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Existence of Weak Solutions for a New Class of Fractional p-Laplacian Boundary Value Systems

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Abstract: In this paper, at least three weak solutions were obtained for a new class of dual non-linear dual-Laplace systems according to two parameters by using variational methods combined with a critical point theory due to Bonano and Marano. Two examples are given to illustrate our main results applications.

Keywords: nonlinear fractional; dirichlet boundary value problems; p-Laplacian type; variational method; critical point theory

1. Introduction

Fractional differential equations have proved to be valuable tools in modeling many phenomena in various fields of physics, chemistry, biology, engineering and economics. There was a significant development in fractional differential equations. We can see the studies of Miller and Ross [1], Samko et al. [2], Podlupny [3], Hilfer [4], Kelpas et al. [5] and papers [6–16] and references therein.

The critical point theory was very useful in determining the existence of solutions to complete differential equations with certain boundary conditions; see for example, in the extensive literature on the subject, classical books [17–19] and references appearing there. However, so far, some problems have been created for fractional marginal value problems (briefly BVP) by exploiting this approach, where it is often very difficult to create a suitable space and a suitable function for fractional problems.

In [20], the authors investigated the following nonlinear fractional differential equation depending on two parameters:

$$\begin{cases} {}_tD_T^{\alpha_i}\left(a_i\left(t\right)\,{}_0D_t^{\alpha_i}u_i\left(t\right)\right) = \lambda F_{u_i}\left(t,u_1\left(t\right),u_2\left(t\right),...,u_n\left(t\right)\right) \\ +\mu G_{u_i}\left(t,u_1\left(t\right),u_2\left(t\right),...,u_n\left(t\right)\right)+h_i\left(u_i\right)\ \text{a.e}\ \left[0,T\right], \\ u_i\left(0\right)=u_i\left(T\right)=0 \end{cases} \tag{1}$$

for $1\leq i\leq n$, where $\alpha_i\in(0;1]$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order α_i respectively, $a_i\in L^\infty([0,T])$ with

$$a_{i0}=ess\inf_{[0,T]}a_i>0 \text{ for } 1\leq i\leq n,\ \lambda,\ \mu$$

are positive parameters, $F, G : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are measurable functions with respect to $t \in [0, T]$ for every $(x_1, ..., x_n) \in \mathbb{R}^n$ and are C^1 with respect to $(x_1, ..., x_n) \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, F_{u_i} and G_{u_i} denotes the partial derivative of F and G with respect to u_i, respectively, and h_i: $\mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i > 0$ for $1 \le i \le n$, i.e.,

$$|h_i(x_1) - h_i(x_2)| \le L_i |x_1 - x_2|$$

for every $x_1, x_2 \in \mathbb{R}$, and $h_i(0) = 0$ for $1 \le i \le n$. Motivated by [21,22], using a three critical points theorem obtained in [23] which we recall in the next section (Theorem 2.6), we ensure the existence of at least three solutions for this system.

For example, according to some assumptions, in [24], by using variational methods the authors obtained the existence of at least one weak solution for the following $p-$Laplacian fractional differential equation [24]

$$\begin{cases} {}_tD_T^{\alpha} \left(\phi_p\left({}_0D_t^{\alpha} u(t)\right)\right) = \lambda f(t, u(t)) \text{ a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives with $0 < \alpha \le 1$, respectively, the function $\phi_p(s) = |s|^{p-2} s$, $p > 1$. Taking a class of fractional differential equation with $p-$Laplacian operator as a model, Li et al. investigated the following equation recently [25]

$$\begin{cases} {}_tD_T^{\alpha} \left(\frac{1}{w(t)^{p-2}} \varphi_p\left({}_0D_t^{\alpha} u(t)\right)\right) + \lambda u(t) \\ = f\left(t, u, {}_0^cD_t^{\alpha} u(t)\right) + h(u(t)) \text{ a.e. } t \in [0, T], \\ u(0) = u(T) = 0 \end{cases} \tag{3}$$

with $\frac{1}{p} < \alpha \le 1$, λ is a non-negative real parameter.

The functions

$$\varphi_p(s) = |s|^{p-2} s, p \ge 2, f : [0; T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is continuous $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

By using the mountain pass theorem combined with iterative technique, the authors obtained the existence of at least one solution for problem (3).

In this paper, we are interested in ensuring the existence of three weak solutions for the following system

$$\begin{cases} {}_tD_T^{\alpha_i} \left(\frac{1}{w_i(t)^{p-2}} \phi_p\left({}_0D_t^{\alpha_i} u_i(t)\right)\right) + \mu |u_i(t)|^{p-2} u_i(t) \\ = \lambda F_{u_i}(t, u_1(t), u_2(t), ..., u_n(t)) \text{ a.e. } t \in [0, T], \\ u_i(0) = u_i(T) = 0, \end{cases} \tag{4}$$

where

$$\phi_p(s) = |s|^{p-2} s, p > 1, w_i(t) \in L^{\infty}[0, T]$$

with $w_i^0 = ess \inf_{[0,T]} w_i(t) > 0, {}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order $0 < \alpha_i \le 1$ respectively, for $1 \le i \le n$, λ is positive parameter, and $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is measurable function with respect to $t \in [0, T]$ for every $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and are C^1 with respect to $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. for a.e. $t \in [0, T]$, F_{u_i} denote the partial derivative of F with respect to u_i,respectively,

(H_0) $\alpha_i \in (0; 1]$ for $1 \le i \le n$.

$(H1)$ $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a function such that $F(.,u_1,u_2,...,u_n)$ is continuous in $[0,T]$ for every $(u_1,u_2,...,u_n)\in\mathbb{R}^n$, $F(t,.,.,...,.)$ is a C^1 function in $\mathbb{R}^2$.

In the present paper, motivated by [26,27], using a three critical points theorem obtained in [23], which we recall in the next section (Theorem 2.6), we ensure the existence of at least three solutions for system (4). This theorem has been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers ([26–30]).

This paper is organized as follows. In Section 2, we present some necessary preliminary facts that will be needed in the paper. In Section 3, we prove our main result. In Section 4, we give two numerical examples in order to support the theory of our contribution.

2. Preliminaries

In this section, we first introduce some necessary definitions and preliminary facts are introduced for fractional calculus which are used in this paper.

For $[0,T]\subseteq\mathbb{R}$, let $C([0,T],\mathbb{R})$ be the real space of all continuous functions with norm $\|x\|_\infty=\max\limits_{t\in[0,T]}|x(t)|$,and $L^p([0,T],\mathbb{R})$ $(1\le p<\infty)$ be the space of functions for which the pth power of the absolute value is Lebesgue integrable with norm $\|x\|_{L^p}=\left(\int\limits_0^T|x(t)|^p\,dt\right)^{\frac{1}{p}}$

Definition 1 (Kilbas et al. [5])**.** *Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha>0$ for a function u are defined by*

$$ {}_aD_t^\alpha u(t):=\frac{d^n}{dt^n}{}_aD_t^{\alpha-n}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_a^t(t-s)^{n-\alpha-1}u(s)\,ds$$

and

$$ {}_tD_b^\alpha u(t):=(-1)^n\frac{d^n}{dt^n}{}_tD_b^{\alpha-n}u(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_t^b(t-s)^{n-\alpha-1}u(s)\,ds$$

for every $t\in[a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1\le\alpha<n$ and $n\in\mathbb{N}$.

Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha):=\int\limits_0^{+\infty}z^{\alpha-1}e^{-z}dz.$$

Setting $AC^n([a,b],\mathbb{R})$ the space of functions $u:[a,b]\to\mathbb{R}$ such that $u\in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)}\in AC^n([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings being $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we denote $AC([a,b],\mathbb{R}):=AC^1([a,b],\mathbb{R})$.

Definition 2 ([25])**.** *Let $0<\alpha_i\le1$, for $1\le i\le n$, $1<p<\infty$. The fractional derivative space*

$$E^p_{\alpha_i}=\left\{u(t)\in L^p([0,T],\mathbb{R})_a\,D_t^{\alpha_i}u(t)\in L^p([0,T],\mathbb{R})\right\}.$$

Then, for any $u\in E^p_{\alpha_i}$, we can define the weighted norm for $E^p_{\alpha_i}$ as

$$\|x\|_{\alpha_i}=\left(\int\limits_0^T|u(t)|^p\,dt+\int\limits_0^T w_i(t)\left|{}_aD_t^{\alpha_i}u(t)\right|^p dt\right)^{\frac{1}{p}}. \quad (5)$$

Definition 3 ([31]). *We mean by a weak solution of system (4), any $u = (u_1, u_2, ..., u_n) \in X$ such that for all $v = (v_1, v_2, ..., v_n) \in X$*

$$\int_0^T \sum_{i=1}^n \frac{1}{w_i(t)^{p-2}} \phi_p \left(w_i(t)_0 D_t^{\alpha_i} u_i(t)\right)_0 D_t^{\alpha_i} v_i(t)\, dt$$
$$+\mu \int_0^T \sum_{i=1}^n |u_i(t)|^{p-2} u_i(t) v_i(t)\, dt$$
$$-\lambda \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), u_2(t), ..., u_n(t)) v_i(t)\, dt = 0.$$

Lemma 1 ([31]). *Let $0 < \alpha_i \leq 1$, for $1 \leq i \leq n$, $1 < p < \infty$.For any $u \in E^p_{\alpha_i}$ we have*

$$\|u_i\|_{L^p} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)} \left\|{}_0 D_t^{\alpha_i} u_i\right\|_{L^p}. \tag{6}$$

Moreover, if $\alpha_i > p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u_i\|_\infty \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i)\Gamma((\alpha_i - 1)q + 1)^{\frac{1}{q}}} \left\|{}_0 D_t^{\alpha_i} u_i\right\|_{L^p}. \tag{7}$$

From Lemma 1, we easily observe that

$$\|u_i\|_{L^p} \leq \frac{T^{\alpha_i} \left(\int_a^b w_i(t) \left|{}_a D_t^{\alpha_i} u(t)\right|^p dt\right)^{1/p}}{\Gamma(\alpha_i + 1)} \tag{8}$$

for $0 < \alpha_i \leq 1$, and

$$\|u_i\|_\infty \leq \frac{T^{\frac{\alpha_i - 1}{p}} \left(\int_a^b w_i(t) \left|{}_a D_t^{\alpha_i} u(t)\right|^p dt\right)^{1/p}}{\Gamma(\alpha_i) \left(w_i^0\right)^{\frac{1}{p}} \Gamma((\alpha_i - 1)q + 1)^{\frac{1}{q}}} \tag{9}$$

for $\alpha_i > p$ and $\frac{1}{p} + \frac{1}{q} = 1$.

By using (8), the norm of (5) is equivalent to

$$\|x\|_{\alpha_i} = \left(\int_0^T w_i(t) \left|{}_a D_t^{\alpha_i} u(t)\right|^p dt\right)^{\frac{1}{p}}, \ \forall u \in E^p_{\alpha_i}. \tag{10}$$

Throughout this paper, we let X be the Cartesian product of the n spaces $E^p_{\alpha_i}$ for $1 \leq i \leq n$, i.e., $X = E^p_{\alpha_1} \times E^p_{\alpha_2} \times ... \times E^p_{\alpha_n}$ equipped with the norm

$$\|u\| = \sum_{i=1}^n \|u_i\|_{E^p_{\alpha_i}}, \ u = (u_1, u_2, ..., u_n),$$

where $\|u_i\|_{E^p_{\alpha_i}}$ is defined in (10) Obviously, X is compactly embedded in $C([0, T], \mathbb{R})^n$.

Lemma 2 ([32]). *For $0 < \alpha_i \leq 1$ and $1 < p < \infty$, the fractional derivative space X is a reflexive separable Banach space.*

Lemma 3 ([33]). *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Assume $\{w_1, w_2...\}$is a basis in X. Then the following assertion holds:*

If A is strictly monotone, then the inverse operator $A^{-1} : X^* \to X$ exists. This operator is strictly monotone, demicontinuous and bounded. If A is uniformly monotone, then A^{-1} is continuous. If A is strongly monotone, then it is Lipschitz continuous.

Theorem 1 ([34]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive ,continuously Gateaux differentiable sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*, bounded on bounded subsets of X, $\Psi : X \to \mathbb{R}$ a continuously Gateaux differentiable functional whose Gateaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

(a_1) $\sup\limits_{\Phi(u)\le r} \frac{\Psi(u)}{r} < \frac{\Phi(\overline{x})}{\Psi(\overline{x})}$.

(a_2) *For each* $\lambda \in \Lambda_\lambda = \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{\Phi(u)\le r} \Psi(u)} \right)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for any $\lambda \in \Lambda_\lambda$, the functional $\Phi - \lambda\Psi$ has at least three critical point in X.

3. The Main Results

In the present section ,the existence of multiple solutions for system (4) is examined by using Theorem 1.

First and foremost, we define the functionals $\Phi, \Psi : X \to \mathbb{R}$ as

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \int_0^T \sum_{i=1}^n \left(w_i(t) \left| {}_0D_t^{\alpha_i} u_i(t) \right|^p + \mu \left| u_i(t) \right|^p \right) dt, \\ u &= (u_1, u_2, ..., u_n) \in X \end{aligned} \tag{11}$$

and

$$\Psi(u) = \int_0^T F(t, u_1(t), u_2(t), ..., u_n(t))\, dt \tag{12}$$

Lemma 4. *Let $0 < \alpha_i \le 1, u = (u_1, u_2, ..., u_n) \in X$. Functionals Φ and Ψ are defined in (11) and (12). Then, $\Phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functionl whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.*

Proof. For each $u = (u_1, u_2, ..., u_n) \in X$, define $\Phi, \Psi : X \to \mathbb{R}$ as

$$\Phi(u) = \frac{1}{p} \int_0^T \sum_{i=1}^n \left(w_i(t) \left| {}_0D_t^{\alpha_i} u_i(t) \right|^p + \mu \left| u_i(t) \right|^p \right) dt$$

and

$$\Psi(u) = \int_0^T F(t, u_1(t), u_2(t), ..., u_n(t))\, dt.$$

Clearly, Φ and Ψ are continuously Gateaux differentiable functionals whose Gateaux derivatives at the point $u \in X$ are given by

$$\begin{aligned}\Phi'(u)(v) = &\int_0^T \sum_{i=1}^n \frac{1}{w_i(t)^{p-2}} \phi_p\left(w_i(t)_0 D_t^{\alpha_i} u_i(t)\right)_0 D_t^{\alpha_i} v_i(t)\,dt \\ &+\mu \int_0^T \sum_{i=1}^n |u_i(t)|^{p-2} u_i(t) v_i(t)\,dt\end{aligned} \tag{13}$$

for every $v = (v_1, v_2, ..., v_n) \in X$.

In addition, according to (11), one has $\Phi(u) \geq \frac{1}{p}\|u\|_X^p$, which means that Φ is a coercive functional.

Next, we claim that Φ' admits a continuous inverse on X^*. Let $u = (u_1, u_2, ..., u_n) \in X, v = (v_1, v_2, ..., v_n) \in X$. Recalling (13),we get

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u - v\rangle = &\int_0^T \sum_{i=1}^n \frac{1}{w_i(t)^{p-2}} \phi_p\left(w_i(t)_0 D_t^{\alpha_i} u_i(t)\right)_0 D_t^{\alpha_i}(u-v)\,dt \\ &+\mu \int_0^T \sum_{i=1}^n |u_i(t)|^{p-2} u_i(t)(u-v)\,dt \\ &-\int_0^T \sum_{i=1}^n \frac{1}{w_i(t)^{p-2}} \phi_p\left(w_i(t)_0 D_t^{\alpha_i} v_i(t)\right)_0 D_t^{\alpha_i}(u-v)\,dt \\ &+\mu \int_0^T \sum_{i=1}^n |v_i(t)|^{p-2} v_i(t)(u-v)\,dt.\end{aligned} \tag{14}$$

According to the well-known inequality

$$\begin{aligned}&\left(|s_1|^{p-2}s_1 - |s_2|^{p-2}s_2\right)(s_1 - s_2) \\ &\geq \begin{cases} |s_1 - s_2|^p, & p \geq 2 \\ \frac{|s_1-s_2|^2}{(|s_1|+|s_2|)^{2-p}}, & 1 < p \leq 2. \end{cases}\end{aligned} \tag{15}$$

We have

$$\begin{aligned}&\left(\phi_p\left(w_i(t)_0 D_t^{\alpha_i} u_i(t)\right) - \phi_p\left(w_i(t)_0 D_t^{\alpha_i} v_i(t)\right)\right) \\ &\geq \begin{cases} \frac{1}{w_i(t)}\left|w_i(t)_0 D_t^{\alpha_i} u_i(t) - w_i(t)_0 D_t^{\alpha_i} v_i(t)\right|^p, & p \geq 2 \\ \frac{1}{w_i(t)} \frac{\left|w_i(t)_0 D_t^{\alpha_i} u_i(t) - w_i(t)_0 D_t^{\alpha_i} v(t)\right|^2}{\left(\left|w_i(t)_0 D_t^{\alpha_i} u_i(t)\right| + \left|w_i(t)_0 D_t^{\alpha_i} v_i(t)\right|\right)^{2-p}}, & 1 < p < 2. \end{cases}\end{aligned}$$

Hence, when $1 < p < 2$, one has

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \left|w_i(t)_0 D_t^{\alpha_i} u_i(t) - w_i(t)_0 D_t^{\alpha_i} u_i(t)\right|^p dt \\ &\leq \left(\int_0^T \sum_{i=1}^n \frac{\left|w_i(t)_0 D_t^{\alpha_i} u_i(t) - w_i(t)_0 D_t^{\alpha_i} v_i(t)\right|^2}{w_i(t)\left(\left|w_i(t)_0 D_t^{\alpha_i} u_i(t)\right| + \left|w_i(t)_0 D_t^{\alpha_i} v_i(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} \\ &\left(\int_0^T \sum_{i=1}^n w_i(t)^{\frac{p}{2-p}}\left(\left|w_i(t)_0 D_t^{\alpha_i} u_i(t)\right| + \left|w_i(t)_0 D_t^{\alpha_i} v_i(t)\right|\right)^p dt\right)^{\frac{2-p}{2}}\end{aligned} \tag{16}$$

which means that

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \frac{\left|w_i(t)_0D_t^{\alpha_i}u_i(t)-w_i(t)_0D_t^{\alpha_i}v_i(t)\right|^2}{w_i(t)\left(\left|w_i(t)_0D_t^{\alpha_i}u_i(t)\right|+\left|w_i(t)_0D_t^{\alpha_i}v_i(t)\right|\right)^{2-p}}dt \\ &\geq \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\overset{\sim}{w_1^0}} \|u_i-v_i\|_{\alpha_i}^2 \left(\|u_i\|_{\alpha_i}^p+\|v_i\|_{\alpha_i}^p\right)^{\frac{p-2}{p}}.\end{aligned} \tag{17}$$

Then, we deduce

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}v_i\left(t\right)\right)_0 D_t^{\alpha_i}\left(u-v\right)\right)dt \\ &\geq \frac{2^{p-2}\left(w_1^0\right)^{\frac{2(p-1)}{p}}}{\overset{\sim}{w_1^0}} \|u_i-v_i\|_{\alpha_i}^2 \left(\|u_i\|_{\alpha_i}^p+\|v_i\|_{\alpha_i}^p\right)^{\frac{p-2}{p}} > 0.\end{aligned} \tag{18}$$

When $p \geq 2$, we get

$$\begin{aligned}&\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}v_i\left(t\right)\right)_0 D_t^{\alpha_i}\left(u-v\right)\right)dt \\ &\geq \left(w_1^0\right)^{p-2}\|u_i-v_i\|_{\alpha_i}^p > 0.\end{aligned} \tag{19}$$

Then, combining with (18), yields

$$\int_0^T \sum_{i=1}^n \left(\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}u_i\left(t\right)\right)-\phi_p\left(w_i\left(t\right)_0 D_t^{\alpha_i}v_i\left(t\right)\right)\left({}_0D_t^{\alpha_i}u_i -_0 D_t^{\alpha_i}v_i\right)\right)dt > 0. \tag{20}$$

For every $1 < p < \infty$
Further, denote

$$A = \int_0^T \sum_{i=1}^n |u_i\left(t\right)|^{p-2} u_i\left(t\right)\left(u-v\right)dt + \int_0^T \sum_{i=1}^n |v_i\left(t\right)|^{p-2} v_i\left(t\right)\left(u-v\right)dt.$$

Then, reapplying inequality (15), we always have

$$A \geq \|u_i-v_i\|_{\alpha_i}^p > 0, \text{ for } p \geq 2$$

and

$$A \geq 2^{p-2}\|u_i-v_i\|_{L^p}^2\left(\|u_i\|_{L^p}^p+\|v_i\|_{L^p}^p\right)^{\frac{p-2}{p}} > 0 \text{ for} 1 < p < 2.$$

That is, $A > 0$ for every $1 < p < \infty$. Therefore, by using (14) and (20), the following inequality holds

$$\left\langle \Phi'\left(u\right)-\Phi'\left(v\right), u-v\right\rangle > 0$$

which means that Φ' is strictly monotone. Furthermore, in view of X being reflexive, for $u_n \to u$ in X strongly, as $n \to \infty$, one has $\Phi'\left(u_n\right) \rightharpoonup \Phi'\left(u\right)$ in X^* as $n \to \infty$.

Thus, we say that Φ' is demicontinuous. Then, according to lemma 2 and 3, we obtain that the inverse operator $\left(\Phi'\right)^{-1}$ of Φ' exists and is continuous.

Moreover , let

$$\|u\|_{\mu,\alpha_i}^p = \int_0^T \sum_{i=1}^n \left(w_i\left(t\right)\left|{}_0D_t^{\alpha_i}u_i\left(t\right)\right|^p + \mu\left|u_i\left(t\right)\right|^p\right)dt$$

owing to the sequentially weakly lower semicontinuity of $\|u\|_{\mu,\alpha_i}^p$ we observe that Φ is sequentially weakly lower semicontinuous in X.

Considering the functional Ψ, we will point out that Ψ is a Gâteaux differentiable, sequentially weakly upper semicontinuous functional on X.

Indeed, for $u_n \subset X$, assume that $u_n \rightharpoonup u$ in X, i.e., u_n uniformly converges to u on $[0, T]$ as $n \to \infty$. By using reverse Fatou's lemma, one has

$$\begin{aligned} \lim_{n\to+\infty} \inf \Psi(u_n) &\leq \int_0^T \lim_{n\to+\infty} \inf F(t, u_n(t))\, dt \\ &= \int_0^T F(t, u_1(t), u_2(t), ..., u_n(t))\, dt = \Psi(u), \end{aligned}$$

whereas $u = (u_1, u_2, ..., u_n) \in X$, which implies that Ψ is sequentially weakly upper semicontinuous. Furthermore, since F is continuously differentiable with respect to u and v for almost every $t \in [0, T]$, then based on the Lebesgue control convergence theorem, we obtain that $\Psi'(u_n) \to \Psi'(u)$ strongly, that is Ψ' is strongly continuous on X. Hence, we confirm that Ψ' is a compact operator.

Moreover, it is easy to prove that the functional with the Gâteaux derivative $\Psi'(u) \in X^*$at the point $u \in X$

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), u_2(t), ..., u_n(t))\, v_i(t)\, dt \tag{21}$$

for any $v = (v_1, v_2, ..., v_n) \in X$.The proof is completed. □

In order to facilitate the proof of our main result, some notations are given.

Putting

$$\begin{aligned} k &:= \max_{1\leq i\leq n} \left\{ \frac{T^{p\alpha_i - 1}}{(\Gamma(\alpha_i))^p w_i^0 ((\alpha_i - 1) q + 1)^{\frac{p}{q}}} \right\}, \\ \tilde{k} &:= \max_{1\leq i\leq n} \left\{ \frac{T^{p\alpha_i}}{(\Gamma(\alpha_i + 1))^p w_i^0} \right\}. \end{aligned}$$

Define

$$\pi(\sigma) = \left\{ u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n : \frac{1}{p} \sum_{i=1}^n |u_i|^p < \sigma \right\}.$$

Theorem 2. *Let $\frac{1}{p} < \alpha_i \leq 1$, for $1 \leq i \leq n$. Assume that there exists a positive constant r and a function $u = (u_1, u_2, ..., u_n) \in X$ such that*

(i)
$$\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \mu \sum_{i=1}^n \|u_i\|_{L^p}^p > pr;$$

(ii)
$$\frac{\int_0^T \sup_{u\in\pi(kr)} F(t, u_1, u_2, ..., u_n)\, dt}{r} < \frac{p \int_0^T F(t, u_1, u_2, ..., u_n)\, dt}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \mu \sum_{i=1}^n \|u_i\|_{L^p}^p};$$

(iii)

$$\liminf_{|u|\to\infty} \frac{F(t,u_1,u_2,...,u_n)}{\sum_{i=1}^{n}|x_i|^p} \le \frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}{pr\tilde{k}}.$$

Then, setting

$$\Lambda = \left(\frac{\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p + \mu\sum_{i=1}^{n}\|u_i\|_{L^p}^p}{p\int_0^T F(t,u_1,u_2,...,u_n)\,dt}, \frac{r}{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt} \right)$$

for each $\lambda \in \Lambda$ system (4) admits at least three weak solutions in X.

Proof. Considering Theorem 1 and Lemma 5, in order to obtain that system (4) possesses at least three weak solutions in X, we only need to guarantee the assumptions (a_1) and (a_2) of Theorem 1 are satisfied. Choose $u_0 = (u_{01}, u_{02}, ..., u_{0n})$ and $u_1 = (u_{11}, u_{12}, ..., u_{1n})$ with $(u_{01}, u_{02}, ..., u_{0n}) = (0, 0, ..., 0)$. Due to (12) and (i), we get $\Psi(u_0) = 0$ and $\Phi(u_1) > r > 0$, which satisfy the requirement of Theorem 1. Then, combining (11) and (9), yields

$$\begin{aligned}
& \{u = (u_1,u_2,...,u_n) \in X : \Phi(u) \le r\} \\
= & \left\{ u = (u_1,u_2,...,u_n) \in X : \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p + \frac{\mu}{p}\sum_{i=1}^{n}\|u_i\|_{L^p}^p \le r \right\} \\
\subseteq & \left\{ u = (u_1,u_2,...,u_n) \in X : \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p \le r \right\} \\
\subseteq & \left\{ u = (u_1,u_2,...,u_n) \in X : \sum_{i=1}^{n}\frac{(\Gamma(\alpha_i))^p\, w_1^0((\alpha_i-1)q+1)^{\frac{p}{q}}}{pT^{p\alpha_i-1}}\|u_i\|_\infty \le r \right\} \\
\subseteq & \left\{ u = (u_1,u_2,...,u_n) \in X : \sum_{i=1}^{n}|u_i|^p \le kpr \right\}
\end{aligned}$$

which implies that

$$\begin{aligned}
\sup_{\Phi(u)\le r} \Psi(u) &= \sup_{\Phi(u)\le r}\int_0^T F(t,u_1,u_2,...,u_n)\,dt \\
&\le \sup_{\Phi(u)\le r}\int_0^T F(t,u_1,u_2,...,u_n)\,dt.
\end{aligned}$$

Then, the following inequality is obtained under condition (ii)

$$\begin{aligned}
\sup_{\Phi(u)\le r} \Psi(u) &\le \frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}{r} \\
&\le \frac{p\int_0^T \sum_{i=1}^n F(t,u_1,u_1,...,u_n)\,dt}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \mu \sum_{i=1}^n \|u_i\|_{L^p}^p} \\
&= \frac{\Psi(u_1)}{\Phi(u_1)}.
\end{aligned}$$

Thus the hypothesis (a_1) of Theorem 1 holds.

On the other hand, taking (iii) into account , there exist constants $C, \varepsilon \in \mathbb{R}$ with

$$C < \frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}{r},$$

such that

$$F(t,u_1,u_2,...,u_n) \le \frac{C}{p\tilde{k}} \sum_{i=1}^n |u_i|^p + \varepsilon \tag{22}$$

for any $t\in[0,T]$ and $u=(u_1,u_2,...,u_n)\in X$, when $C>0$ by using (11), (22) and (8) yields

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \frac{\mu}{p}\sum_{i=1}^n \|u_i\|_{L^p}^p - \lambda\int_0^T F(t,u_1,u_2,...,u_n)\,dt \\
&\ge \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \lambda\int_0^T F(t,u_1,u_2,...,u_n)\,dt \\
&\ge \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p\tilde{k}}\int_0^T \sum_{i=1}^n \|u_i\|_{L^p}^p\,dt - \lambda T\varepsilon \\
&\ge \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p\tilde{k}}\left(\sum_{i=1}^n \frac{T^{\alpha_i}}{(\Gamma(\alpha_i+1))^p w_i^0}\|u_i\|_{\alpha_i}^p\right) - \lambda T\varepsilon \\
&\ge \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \frac{\lambda C}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \lambda T\varepsilon \\
&\ge \left(\frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p\right)\left(1 - C\frac{r}{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)\,dt}\right) - \lambda T\varepsilon.
\end{aligned}$$

That is

$$\lim_{\|u\|_X\to+\infty} \Phi(u) - \lambda\Psi(u) = +\infty.$$

Furthermore, analogous to the case of $C>0$, we can deduce that $\Phi(u)-\lambda\Psi(u)\to+\infty$ as $\|u\|_X \to +\infty$ with $C\le 0$. Hence, all the hypotheses of Theorem 1 hold, then , system (4) admits at least three weak solutions in X. The proof is completed.

For simplicity, before giving a corollary of Theorem 2, some notations are presented.

Let $0 < h < \frac{1}{2}$ we put

$$A_i(\alpha_i, h) = \frac{1}{(hT)}\left[\begin{array}{l} \int_0^{hT}\sum_{i=1}^{n} w_i(t)\, t^{(1-\alpha_i)p} dt + \int_{hT}^{(1-h)T}\sum_{i=1}^{n} w_i(t)\left[t^{1-\alpha_i} - (t-hT)^{1-\alpha_i}\right]^p dt \\ + \int_{(1-h)T}^{T}\sum_{i=1}^{n} w_i(t)\left[t^{1-\alpha_i} - (t-hT)^{1-\alpha_i} - (t-((1-h)T))^{1-\alpha_i}\right]^p dt. \end{array}\right] \quad (23)$$

□

Corollary 1. *Let* $\frac{1}{p} < \alpha_i \le 1$. *Assume that there exist* $\tau > 0$ *and* $\theta = (\theta_1, \theta_2, ..., \theta_n) \in \mathbb{R}^n$ *with* $\theta_1 > 0, \theta_2 > 0, ..., \theta_n > 0$ *and* $\tau \le k\,\frac{\sum_{i=1}^{n} A_i(\alpha_i,h)\theta_i^p}{p}$, *such that*

$(i)'$

$$\begin{aligned} F(t, u_1, u_2, ..., u_n) &\ge 0 \text{ for} \\ (t, u_1, u_2, ..., u_n) &\in ([0, hT] \cup [(1-h)T, T] \times [0, \theta]) \end{aligned}$$

$(ii)'$

$$\begin{aligned} &\frac{\int_0^T \sup_{u\in\pi(kr)} F(t, u_1, u_2, ..., u_n)\, dt}{r} \\ < &\frac{p\int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\theta_1, \Gamma(2-\alpha_2)\theta_2, ..., \Gamma(2-\alpha_n)\theta_n)\, dt}{k\left(1+\mu\tilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i, h)\,\theta_i^p}; \end{aligned}$$

$(iii)'$

$$\lim_{|u|\to\infty}\inf \frac{F(t, u_1, u_2, ..., u_n)}{\sum_{i=1}^{n} |u_i|^p} \le 0$$

for each

$$\lambda \in \Lambda' = \left(\begin{array}{c} \frac{\left(1+\mu\tilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i,h)\theta_i^p}{p\int_{hT}^{(1-h)T} F(t,\Gamma(2-\alpha_1)\theta_1,\Gamma(2-\alpha_2)\theta_2,...,\Gamma(2-\alpha_n)\theta_n)dt}, \\ \\ \frac{\tau}{k\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,...,u_n)dt} \end{array}\right). \quad (24)$$

Thus, system (4) admits at least three weak solutions in X.

Proof. Choose

$$U_i(t) = \begin{cases} \frac{\Gamma(2-\alpha_i)\theta_i}{hT} t\,,\ t \in [0, hT[\\ \Gamma(2-\alpha_i)\theta_i,\ t \in [hT, (1-h)T] \\ \frac{\Gamma(2-\alpha_i)\theta_i}{hT}(T-t)\,,\ t \in](1-h)T, T]\,. \end{cases}$$

Obviously $U_i(0) = U_i(T) = 0, U(t) \in L^p[0,T]$. Owing to Definition 1 , we derive,

$$ {}_0D_t^{\alpha_i}U(t) = \begin{cases} a_1(t), t \in [0, hT[, \\ a_2(t), t \in [hT, (1-h)T], \\ a_3(t), t \in](1-h)T, T], \end{cases} $$

where

$$ a_1(t) = \frac{\theta_i}{hT} t^{1-\alpha_i},\ a_2(t) = \frac{\theta_i}{hT}\left[t^{1-\alpha_i} - (t-(hT)^{1-\alpha_i}\right] $$

and

$$ a_3(t) = \frac{\theta_i}{hT}\left[t^{1-\alpha_i} - (t-(hT)^{1-\alpha_i} - (t-(T-hT)^{1-\alpha_i}\right]. $$

That is

$$ \begin{aligned} \|U\|_{\alpha_i}^p &= \int_0^T \sum_{i=1}^n w_i(t) \left|{}_0D_t^{\alpha_i}U_i(t)\right|^p dt \\ &= \int_0^{hT} + \int_{hT}^{(1-h)T} + \int_{(1-h)T}^{T} \sum_{i=1}^n w_i(t) \left|{}_0D_t^{\alpha_i}U(t)\right|^p dt \\ &= \sum_{i=1}^n A_i(\alpha_i, h)\theta_i^p, \end{aligned} $$

where (23) is used. Hence $U = (U_1, U_2, ..., U_n) \in X$.

Take $r = \frac{\tau}{k}$, then

$$ \begin{aligned} rk &= \tau \le k \frac{\sum_{i=1}^n A_i(\alpha_i, h)\theta_i^p}{p} \\ &= k \frac{\sum_{i=1}^n \|U\|_{\alpha_i}^p}{p} \le k\Phi(U) \end{aligned} $$

for every $U = (U_1, U_2, ..., U_n) \in X$ which means that

$$ r \le \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \frac{\mu}{p}\sum_{i=1}^n \|u_i\|_{L^p}^p. $$

Thus, the assumption (ii) of Theorem 2 holds.

On the other hand, based on (7) and (23), yields

$$ \begin{aligned} \Phi(U) &\le \tfrac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \tfrac{\mu}{p}\sum_{i=1}^n \tfrac{T^{\alpha_i}}{(\Gamma(\alpha_i+1))^p w_i^0}\|u_i\|_{\alpha_i}^p \\ &\le \tfrac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \tfrac{\mu}{p}\tilde{k}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p \\ &\le \frac{(1+\mu\tilde{k})\sum_{i=1}^n A_i(\alpha_i,h)\theta_i^p}{p}. \end{aligned} \tag{25} $$

Then, from (25) and $(ii)'$, we can obtain the following inequality

$$
\begin{aligned}
\frac{\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,\dots,u_n)\,dt}{r} &= \frac{k\int_0^T \sup_{u\in\pi(kr)} F(t,u_1,u_2,\dots,u_n)\,dt}{\tau} \\
&< \frac{kp\int_{hT}^{(1-h)T} F(t,\Gamma(2-\alpha_1)\theta_1,\Gamma(2-\alpha_2)\theta_2,\dots,\Gamma(2-\alpha_n)\theta_n)\,dt}{k\left(1+\mu\tilde{k}\right)\sum_{i=1}^{n} A_i(\alpha_i,h)\,\theta_i^p} \\
&\le \frac{\int_{hT}^{(1-h)T} F(t,\Gamma(2-\alpha_1)\theta_1,\Gamma(2-\alpha_2)\theta_2,\dots,\Gamma(2-\alpha_n)\theta_n)\,dt}{\Phi(U)} \\
&\le \frac{k\int_0^T F(t,u_1,u_2,\dots,u_n)\,dt}{\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^p+\mu\sum_{i=1}^{n}\|u_i\|_{L^p}^p}
\end{aligned}
$$

which means that the hypothesis (ii) of Theorem 2 is satisfied.

Furthermore, the condition (iii) of Theorem 2 holds under $(iii)'$ since $\Lambda' \subseteq \Lambda$ Theorem 2 is successfully employed to ensure the existence of at least three weak solutions for system (4), the proof is completed. □

4. Numerical Examples

Now, we give the following two examples to illustrate the applications of our result:

Example 1. *Let* $p=2, \alpha_1=0.8, \alpha_2=0.65, \mu=1, w_1(t)=1+t^2, w_2(t)=0.5+t, T=1$. *Then, system (4) gets the following form*

$$
\begin{cases}
{}_tD_1^{0.8}\left((1+t^2)\,{}_0D_t^{0.8}u_1(t)\right)+u_1(t)=\lambda F_{u_1}(t,u_1(t),u_2(t)),\ t\in[0,1], \\
{}_tD_1^{0.65}\left((0.5+t)\,{}_0D_t^{0.65}u_2(t)\right)+u_2(t)=\lambda F_{u_2}(t,u_1(t),u_2(t)),\ t\in[0,1], \\
u_1(o)=u_1(1)=0, u_2(o)=u_2(1)=0.
\end{cases}
$$

Taking

$$U_1(t)=\Gamma(1.2)\,t(1-t), U_2(t)=\Gamma(1.35)\,t(1-t)$$

and

$$F(t,u_1(t),u_2(t))=\left(1+t^2\right)G(u_1,u_2),$$

where

$$
G(u_1,u_2)=\begin{cases}
\left(u_1^2+u_2^2\right)^2,\ u_1^2+u_2^2\le 1, \\
10\left(u_1^2+u_2^2\right)^{\frac{1}{2}}-9\left(u_1^2+u_2^2\right)^{\frac{1}{3}}, u_1^2+u_2^2>1.
\end{cases}
$$

Clearly, $F(t,0,0)=0, w_1^0=1$ and $w_2^0=0.5$ for any $t\in[0,1]$.

By the direct calculation, we have

$$\begin{aligned} & \max\left\{\frac{1}{(\Gamma(0.8))^2(2\times 0.8-1)},\frac{1}{(\Gamma(0.65))^2(2\times 0.65-1)}\right\} \\ = \; & k \approx 3.4764, \end{aligned}$$

$$\max\left\{\frac{1}{(\Gamma(0.8)+1)^2},\frac{1}{(\Gamma(0.65)+1)^2\times 0.5}\right\} = \tilde{k} \approx 2.4684$$

and

$$\begin{aligned} {}_0D_t^{0.8}U_1(t) &= t^{0.2} - \frac{2\Gamma(1.2)}{\Gamma(2.2)}t^{1.2}, \\ {}_0D_t^{0.65}U_2(t) &= t^{0.35} - \frac{2\Gamma(1.35)}{\Gamma(2.35)}t^{1.35}. \end{aligned}$$

So that

$$\begin{aligned} \|U_1(t)\|_{0.8}^2 &\approx 0.19333, \|U_2(t)\|_{0.65}^2 \approx 0.078559 \\ \|U_1(t)\|_{L^2}^2 &\approx 0.028101, \|U_2(t)\|_{L^2}^2 \approx 0.026716. \end{aligned}$$

Take $r = 1\times 10^{-4}$. We easily obtain that

$$\frac{1}{2}\left(\|U_1(t)\|_{0.8}^2 + \|U_2(t)\|_{0.65}^2\right) + \frac{1}{2}\left(\|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2\right) \approx 0.1632 > r$$

which implies that the condition (i) holds, and

$$\begin{aligned} \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{r} &= \frac{16k^2r}{3} \approx 0.006445 \\ &< \frac{2\int_0^1 F(t,U_1,U_2)\,dt}{\sum_{i=1}^2 \|U_i\|_{\alpha_i}^2 + \sum_{i=1}^2 \|U_i\|_{L^2}^2} \approx 0.0320085949 \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{|u_1|\to\infty,\,|u_2|\to\infty} \inf \frac{F(t,u_1,u_2)}{|u_1|^2+|u_2|^2} \le \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{pr\tilde{k}} \\ &\approx 0.001305. \end{aligned}$$

Thus, conditions (ii) and (iii) are satisfied. Then, in view of Theorem 2 for each $\lambda \in (31.241, 155.159)$, the system (4) has at least three weak solutions in X.

Example 2. *Let $p = 3$, $\alpha_1 = 0.8$, $\alpha_2 = 0.6$, $\mu = 1$, $w_1(t) = 1 + t^2$, $w_2(t) = 0.5 + t$ and $T = 1$. Then , system (4) gets the following form*

$$\begin{cases} {}_tD_1^{0.8}\left(\left(1+t^2\right)_0 D_t^{0.8} u_1(t)\right) + u_1(t) = \lambda F_{u_1}(t, u_1(t), u_2(t)), \ t \in [0,1], \\ {}_tD_1^{0.6}\left((0.5+t)_0 D_t^{0.6} u_2(t)\right) + u_2(t) = \lambda F_{u_2}(t, u_1(t), u_2(t)), \ t \in [0,1], \\ u_1(o) = u_1(1) = 0, u_2(o) = u_2(1) = 0. \end{cases}$$

Taking

$$U_1(t) = \Gamma(1.2)\, t(1-t), U_2(t) = \Gamma(1.4)\, t(1-t)$$

and

$$F(t, u_1(t), u_2(t)) = (1+t) H(u_1, u_2),$$

where

$$H(u_1, u_2) = \begin{cases} \left(u_1^3 + u_2^3\right)^2, \ u_1^3 + u_2^3 \le 1, \\ 10\left(u_1^3 + u_2^3\right)^{\frac{1}{2}} - 9\left(u_1^3 + u_2^3\right)^{\frac{1}{3}}, u_1^3 + u_2^3 > 1. \end{cases}$$

Clearly, $F(t, 0, 0) = 0, w_1^0 = 1$ and $w_2^0 = 0.5$ for any $t \in [0,1]$
By the direct calculation, we have

$$\max\left\{\frac{1}{(\Gamma(0.8))^3((0.8-1)\times 1.5+1)^2}, \frac{1}{((\Gamma(0.6))^3(0.6-1)\times 1.5+1)^2 \times 0.5}\right\}$$

$$= \quad k \approx 3.7849,$$

$$\max\left\{\frac{1}{(\Gamma(0.8+1))^3}, \frac{1}{(\Gamma(0.6+1))^3 \times 0.5}\right\} = \tilde{k} \approx 2.803$$

and

$$\begin{aligned} \left|{}_0D_t^{0.8} U_1(t)\right| &= \begin{cases} t^{0.2} - \frac{2\Gamma(1.2)}{\Gamma(2.2)} t^{1.2}, t \in [0, 0.6] \\ -\left(t^{0.2} - \frac{2\Gamma(1.2)}{\Gamma(2.2)} t^{1.2}\right), \ t \in (0.6, 1], \end{cases} \\ \left|{}_0D_t^{0.6} U_2(t)\right| &= \begin{cases} t^{0.4} - \frac{2\Gamma(1.35)}{\Gamma(2.4)} t^{1.4}, t \in [0, 0.7] \\ -\left(t^{0.4} - \frac{2\Gamma(1.35)}{\Gamma(2.35)} t^{1.4}\right), \ t \in (0.7, 1]. \end{cases} \end{aligned}$$

So that

$$\begin{aligned} \|U_1(t)\|_{0.8}^3 &\approx 0.09228, \|U_2(t)\|_{0.6}^3 \approx 0.0212 \\ \|U_1(t)\|_{L^3}^3 &\approx 0.0053, \|U_2(t)\|_{L^3}^3 \approx 0.005 \end{aligned}$$

Take $r = 1 \times 10^{-5}$. We easily obtain that

$$\frac{1}{3}\left(\|U_1(t)\|_{0.8}^3 + \|U_2(t)\|_{0.65}^3\right) + \frac{1}{3}\left(\|U_1(t)\|_{L^3}^3 + \|U_2(t)\|_{L^3}^3\right)$$

$$\approx \quad 0.041 > r$$

which implies that the condition (i) holds, and

$$\begin{aligned}\frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{r} &= \frac{27k^2r}{2} \approx 0.00193 \\ &< \frac{3\int_0^1 F(t,U_1,U_2)\,dt}{\sum_{i=1}^{2}\|U_i\|_{\alpha_i}^3 + \sum_{i=1}^{2}\|U_i\|_{L^3}^3} \approx 0.00656\end{aligned}$$

and

$$0 = \lim_{u_1\to+\infty,u_2\to+\infty} \inf \frac{F(t,u_1,u_2)}{u_1^3+u_2^3} \leq \frac{\int_0^1 \sup_{(u_1,u_2)\in\pi(kr)} F(t,u_1,u_2)\,dt}{3r\tilde{k}} \approx 0.00023.$$

Thus, conditions (ii) and (iii) are satisfied. Then, in view of Theorem 2 for each $\lambda \in (152.43, 518.13)$, the system (4) admits at least three weak solutions in X.

5. Conclusions

Fractional differential equations have been carefully investigated. Such problems were studied in many scientific and engineering applications such as models for various processes in plasma physics, biology, medical science, chemistry, chemical engineering, as well as population dynamics, and control theory. In the current contribution, motivated by work in ([21,22]) and using a three critical points theorem obtained in [23], we could ensure the existence of at least three solutions for system (4). Note that some appropriate function spaces and variational frameworks were successfully created for the system (4). Furthermore, we have given two examples to illustrate the application of Theorem 2 we have discussed for the special case of $p = 2$, and the discussions presented with respect to the case of $p \neq 2$, which highlighted the superiority of our results. In next work we will use two control parameters to study a class of perturbed nonlinear fractional p-Laplacian differential systems where we will try to prove the existence of three weak solutions by using the variational method and Ricceri's critical points theorems respecting some necessary conditions.

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