac{\Theta\|F\|(T+\alpha)^{\underline{\alpha-1}}}{|\Lambda|\Gamma(\beta+1)}\left|(t_2-\alpha+2)^{\overline{\beta}}-(t_1-\alpha+2)^{\overline{\beta}}\right|+\frac{\|F\|(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\times\\
&\left|(t_2-\alpha+1)^{\overline{\beta}}-(t_1-\alpha+1)^{\overline{\beta}}\right|\\
<\;& \frac{\epsilon}{6}+\frac{\epsilon}{6}=\frac{\epsilon}{3}.
\end{aligned}\tag{32}$$

Similarly to the above, we have

$$\begin{aligned}
\left|(\nabla^{-\theta}\mathcal{F}u)(t_2)-(\nabla^{-\theta}\mathcal{F}u)(t_1)\right| &< \tfrac{\Theta\|F\|(T+\alpha)^{\underline{\alpha-1}}(T+2)^{\overline{\beta}}}{|\Lambda|\Gamma(\beta+1)\Gamma(\theta+1)}\left|(t_2-\alpha+2)^{\overline{\theta}}-(t_1-\alpha+2)^{\overline{\theta}}\right|\\
&\quad+\tfrac{\|F\|(T+\alpha)^{\underline{\alpha}}(T+1)^{\overline{\beta}}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\theta+1)}\left|(t_2-\alpha+1)^{\overline{\theta}}-(t_1-\alpha+1)^{\overline{\theta}}\right|\\
&< \tfrac{\epsilon}{6}+\tfrac{\epsilon}{6}=\tfrac{\epsilon}{3},
\end{aligned}\tag{33}$$

$$\begin{aligned}
\left|(\Delta^{-\phi}\mathcal{F}u)(t_2+\phi)-(\Delta^{-\phi}\mathcal{F}u)(t_1+\phi)\right| &< \tfrac{\Theta\|F\|(T+\alpha)^{\underline{\alpha-1}}(T+2)^{\overline{\beta}}}{|\Lambda|\Gamma(\beta+1)\Gamma(\phi+1)}\left|(t_2-\alpha+2)^{\overline{\phi}}-(t_1-\alpha+2)^{\overline{\phi}}\right|\\
&\quad+\tfrac{\|F\|(T+\alpha)^{\underline{\alpha}}(T+1)^{\overline{\beta}}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\phi+1)}\left|(t_2-\alpha)^{\overline{\phi}}-(t_1-\alpha)^{\overline{\phi}}\right|\\
&< \tfrac{\epsilon}{6}+\tfrac{\epsilon}{6}=\tfrac{\epsilon}{3}.
\end{aligned}\tag{34}$$

Hence

$$\left\|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\right\|_{\mathcal{C}} < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.\tag{35}$$

It implies that the set $\mathcal{F}(B_R)$ is equicontinuous. By the results of Steps I to III and the Arzelá-Ascoli theorem, $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is completely continuous. By Schauder fixed point theorem, the boundary value problem (1) has at least one solution. □

5. An Example

Here, we provide a sequential fractional delta-nabla sum-difference equations with nonlocal fractional delta-nabla sum boundary conditions

$$\begin{aligned} \Delta^{\frac{3}{2}}\nabla^{\frac{2}{3}}u(t) &= \frac{e^{-\sin^2\left(t+\frac{1}{2}\right)}}{\left(t+\frac{201}{2}\right)^2}\cdot\left[\frac{2\left|u\left(t+\frac{1}{2}\right)\right|}{\left|u\left(t+\frac{1}{2}\right)\right|+1}+\frac{\left|\left(\mathcal{S}^{\frac{1}{5}}u\right)\left(t+\frac{1}{2}\right)\right|}{\left|\left(\mathcal{S}^{\frac{1}{5}}u\right)\left(t+\frac{1}{2}\right)\right|+1}+\frac{3\left|\left(\mathcal{T}^{\frac{3}{4}}u\right)\left(t+\frac{1}{2}\right)\right|}{\left|\left(\mathcal{T}^{\frac{3}{4}}u\right)\left(t+\frac{1}{2}\right)\right|+1}\right] \\ \Delta^{-\frac{1}{3}}u\left(-\frac{11}{2}\right) &= 2\nabla^{-\frac{2}{5}}u\left(-\frac{1}{2}\right) \\ u\left(\frac{15}{2}\right) &= 3u\left(-\frac{7}{2}\right), \end{aligned} \tag{36}$$

where

$$\left(\mathcal{S}^{\theta}u\right)\left(t+\frac{1}{2}\right)=\frac{1}{\Gamma\left(\frac{1}{5}\right)}\sum_{s=-\frac{1}{2}}^{t+\frac{1}{2}}\left(t+\frac{1}{2}-\rho(s)\right)^{\overline{-\frac{4}{5}}}\frac{e^{-(s+1)}}{\left(t+\frac{21}{2}\right)^4}u(s),$$

$$\left(\mathcal{T}^{\phi}u\right)\left(t+\frac{1}{2}\right)=\frac{1}{\Gamma\left(\frac{3}{4}\right)}\sum_{s=-\frac{5}{4}}^{t-\frac{1}{4}}\left(t+\frac{1}{2}-\sigma(s)\right)^{\underline{-\frac{1}{4}}}\frac{e^{-\left(s+\frac{3}{4}\right)}}{\left(t+\frac{201}{2}\right)^2}u\left(s+\frac{3}{4}\right).$$

Letting $\alpha=\frac{3}{2}$, $\beta=\frac{2}{3}$, $\theta=\frac{1}{5}$, $\phi=\frac{3}{4}$, $\omega=\frac{1}{3}$, $\gamma=\frac{2}{5}$, $\eta=\frac{7}{2}$, $\kappa=2$, $\lambda=3$, $T=6$, $\varphi(t,s)=\frac{e^{-(s+1)}}{(t+10)^4}$, $\psi(t,s+\phi)=\frac{e^{-\left(s+\frac{3}{4}\right)}}{(t+100)^2}$, and $F\left[t,u(t),(\mathcal{S}^{\theta}u)(t),(\mathcal{T}^{\phi}u)(t)\right]=\frac{e^{-\sin^2 t}}{(t+100)^2}\cdot\left[\frac{2|u(t)|}{1+|u(t)|}+\frac{\left|(\mathcal{S}^{\frac{1}{5}}u)(t)\right|}{\left|(\mathcal{S}^{\frac{1}{5}}u)(t)\right|+1}+3\frac{\left|(\mathcal{T}^{\frac{3}{4}}u)(t)\right|}{\left|(\mathcal{T}^{\frac{3}{4}}u)(t)\right|+1}\right]$, we find that

$$|\Lambda|=3.29500,\ \ \Theta=29.34125,\ \ \Omega_1=57.88022,\ \ \Omega_2=246.53142,\ \ \Omega_3=699.50153.$$

where $|\Lambda|,\Theta,\Omega_1,\Omega_2,\Omega_3$ are defined by (7), (15)–(18), respectively. In addition, for $(t,s)\in\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}\times\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we find that

$$\varphi_0=\max\left\{\varphi(t-1,s)\right\}=0.0000745 \quad\text{and}\quad \psi_0=\max\left\{\psi(t-1,s)\right\}=0.0000787.$$

For each $t\in\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we obtain

$$\begin{aligned} &\left|F\left[t,u(t),\left(\mathcal{S}^{\theta}u\right)(t),\left(\mathcal{T}^{\phi}u\right)(t)\right]-F\left[t,v(t),\left(\mathcal{S}^{\theta}v\right)(t),\left(\mathcal{T}^{\phi}v\right)(t)\right]\right| \\ <\ &\frac{1}{\left(-\frac{1}{2}+100\right)^2}\cdot\left[2\frac{|u|-|v|}{[1+|u|]\,[1+|v|]}+\frac{|\left(\mathcal{S}^{\theta}u\right)|-|\left(\mathcal{S}^{\theta}v\right)|}{[|\left(\mathcal{S}^{\theta}u\right)|+1]\,[|\left(\mathcal{S}^{\theta}v\right)|+1]}\right. \\ &\left.+3\frac{|\left(\mathcal{T}^{\phi}u\right)|-|\left(\mathcal{T}^{\phi}v\right)|}{[|\left(\mathcal{T}^{\phi}u\right)|+1]\,[|\left(\mathcal{T}^{\phi}v\right)|+1]}\right] \\ \le\ &\frac{8}{39601}|u-v|+\frac{4}{39601}\left|\left(\mathcal{S}^{\theta}u\right)-\left(\mathcal{S}^{\theta}v\right)\right|+\frac{12}{39601}\left|\left(\mathcal{T}^{\phi}u\right)-\left(\mathcal{T}^{\phi}v\right)\right|. \end{aligned}$$

Therefore, (H_1) holds with $L_1=\dfrac{8}{39601}$, $L_2=\dfrac{4}{39601}$, and $L_3=\frac{12}{39601}$.

Then, we can show that (25) is true as follows

$$\chi=\left\{L_1+L_2\varphi_0\frac{(9)^{\overline{\frac{1}{5}}}}{\Gamma(\frac{6}{5})}+L_3\psi_0\frac{(8)^{\underline{\frac{3}{4}}}}{\Gamma(\frac{7}{4})}\right\}[\Omega_1+\Omega_2+\Omega_3]\approx 0.20294<1.$$

Hence, by Theorem 1, the problem (36) has a unique solution.

6. Conclusions

We estabish the conditions for the existence and unique results of the solution for a fractional delta-nabla difference equations with fractional delta-nabla sum-difference boundary value conditions by using Banach contraction principle and the conditions for result of at least one solution by using the Schauder's fixed point theorem.

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