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The Derived Subgroups of Sylow 2-Subgroups of the Alternating Group, Commutator Width of Wreath Product of Groups

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Abstract: The structure of the commutator subgroup of Sylow 2-subgroups of an alternating group A_{2^k} is determined. This work continues the previous investigations of me, where minimal generating sets for Sylow 2-subgroups of alternating groups were constructed. Here we study the commutator subgroup of these groups. The minimal generating set of the commutator subgroup of A_{2^k} is constructed. It is shown that $(Syl_2 A_{2^k})^2 = Syl_2' A_{2^k}$, $k > 2$. It serves to solve quadratic equations in this group, as were solved by Lysenok I. in the Grigorchuk group. It is proved that the commutator length of an arbitrary element of the iterated wreath product of cyclic groups C_{p_i}, $p_i \in \mathbb{N}$ equals to 1. The commutator width of direct limit of wreath product of cyclic groups is found. Upper bounds for the commutator width $(cw(G))$ of a wreath product of groups are presented in this paper. A presentation in form of wreath recursion of Sylow 2-subgroups $Syl_2(A_{2^k})$ of A_{2^k} is introduced. As a result, a short proof that the commutator width is equal to 1 for Sylow 2-subgroups of alternating group A_{2^k}, where $k > 2$, the permutation group S_{2^k}, as well as Sylow p-subgroups of $Syl_2 A_{p^k}$ as well as $Syl_2 S_{p^k}$) are equal to 1 was obtained. A commutator width of permutational wreath product $B \wr C_n$ is investigated. An upper bound of the commutator width of permutational wreath product $B \wr C_n$ for an arbitrary group B is found. The size of a minimal generating set for the commutator subgroup of Sylow 2-subgroup of the alternating group is found. The proofs were assisted by the computer algebra system GAP.

Keywords: commutator subgroup; alternating group; minimal generating set; Sylow 2-subgroups; Sylow p-subgroups; commutator width; permutational wreath product

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1. Introduction

The object of our study is the commutatorwidth [1] of Sylow 2-subgroups of alternating group A_{2^k}. As an intermediate goal, we have a structural description of the derived subgroup of this subgroup. The commutator width of G is the minimal n such that for arbitrary $g \in [G, G]$ there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G such that $g = [x_1, y_1] \ldots [x_n, y_n]$.

Our study of the width of the commutator is somewhat similar to the study of equations in simple matrix groups [2], and is also associated with verbal subgroups. Additionally, in related work [3], it was established that the commutator width of the first Grigorchuk group is 2.

Commutator width of groups, and of elements, has proven to be an important group property, in particular via its connections with stable commutator length and bounded cohomology [4,5]. It is also related to solvability of quadratic equations in groups [6]: a group G has commutator width $\leq n$ if and only if the equation $[X_1, X_2] \ldots [X_{2n-1}, X_{2n}]g = 1$ is solvable for all $g \in G'$.

As it is well known, the first example of a group G with commutator width greater than 1 ($cw(G) > 1$) was given by Fite [7]. The smallest finite examples of such groups are groups of order 96; there are two of them, nonisomorphic to each other, which were given by Guralnick [8].

We obtain an upper bound for commutator width of wreath product $C_n \wr B$, where C_n is cyclic group of order n, in terms of the commutator width $cw(B)$ of passive group B. A form of commutators of wreath product $A \wr B$ was briefly considered in [9]. The form of commutator presentation [9] is proposed by us as wreath recursion [10], and the commutator width of it was studied. We imposed a weaker condition on the presentation of wreath product commutator than was proposed by J. Meldrum.

In this paper we continue investigations started in [11–17]. We find a minimal generating set and the structure for commutator subgroup of $Syl_2 A_{2^k}$.

Research of commutator-group serves the decision of inclusion problem [18] for elements of $Syl_2 A_{2^k}$ in its derived subgroup $(Syl_2 A_{2^k})'$. Knowledge of the method for solving the of inclusion problem in a subgroup H facilitates the solution of the problem of finding the conjugate elements in the whole group (conjugacy search problem) [19]. Because by the characterization of the conjugated elements g and $h^{-1}gh$, we can determine which subgroups they belong to and which do not belong.

It is known that the commutator width of iterated wreath products of nonabelian finite simple groups is bounded by an absolute constant [7,20]. But it has not been proven that commutator subgroup of $\mathop{\wr}\limits_{i=1}^{k} \mathcal{C}_{p_i}$ consists of commutators. We generalize the passive group of this wreath product to any group B instead of only wreath product of cyclic groups and obtain an exact commutator width.

Additionally, we are going to prove that the commutator width of Sylow p-subgroups of symmetric and alternating groups for $p \geq 2$ is 1.

2. Preliminaries

Let G be a group acting (from the right) by permutations on a set X and let H be an arbitrary group. Then the (permutational) wreath product $H \wr G$ is the semidirect product $H^X \leftthreetimes G$, where G acts on the direct power H^X by the respective permutations of the direct factors. The cyclic group C_p or (C_p, X) is equipped with a natural action by the left shift on $X = \{1, ..., p\}$, $p \in \mathbb{N}$. It is well known that a wreath product of permutation groups is associative construction [9].

The multiplication rule of automorphisms g and h, which are presented in form of the wreath recursion [21] $g = (g_{(1)}, g_{(2)}, \ldots, g_{(d)})\sigma_g$, $h = (h_{(1)}, h_{(2)}, \ldots, h_{(d)})\sigma_h$, is given by the formula:

$$g \cdot h = (g_{(1)} h_{(\sigma_g(1))}, g_{(2)} h_{(\sigma_g(2))}, \ldots, g_{(d)} h_{(\sigma_g(d))})\sigma_g \sigma_h.$$

We define σ as $(1, 2, \ldots, p)$ where p is defined by context.

The set X^* is naturally a vertex set of a regular rooted tree; i.e., a connected graph without cycles and a designated vertex v_0 called the root, in which two words are connected by an edge if and only if they are of form v and vx, where $v \in X^*$, $x \in X$. The set $X^n \subset X^*$ is called the n-th level of the tree X^* and $X^0 = \{v_0\}$. We denote by v_{ji} the vertex of X^j, which has the number i, where $1 \leq i \leq X^{2^j}$ and the numeration starts from 1. Note that the unique vertex $v_{k,i}$ corresponds to the unique word v in alphabet X. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ determine the section (state) $g_{(v)} \in AutX^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The subtree of X^* induced by the set of vertices $\cup_{i=0}^{k} X^i$ is denoted by $X^{[k]}$. The restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$. The restriction $g_{(v_{ij})}|_{X^{[1]}}$ is called the vertex permutation (v.p.) of g at a vertex v_{ij} and denoted by g_{ij}. For example, if $|X| = 2$ then we just have to distinguish active vertices; i.e. the vertices for which g_{ij} is non-trivial [21].

We label every vertex of X^l, $0 \leq l < k$ by 0 or 1 depending on the action of v.p. on it. The resulting vertex-labeled regular tree is an element of $AutX^{[k]}$. All undeclared terms are from [22–24].

Let us fix some notation. For convenience the commutator of two group elements a and b is denoted by $[a,b] = aba^{-1}b^{-1}$, conjugation by an element b we denote by

$$a^b = bab^{-1}.$$

We define G_k and B_k recursively; i.e.,

$$B_1 = C_2,\ B_k = B_{k-1} \wr C_2 \text{ for } k > 1,$$
$$G_1 = \langle e \rangle,\ G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\} \text{ for } k > 1.$$

Note that $B_k = \mathop{\wr}\limits_{i=1}^{k} C_2$.

The commutator length of an element g of a derived subgroup of a group G, is the minimal n such that there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G such that $g = [x_1, y_1] \ldots [x_n, y_n]$. The commutator length of the identity element is 0. Let *clG(g)* denotes the commutator length of an element g of a group G. The commutator width of a group G is the maximum of *clG(g)* of the elements of its derived subgroup $[G, G]$. We denote by $d(G)$ the minimal number of generators of the group G.

3. Commutator Width of Sylow 2-Subgroups of A_{2^k} and S_{2^k}

The the following lemma improves the result stated as Corollary 4.9 in of [9]. Our proof uses arguments similar to those of [9].

Lemma 1. *An element of form $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ iff product of all r_i (in any order) belongs to B', where $p \in \mathrm{N}$, $p \geq 2$.*

Proof. More details of our argument may be given as follows. If we multiply elements from a tuple $(r_1, \ldots, r_{p-1}, r_p) = w$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, h_i, $g_i \in B$ and $a, b \in C_p$, then we get a product

$$x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B', \tag{1}$$

where x is a product of appropriate commutators. Therefore, we can write $r_p = r^{-1}_{p-1} \ldots r_1^{-1} x$. We can rewrite element $x \in B'$ as the product $x = \prod\limits_{j=1}^{m} [h_j, g_j]$, $m \leq cw(B)$.

Note that we impose a weaker condition on the product of all r_i which belongs to B' than in Definition 4.5 of form $P(L)$ in [9], where the product of all r_i belongs to a subgroup L of B such that $L > B'$.

In more detail, deducing of our representation construct can be reported in the following way. If we multiply elements having form of a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, h_i, $g_i \in B$ and $a, b \in C_p$, then we obtain a product

$$\prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B'. \tag{2}$$

Note that if we rearrange elements in (1) as $h_1 h_1^{-1} g_1 g_2^{-1} h_2 h_2^{-1} g_1 g_2^{-1} \ldots h_p h_p^{-1} g_p g_p^{-1}$ then by the reason of such permutations we obtain a product of appropriate commutators. Therefore, the following equality holds

$$\prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} = \prod_{i=1}^{p} h_i g_i h_i^{-1} g_i^{-1} x_0 = \prod_{i=1}^{p} h_i h_i^{-1} g_i g_i^{-1} x \in B', \tag{3}$$

where x_0, x are the products of appropriate commutators. Therefore,

$$(r_1, \ldots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'. \tag{4}$$

Thus, one element from states of wreath recursion $(r_1, \ldots, r_{p-1}, r_p)$ depends on rest of r_i. This implies that the product $\prod_{j=1}^{p} r_j$ for an arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to B'. Thus, r_p can be expressed as:

$$r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x.$$

Denote a j-th tuple consisting of wreath recursion elements by $(r_{j_1}, r_{j_2}, ..., r_{j_p})$. The fact that the set of forms $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ is closed under multiplication follows from the identity follows from

$$\prod_{j=1}^{k} (r_{j1} \ldots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{j_i} = R_1 R_2 ... R_k \in B', \tag{5}$$

where r_{ji} is i-th element of the tuple number j, $R_j = \prod_{i=1}^{p} r_{ji}$, $1 \le j \le k$. As it was shown above $R_j = \prod_{i=1}^{p-1} r_{ji} \in B'$. Therefore, the product (5) of R_j, $j \in \{1, ..., k\}$ which is similar to the product mentioned in [9], has the property $R_1 R_2 ... R_k \in B'$ too, because of B' is subgroup. Thus, we get a product of form (1) and the similar reasoning as above is applicable.

Let us prove the sufficiency condition. If the set K of elements satisfying the condition of this theorem, that all products of all r_i, where every i occurs in this form once, belong to B'. Then using the elements of the form

$(r_1, e, ..., e, r_1^{-1})$, ... , $(e, e, ..., e, r_i, e, r_i^{-1})$, ... ,$(e, e, ..., e, r_{p-1}, r_{p-1}^{-1})$, $(e, e, ..., e, r_1 r_2 \cdot ... \cdot r_{p-1})$

we can express any elements of the form $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$. We need to prove that in such a way we can express all element from W and only elements of W. All elements of W can be generated by elements of K since r_i, $i < p$ are arbitrary and the fact that equality (1) holds, so r_p is well determined. □

Lemma 2. *Assume a group B and an integer $p \ge 2$. If $w \in (B \wr C_p)'$ then w can be represented as the following wreath recursion*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$ and $k \le cw(B)$.

Proof. According to Lemma 1 we have the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$ and $r_{p-1} r_{p-2} \ldots r_2 r_1 r_p = x \in B'$. Therefore, we can write $r_p = r_1^{-1} \ldots r_{p-1}^{-1} x$. We can also rewrite an element $x \in B'$ as a product of commutators $x = \prod_{j=1}^{k} [f_j, g_j]$ where $k \le cw(B)$. □

Lemma 3. *For any group B and integer $p \ge 2$, suppose $w \in (B \wr C_p)'$ is defined by the following wreath recursion:*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} [f, g]),$$

where $r_1, \ldots, r_{p-1}, f, g \in B$. Then we can represent w as the following commutator

$$w = [(a_{1,1}, \ldots, a_{1,p})\sigma, (a_{2,1}, \ldots, a_{2,p})],$$

where

$$\begin{aligned} &a_{1,i} = e, \text{ for } 1 \le i \le p-1\,, \\ &a_{2,1} = (f^{-1})^{r_1^{-1} \ldots r_{p-1}^{-1}}, \\ &a_{2,i} = r_{i-1}a_{2,i-1}, \text{ for } 2 \le i \le p, \\ &a_{1,p} = g^{a_{2,p}^{-1}}. \end{aligned}$$

Proof. Consider the following commutator

$$\begin{aligned} \kappa &= (a_{1,1}, \ldots, a_{1,p})\sigma \cdot (a_{2,1}, \ldots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}^{-1}, \ldots, a_{1,p-1}^{-1})\sigma^{-1} \cdot (a_{2,1}^{-1}, \ldots, a_{2,p}^{-1}) \\ &= (a_{3,1}, \ldots, a_{3,p}), \end{aligned}$$

where

$$a_{3,i} = a_{1,i}a_{2,1+(i \bmod p)}a_{1,i}^{-1}a_{2,i}^{-1}.$$

At first we compute the following

$$a_{3,i} = a_{1,i}a_{2,i+1}a_{1,i}^{-1}a_{2,i}^{-1} = a_{2,i+1}a_{2,i}^{-1} = r_i a_{2,i}a_{2,i}^{-1} = r_i, \text{ for } 1 \le i \le p-1.$$

Then we make some transformation of $a_{3,p}$:

$$\begin{aligned} a_{3,p} &= a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\ &= (a_{2,1}a_{2,1}^{-1})a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\ &= a_{2,1}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}^{-1} \\ &= a_{2,1}a_{2,p}^{-1}a_{2,p}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}^{-1} \\ &= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,p}}, a_{1,p}^{a_{2,p}}] \\ &= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,p}a_{2,1}^{-1}}, a_{1,p}^{a_{2,p}}]. \end{aligned}$$

Now we can see that the form of the commutator κ is similar to the form of w.

Introduce the following notation

$$r' = r_{p-1} \ldots r_1.$$

We note that from the definition of $a_{2,i}$ for $2 \le i \le p$ it follows that

$$r_i = a_{2,i+1}a_{2,i}^{-1}, \text{ for } 1 \le i \le p-1.$$

Therefore

$$\begin{aligned} r' &= (a_{2,p}a_{2,p-1}^{-1})(a_{2,p-1}a_{2,p-2}^{-1}) \ldots (a_{2,3}a_{2,2}^{-1})(a_{2,2}a_{2,1}^{-1}) \\ &= a_{2,p}a_{2,1}^{-1}. \end{aligned}$$

Then

$$(a_{2,p}a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \ldots r_{p-1}^{-1}.$$

Now we compute the following

$$(a_{2,1}^{-1})^{a_{2,p}a_{2,1}^{-1}} = (((f^{-1})^{r_1^{-1}\dots r_{p-1}^{-1}})^{-1})^{r'} = (f^{(r')^{-1}})^{r'} = f,$$
$$a_{1,p}^{a_{2,p}} = (g^{a_{2,p}^{-1}})^{a_{2,p}} = g.$$

Finally, we conclude that

$$a_{3,p} = r_1^{-1}\dots r_{p-1}^{-1}[f,g].$$

Thus, the commutator κ has the same form as w. □

For future using we formulate previous Lemma for the case $p = 2$.

Corollary 1. *For any group B, suppose $w \in (B \wr C_2)'$ is defined by the following wreath recursion*

$$w = (r_1, r_1^{-1}[f,g]),$$

where $r_1, f, g \in B$. Then we can represent w as commutator

$$w = [(e, a_{1,2})\sigma, (a_{2,1}, a_{2,2})],$$

where

$$a_{2,1} = (f^{-1})^{r_1^{-1}},$$
$$a_{2,2} = r_1 a_{2,1},$$
$$a_{1,2} = g^{a_{2,2}^{-1}}.$$

Lemma 4. *For any group B and integer $p \geq 2$ the inequality*

$$cw(B \wr C_p) \leq \max(1, cw(B))$$

holds.

Proof. By Lemma 1, we can represent any $w \in (B \wr C_p)'$ as the following wreath recursion

$$w = (r_1, r_2, \dots, r_{p-1}, r_1^{-1}\dots, r_{p-1}^{-1}\prod_{j=1}^{k}[f_j, g_j])$$
$$= (r_1, r_2, \dots, r_{p-1}, r_1^{-1}\dots, r_{p-1}^{-1}[f_1, g_1]) \cdot \prod_{j=2}^{k}[(e, \dots, e, f_j), (e, \dots, e, g_j)],$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$ and $k \leq cw(B)$. Now by the Lemma 3 we can see that w can be represented as a product of $\max(1, cw(B))$ commutators. □

Corollary 2. *If $W = C_{p_k} \wr \dots \wr C_{p_1}$ then $cw(W) = 1$ for $k \geq 2$.*

Proof. If $B = C_{p_k} \wr C_{p_{k-1}}$, then take into consideration that $cw(B) > 0$ (because $C_{p_k} \wr C_{p_{k-1}}$ is not commutative group). Lemma 4 implies that $cw(C_{p_k} \wr C_{p_{k-1}}) = 1$, and using the inequality $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) \leq \max(1, cw(B))$ from Lemma 4 we obtain $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) = 1$. Similarly, if $W = C_{p_k} \wr \dots \wr C_{p_1}$ we use inductive assumption for $C_{p_k} \wr \dots \wr C_{p_2}$ the associativity of a permutational wreath product, the inequality of Lemma 4 and the equality $cw(C_{p_k} \wr \dots \wr C_{p_2}) = 1$ to conclude that $cw(W) = 1$. □

We define our partially ordered set M and directed system of finite wreath products of cyclic groups as the set of all finite wreath products of cyclic groups. We make of use directed set $\mathbb{N}$.

$$H_k = \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \tag{6}$$

Moreover, it has already been proved in Corollary 3 that each group of the form $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ has a commutator width equal to 1; i.e., $cw(\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}) = 1$. A partially ordered set of a subgroups is ordered by relation of inclusion group as a subgroup. Define the injective homomorphism $f_{k,k+1}$ from the $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ into $\underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i}$ by mapping a generator of active group $\mathcal{C}_{p_i}$ of H_k in a generator of active group $\mathcal{C}_{p_i}$ of H_{k+1}.

In more detail, the injective homomorphism $f_{k,k+1}$ is defined as $g \mapsto g(e, ..., e)$, where a generator $g \in \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$, $g(e, ..., e) \in \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i}$.

We therefore obtain an injective homomorphism from H_k onto the subgroup $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ of H_{k+1}.

Corollary 3. *The direct limit* $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ *of the direct system* $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ *has commutator width 1.*

Proof. We make the transition to the direct limit in the direct system $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ of injective mappings from chain $e \rightarrow \ ... \ \rightarrow \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \rightarrow \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i} \rightarrow \underset{i=1}{\overset{k+2}{\wr}} \mathcal{C}_{p_i} \rightarrow ....$

Since all mappings in chains are injective homomorphisms, they have a trivial kernel. Therefore, the transition to a direct limit boundary preserves the property $cw(H) = 1$, because each group H_k from the chain is endowed by $cw(H_k) = 1$.

The direct limit of the direct system is denoted by $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ and is defined as disjoint union of the H_k's modulo a certain equivalence relation:

$$\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} = {}^{\coprod_k \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}} / \sim .$$

Since every element g of $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ coincides with a correspondent element from some H_k of direct system, then by the injectivity of the mappings for g the property $cw(\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}) = 1$ also holds. Thus, it holds for the whole $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$. $\square$

Corollary 4. *For prime* p *and* $k \geq 2$ *we have* $cw(Syl_p(S_{p^k})) = 1$. *For prime* $p > 2$ *and* $k \geq 2$ *we have* $cw(Syl_p(A_{p^k})) = 1$.

Proof. Since $Syl_p(S_{p^k}) \simeq \underset{i=1}{\overset{k}{\wr}} C_p$ (see [25,26]), we have $cw(Syl_p(S_{p^k})) = 1$. It is well known that in a case $p > 2$ where we have $Syl_p S_{p^k} \simeq Syl_p A_{p^k}$ (see [15,23]), so we obtain $cw(Syl_p(A_{p^k})) = 1$. $\square$

Proposition 1. *There is an inclusion* $B'_k < G_k$ *holds.*

Proof. We use induction on k. For $k = 1$ we have $B'_k = G_k = \{e\}$. Fix some $g = (g_1, g_2) \in B'_k$. Then $g_1 g_2 \in B'_{k-1}$ by Lemma 1. As $B'_{k-1} < G_{k-1}$ by the induction hypothesis therefore $g_1 g_2 \in G_{k-1}$ and by definition of G_k it follows that $g \in G_k$. □

Corollary 5. *The set G_k is a subgroup in the group B_k.*

Proof. According to the recursively definition of G_k and B_k, where $G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ $k > 1$, i.e. G_k is subset of B_k with condition $g_1 g_2 \in G_{k-1}$. The result follows from the fact that G_{k-1} is a subgroup of G_k. It is easy to check the closedness by multiplication elements of G_k with condition $g_1 g_2, h_1 h_2 \in G_{k-1}$ because G_{k-1} is subgroup so $g_1 g_2 h_1 h_2 \in G_{k-1}$ too. The inverses can be verified easily. □

Lemma 5. *For any $k \geq 1$ we have $|G_k| = |B_k|/2$.*

Proof. Induction on k. For $k = 1$ we have $|G_1| = 1 = |B_1/2|$. Every element $g \in G_k$ can be uniquely written as the following wreath recursion

$$g = (g_1, g_2)\pi = (g_1, g_1^{-1}x)\pi$$

where $g_1 \in B_{k-1}$, $x \in G_{k-1}$ and $\pi \in C_2$. Elements g_1, x and π are independent; therefore, $|G_k| = 2|B_{k-1}| \cdot |G_{k-1}| = 2|B_{k-1}| \cdot |B_{k-1}|/2 = |B_k|/2$. □

Corollary 6. *The group G_k is a normal subgroup in the group B_k; i.e., $G_k \lhd B_k$.*

Proof. There exists normal embedding (normal injective monomorphism) $\varphi : G_k \to B_k$ [27] such that $G_k \lhd B_k$. Indeed, according to Lemma index $|B_k : G_k| = 2$, so it is a normal subgroup; that is, a quotient subgroup ${}^{B_k}/_{C_2} \simeq G_k$. □

Theorem 1. *For any $k \geq 1$ we have $G_k \simeq Syl_2 A_{2^k}$.*

Proof. Group C_2 acts on the set $X = \{1, 2\}$. Therefore, we can recursively define sets X^k on which group B_k acts $X^1 = X$, $X^k = X^{k-1} \times X$ for k>1. At first we define $S_{2^k} = Sym(X^k)$ and $A_{2^k} = Alt(X^k)$ for all integers $k \geq 1$. Then $G_k < B_k < S_{2^k}$ and $A_{2^k} < S_{2^k}$.

We already know [15] that $B_k \simeq Syl_2(S_{2^k})$. Since $|A_{2^k}| = |S_{2^k}|/2$, $|Syl_2 A_{2^k}| = |Syl_2 S_{2^k}|/2 = |B_k|/2$. By Lemma 3 it follows that $|Syl_2 A_{2^k}| = |G_k|$. Therefore, it remains to show that $G_k < Alt(X^k)$.

Let us fix some $g = (g_1, g_2)\sigma^i$ where $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$ and $g_1 g_2 \in G_{k-1}$. Then we can represent g as follows

$$g = (g_1 g_2, e) \cdot (g_2^{-1}, g_2) \cdot (e, e,)\sigma^i.$$

In order to prove this theorem it is enough to show that $(g_1 g_2, e), (g_2^{-1}, g_2), (e, e,)\sigma \in Alt(X^k)$.

Elements $(e, e,)\sigma$ just switch letters x_1 and x_2 for all $x \in X^k$. Therefore, $(e, e,)\sigma$ is product of $|X^{k-1}| = 2^{k-1}$ transpositions, and therefore, $(e, e,)\sigma \in Alt(X^k)$.

Elements g_2^{-1} and g_2 have the same cycle type. Therefore, elements (g_2^{-1}, e) and (e, g_2) also have the same cycle type. Let us fix the following cycle decompositions

$$(g_2^{-1}, e) = \sigma_1 \cdot \ldots \cdot \sigma_n,$$
$$(e, g_2) = \pi_1 \cdot \ldots \cdot \pi_n.$$

Note that element (g_2^{-1}, e) acts only on letters like x_1, and element (e, g_2) acts only on letters like x_2. Therefore, we have the following cycle decomposition

$$(g_2^{-1}, g_2) = \sigma_1 \cdot \ldots \cdot \sigma_n \cdot \pi_1 \cdot \ldots \cdot \pi_n.$$

So, element (g_2^{-1}, g_2) has even number of odd permutations and then $(g_2^{-1}, g_2) \in Alt(X^k)$.

Note that $g_1 g_2 \in G_{k-1}$ and $G_{k-1} = Alt(X^{k-1})$ by induction hypothesis. Therefore, $g_1 g_2 \in Alt(X^{k-1})$. As elements $g_1 g_2$ and $(g_1 g_2, e)$ have the same cycle type, $(g_1 g_2, e) \in Alt(X^k)$. □

As it was proven by the author in [15], the Sylow 2-subgroup has structure $B_{k-1} \ltimes W_{k-1}$, where the definition of B_{k-1} is the same that which was given in [15].

Recall that it was denoted by W_{k-1} the subgroup of $AutX^{[k]}$ such that it had active states only on X^{k-1} and a number of such states that was even; i.e., $W_{k-1} \lhd St_{G_k}(k-1)$ [21]. It was proven that the size of W_{k-1} is equal to $2^{2^{k-1}-1}$, $k > 1$ and its structure is $(C_2)^{2^{k-1}-1}$. The following structural theorem characterizing the group G_k was proven by us [15].

Theorem 2. *A maximal 2-subgroup of $AutX^{[k]}$ that acts by even permutations on X^k has the structure of the semidirect product $G_k \simeq B_{k-1} \ltimes W_{k-1}$ and isomorphic to $Syl_2 A_{2^k}$.*

Note that W_{k-1} is subgroup of stabilizer of X^{k-1}, i.e., $W_{k-1} < St_{AutX^{[k]}}(k-1) \lhd AutX^{[k]}$ and is normal to $W_{k-1} \lhd AutX^{[k]}$, because conjugation keeps a cyclic structure of permutation, so even permutation maps are even. Therefore, such conjugation induce an automorphism of W_{k-1} and $G_k \simeq B_{k-1} \ltimes W_{k-1}$.

Remark 1. *As a consequence, the structure founded by us in [15] is fully consistent with the recursive group representation (used in this paper) based on the concept of wreath recursion [10].*

Theorem 3. *Elements of B_k' have the following form $B_k' = \{[f,l] \mid f \in B_k, l \in G_k\} = \{[l,f] \mid f \in B_k, l \in G_k\}$.*

Proof. It is enough to show either $B_k' = \{[f,l] \mid f \in B_k, l \in G_k\}$ or $B_k' = \{[l,f] \mid f \in B_k, l \in G_k\}$, because if $f = [g,h]$, then $f^{-1} = [h,g]$.

We prove the proposition by induction on k. For the case $k = 1$ we have $B_1' = \langle e \rangle$.

Consider case $k > 1$. According to Lemma 2 and Corollary 1 every element $w \in B_k'$ can be represented as

$$w = (r_1, r_1^{-1}[f,g])$$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By the Corollary 1 we can represent w as commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where

$$\begin{aligned} a_{2,1} &= (f^{-1})^{r_1^{-1}}, \\ a_{2,2} &= r_1 a_{2,1}, \\ a_{1,2} &= g^{a_{2,2}^{-1}}. \end{aligned}$$

If $g \in G_{k-1}$, then by the definition of G_k and Corollary 6 we obtain $(e, a_{1,2})\sigma \in G_k$. □

Remark 2. *Let us to note that Theorem 3 improve Corollary 4 for the case $Syl_2 S_{2^k}$.*

Proposition 2. *If g is an element of the group B_k then $g^2 \in B_k'$.*

Proof. Induction on k. We note that $B_k = B_{k-1} \wr C_2$. Therefore, we fix some element

$$g = (g_1, g_2)\sigma^i \in B_{k-1} \wr C_2,$$

where $g_1, g_2 \in B_{k-1}$ and $i \in \{0,1\}$. Let us to consider g^2. Then, two cases are possible:

$$g^2 = (g_1^2, g_2^2) \text{ or } g^2 = (g_1 g_2, g_2 g_1).$$

In the second case we consider a product of coordinates $g_1g_2 \cdot g_2g_1 = g_1^2 g_2^2 x$. Since according to the induction hypothesis $g_i^2 \in B_k'$, $i \leq 2$ then $g_1g_2 \cdot g_2g_1 \in B_k'$ also according to Lemma 1 $x \in B_k'$. Therefore, a following inclusion holds $(g_1g_2, g_2g_1) = g^2 \in B_k'$. In first case the proof is even simpler because $g_1^2, g_2^2 \in B'$ by the induction hypothesis. □

Lemma 6. *If an element $g = (g_1, g_2) \in G_k'$ then $g_1, g_2 \in G_{k-1}$ and $g_1g_2 \in B_{k-1}'$.*

Proof. As $B_k' < G_k$, it is therefore enough to show that $g_1 \in G_{k-1}$ and $g_1g_2 \in B_{k-1}'$. Let us fix some $g = (g_1, g_2) \in G_k' < B_k'$. Then, Lemma 1 implies that $g_1g_2 \in B_{k-1}'$.

In order to show that $g_1 \in G_{k-1}$, we firstly consider just one commutator of arbitrary elements from G_k

$$f = (f_1, f_2)\sigma, \ h = (h_1, h_2)\pi \in G_k,$$

where $f_1, f_2, h_1, h_2 \in B_{k-1}, \sigma, \pi \in C_2$. The definition of G_k implies that $f_1f_2, h_1h_2 \in G_{k-1}$.

If $g = (g_1, g_2) = [f, h]$, then

$$g_1 = f_1 h_i f_j^{-1} h_k^{-1}$$

for some $i, j, k \in \{1, 2\}$. Then

$$g_1 = f_1 h_i f_j (f_j^{-1})^2 h_k (h_k^{-1})^2 = (f_1 f_j)(h_i h_k) x (f_j^{-1} h_k^{-1})^2,$$

where x is product of commutators of f_i, h_j and f_i, h_k; hence, $x \in B_{k-1}'$.

It is enough to consider the first product $f_1 f_j$. If $j = 1$, then $f_1^2 \in B_{k-1}'$ by Proposition 2 if $j = 2$ then $f_1 f_2 \in G_{k-1}$ according to definition of G_k; the same is true for $h_i h_k$. Thus, for any i, j, k it holds that $f_1 f_j, h_i h_k \in G_{k-1}$. Besides that, a square $(f_j^{-1} h_k^{-1})^2 \in B_k'$ according to Proposition 2. Therefore, $g_1 \in G_{k-1}$ because of Propositions 1 and 2, the same is true for g_2.

Now it remains to consider the product of some $f = (f_1, f_2), h = (h_1, h_2)$, where $f_1, h_1 \in G_{k-1}$, $f_1 h_1 \in G_{k-1}$ and $f_1 f_2, h_1 h_2 \in B_{k-1}'$

$$fh = (f_1 h_1, f_2 h_2).$$

Since $f_1 f_2, h_1 h_2 \in B_{k-1}'$ by imposed condition in this item and taking into account that $f_1 h_1 f_2 h_2 = f_1 f_2 h_1 h_2 x$ for some $x \in B_{k-1}'$, then $f_1 h_1 f_2 h_2 \in B_{k-1}'$ by Lemma 1. In other words, closedness by multiplication holds, and so according to Lemma 1, we have element of commutator G_k'. □

In the following theorem we prove two facts at once.

Theorem 4. *The following statements are true.*

1. *An element $g = (g_1, g_2) \in G_k'$ iff $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B_{k-1}'$.*
2. *Commutator subgroup G_k' coincides with set of all commutators for $k \geq 3$*

$$G_k' = \{[f_1, f_2] \mid f_1 \in G_k, f_2 \in G_k\}.$$

Proof. For the case $k = 1$ we have $G'_1 = \langle e \rangle$. So, further we consider the case $k \geq 2$. If $k = 2$ then we have $G_2 \simeq V_4$, where V_4 is the Klein four group. But $cw(V_4) = 0$.

Sufficiency of the first statement of this theorem follows from the Lemma 6. So, in order to prove the necessity of the both statements it is enough to show that element

$$w = (r_1, r_1^{-1}x),$$

where $r_1 \in G_{k-1}$ and $x \in B'_{k-1}$, can be represented as a commutator of elements from G_k. By Proposition 3 we have $x = [f, g]$ for some $f \in B_{k-1}$ and $g \in G_{k-1}$. Therefore,

$$w = (r_1, r_1^{-1}[f, g]).$$

By the Corollary 1 we can represent w as a commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where $a_{2,1} = (f^{-1})^{r_1^{-1}}, a_{2,2} = r_1 a_{2,1}, a_{1,2} = g^{a_{2,2}^{-1}}$. It only remains to show that $(e, a_{1,2})\sigma$, $(a_{2,1}, a_{2,2}) \in G_k$. Note the following

$$a_{1,2} = g^{a_{2,2}^{-1}} \in G_{k-1} \text{ by Corollary 6.}$$
$$a_{2,1}a_{2,2} = a_{2,1}r_1a_{2,1} = r_1[r_1, a_{2,1}]a_{2,1}^2 \in G_{k-1} \text{ by Propositions 1 and 2.}$$

So we have $(e, a_{1,2})\sigma \in G_k$ and $(a_{2,1}, a_{2,2}) \in G_k$ by the definition of G_k. □

Proposition 3. *For arbitrary $g \in G_k$ the inclusion $g^2 \in G'_k$ holds.*

Proof. Induction on k: elements of G_1^2 have form $(\sigma)^2 = e$, where $\sigma = (1, 2)$, so the statement holds. In a general case, when $k > 1$, the elements of G_k have the form $g = (g_1, g_2)\sigma^i$, $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$. Then we have two possibilities: $g^2 = (g_1^2, g_2^2)$ or $g^2 = (g_1g_2, g_2g_1)$.

Firstly we show that $g_1^2 \in G_{k-1}, g_2^2 \in G_{k-1}$. According to Proposition 2, we have $g_1^2, g_2^2 \in B'_{k-1}$ and according to Proposition 1, we have $B'_{k-1} < G_{k-1}$. Then, using Theorem 4 $g^2 = (g_1^2, g_2^2) \in G_k$.

Consider the second case $g^2 = (g_1g_2, g_2g_1)$. Since $g \in G_k$, then, according to the definition of G_k, we have that $g_1g_2 \in G_{k-1}$. By Proposition 1, and the definition of G_k, we obtain

$$g_2g_1 = g_1g_2g_2^{-1}g_1^{-1}g_2g_1 = g_1g_2[g_2^{-1}, g_1^{-1}] \in G_{k-1},$$
$$g_1g_2 \cdot g_2g_1 = g_1g_2^2g_1 = g_1^2g_2^2[g_2^{-2}, g_1^{-1}] \in B'_{k-1}.$$

Note that $g_1^2, g_2^2 \in B'_{k-1}$ according to Proposition 2, $g_1^2g_2^2[g_2^{-2}, g_1^{-1}] \in B'_{k-1}$. Since $g_1g_2 \cdot g_2g_1 \in B'_{k-1}$ and $g_1g_2, g_2g_1 \in G_{k-1}$, then, according to Lemma 6, we obtain $g^2 = (g_1g_2, g_2g_1) \in G'_k$. □

Statement 1. *The commutator subgroup is a subgroup of G_k^2; i.e., $G'_k < G_k^2$.*

Proof. Indeed, an arbitrary commutator presented as the product of squares. Let $a,\ b \in G$ and set that $x = a,\ \ y = a^{-1}ba,\ \ z = a^{-1}b^{-1}$. Then $x^2y^2z^2 = a^2(a^{-1}ba)^2(a^{-1}b^{-1})^2 = aba^{-1}b^{-1}$. In more detail: $a^2(a^{-1}ba)^2(a^{-1}b^{-1})^2 = a^2a^{-1}ba\,a^{-1}ba\ a^{-1}b^{-1}a^{-1}b^{-1} = abbb^{-1}a^{-1}b^{-1} = [a, b]$. In such way we obtain all commutators and their products. Thus, we generate by squares the whole G'_k. □

Corollary 7. *For the Syllow subgroup $(Syl_2 A_{2^k})$ the following equalities $Syl'_2(A_{2^k}) = (Syl_2(A_{2^k}))^2$, $\Phi(Syl_2 A_{2^k}) = Syl'_2(A_{2^k})$, which are characteristic properties of special p-groups [28], are true.*

Proof. As is well known, for an arbitrary group (also by Statement 1) the following embedding $G' \triangleleft G^2$ holds. In view of the above Proposition 3, a reverse embedding for G_k is true. Thus, the group $Syl_2 A_{2^k}$ has some properties of special p-groups; that is, $P' = \Phi(P)$ [28] because $G_k^2 = G_k'$ and so $\Phi(Syl_2 A_{2^k}) = Syl_2'(A_{2^k})$. □

Corollary 8. *Commutator width of the group $Syl_2 A_{2^k}$ is equal to 1 for $k \geq 3$, also $cw(Syl_2 A_4) = 0$.*

It immediately follows from item 2 of Theorem 4 and the fact that $Syl_2 A_4 \simeq V_4$.

4. Minimal Generating Set

For the construction of minimal generating set, we used the representation of elements of group G_k by portraits of automorphisms at restricted binary tree $AutX^k$. For convenience we will identify elements of G_k with their faithful representations by portraits of automorphisms from $AutX^{[k]}$.

We denote by $A|_l$, a set of all functions a_l, such that $[\varepsilon, \dots, \varepsilon, a_l, \varepsilon, \dots] \in [A]_l$. Recall that according to [29], l-coordinate subgroup $U < G$ is the following subgroup.

Definition 1. *For an arbitrarry $k \in \mathbb{N}$ we call a $k-$*coordinate *subgroup $U < G$ a subgroup, which is determined by k-coordinate sets $[U]_l$, $l \in \mathbb{N}$, if this subgroup consists of all Kaloujnine's tableaux $a \in I$ for which $[a]_l \in [U]_l$.*

We denote by $G_k(l)$ a level subgroup of G_k, which consists of the tuples of v.p. from X^l, $l < k-1$ of any $\alpha \in G_k$. We denote as $G_k(k-1)$ such subgroups of G_k that are generated by v.p., which are located on X^{k-1} and isomorphic to W_{k-1}. Note that $G_k(l)$ is in bijective correspondence (and isomorphism) with l-coordinate subgroup $[U]_l$ [29].

For any v.p. g_{li} in v_{li} of X^l we set in correspondence with g_{li} the permutation $\varphi\,(g_{li}) \in S_2$ by the following rule:

$$\varphi(g_{li}) = \begin{cases} (1,2), & \text{if } g_{li} \neq e, \\ e, & \text{if } g_{li} = e. \end{cases} \tag{7}$$

Define a homomorphic map from $G_k(l)$ onto S_2 with the kernel consisting of all products of even number of transpositions that belong to $G_k(l)$. For instance, the element $(12)(34)$ of $G_k(2)$ belongs to $ker\varphi$. Hence, $\varphi\,(g_{li}) \in S_2$.

Definition 2. *We define the subgroup of l-th level as a subgroup generated by all possible vertex permutation of this level.*

Statement 2. *In ${G_k}'$, the following k equalities are true:*

$$\prod_{l=1}^{2^l} \varphi(g_{lj}) = e, \quad 0 \leq l < k-1. \tag{8}$$

For the case $i = k-1$, the following condition holds:

$$\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1j}) = \prod_{j=2^{k-2}+1}^{2^{k-1}} \varphi(g_{k-1j}) = e. \tag{9}$$

Thus, ${G'}_k$ has k new conditions on a combination of level subgroup elements, except for the condition of last level parity from the original group.

Proof. Note that the condition (8) is compatible with those which were founded by R. Guralnik in [8], because as it was proven by author [15] $G_{k-1} \simeq B_{k-2} \rtimes \mathcal{W}_{k-1}$, where $B_{k-2} \simeq \underset{i=1}{\overset{k-2}{\wr}} C_2^{(i)}$.

According to Property 1, $G'_k \leq G_k^2$, so it is enough to prove the statement for the elements of G_k^2. Such elements, as it was described above, can be presented in the form $s = (s_{l1}, ..., s_{l2^l})\sigma$, where $\sigma \in G_{l-1}$ and s_{li} are states of $s \in G_k$ in v_{li}, $i \leq 2^l$. For convenience we will make the transition from the tuple $(s_{l1}, ..., s_{l2^l})$ to the tuple $(g_{l1}, ..., g_{l2^l})$. Note that there is the trivial vertex permutation $g_{lj}^2 = e$ in the product of the states $s_{lj} \cdot s_{lj}$.

Since in G'_k v.p. on X^0 are trivial, so σ can be decomposed as $\sigma = (\sigma_{11}, \sigma_{21})$, where σ_{21}, σ_{22} are root permutations in v_{11} and v_{12}.

Consider the square of s. We calculate squares $((s_{l1}, s_{l2}, ..., s_{l2^{l-1}})\,\sigma)^2$. The condition (8) is equivalent to the condition that s^2 has even index on each level. Two cases are feasible: if permutation $\sigma = e$, then $((s_{l1}, s_{l2}, ..., s_{l2^{l-1}})\,\sigma)^2 = \left(s_{l1}^2, s_{l2}^2, ..., s_{l2^{l-1}}^2\right) e$, so after the transition from $\left(s_{l1}^2, s_{l2}^2, ..., s_{l2^{l-1}}^2\right)$ to $\left(g_{l1}^2, g_{l2}^2, ..., g_{l2^{l-1}}^2\right)$, we get a tuple of trivial permutations $(e, \ \dots \ , e)$ on X^l, because $g_{lj}^2 = e$. In the general case, if $\sigma \neq e$, after such transition we obtain $\left(g_{l1}g_{l\sigma(2)}, \ \dots \ , g_{l2^{l-1}}g_{l\sigma(2^{l-1})}\right)\sigma^2$. Consider the product of form

$$\prod_{j=1}^{2^l} \varphi(g_{lj}g_{l\sigma(j)}), \tag{10}$$

where σ and $g_{li}g_{l\sigma(i)}$ are from $\left(g_{l1}g_{l\sigma(2)}, \ \dots \ , g_{l2^{l-1}}g_{l\sigma(2^{l-1})}\right)\sigma^2$.

Note that each element g_{lj} occurs twice in (10) regardless of the permutation σ; therefore, considering the commutativity of homomorphic images $\varphi(g_{lj})$, $1 \leq j \leq 2^l$ we conclude that $\prod_{j=1}^{2^l} \varphi(g_{lj}g_{l\sigma(j)}) = \prod_{j=1}^{2^l} \varphi(g_{lj}^2) = e$, because of $g_{lj}^2 = e$. We rewrite $\prod_{j=1}^{2^l} \varphi(g_{lj}^2) = e$ as characteristic condition: $\prod_{j=1}^{2^{l-1}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+1}^{2^l} \varphi(g_{lj}) = e$.

According to Property 1, any commutator from G'_k can be presented as a product of some squares s^2, $s \in G_k$, $s = ((s_{l1}, ..., s_{l2^l})\sigma)$.

A product of elements of $G_k(k-1)$ satisfies the equation $\prod_{j=1}^{2^l} \varphi(g_{lj}) = e$, because any permutation of elements from X^k, which belongs to G_k is even. Consider the element $s = (s_{k-1,1}, ..., s_{k-1,2^{k-1}})\sigma$, where $(s_{k-1,1}, ..., s_{k-1,2^{k-1}}) \in G_k(k-1)$, $\sigma \in G_{k-1}$. If $g_{01} = (1,2)$, where g_{01} is root permutation of σ, then $s^2 = (s_{k-1,1}s_{k-1\sigma(1)}, ..., s_{k-1,(2^{k-1})}s_{k-1,\sigma(2^{k-1})})$, where $\sigma(j) > 2^{k-1}$ for $j \leq 2^{k-1}$. And if $j < 2^{k-1}$ then $\sigma(j) \geq 2^{k-1}$. Because of $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}) = e$ holds in G_k and the property $\sigma(j) \leq 2^{k-1}$ hold for $j > 2^{k-1}$, then the product $\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1,j}g_{k-1,\sigma(j)})$ of images of v.p. from $(g_{k-1,1}g_{k-1,\sigma(1)}, ..., g_{k-1,(2^{k-1})}g_{k-1,\sigma(2^{k-1})})$ is equal to $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}) = e$. Indeed, the products $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j})$ and $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j}g_{k-1,\sigma(j)})$ have the same v.p. from X^{k-1} which do not depend on such σ as described above.

The same is true for right half of X^{k-1}. Therefore, the equality (9) holds.

Note that such product $\prod_{j=1}^{2^{k-1}} \varphi(g_{k-1,j})$ is homomorphic image of $(g_{l,1}g_{l,\sigma(1)}, ..., g_{l,(2^l)}g_{l\sigma(2^l)})$, where $l = k-1$, as an element of $G'_k(l)$ after mapping (7).

If $g_{01} = e$, where g_{01} is root permutation of σ, then σ can be decomposed as $\sigma = (\sigma_{11}, \sigma_{12})$, where σ_{11}, σ_{12} are root permutations in v_{11} and v_{12}. As a result s^2 has a form

$\Big((s_{l1}s_{l\sigma(1)}, ..., s_{l\sigma(2^{l-1})})\sigma_1^2, (s_{l2^{l-1}+1}s_{l\sigma(2^{l-1}+1)}, ..., s_{l(2^l)}s_{l\sigma(2^l)})\sigma_2^2\Big)$, where $l = k-1$. As a result of action of σ_{11} all states of l-th level with number $1 \leq j \leq 2^{k-2}$ permutes in the set of coordinate from 1 to 2^{k-2}. The others are fixed. The action of σ_{11} is analogous.

It corresponds to the next form of element from $G'_k(l)$: $(g_{l1}g_{l\sigma_1(1)}, ..., g_{l\sigma_1(2^{l-1})})$, $(g_{l2^{l-1}+1}g_{l\sigma_2(2^{l-1}+1)}, ..., g_{l(2^l)}g_{l\sigma_2(2^l)})$. Therefore, the equality of form $\prod_{j=1}^{2^{k-2}} \varphi(g_{k-1,j}g_{l\sigma(j)}) = \prod_{j=2^{k-2}+1}^{2^{k-1}} \varphi(g_{k-1,j}^2) = e$, because of $g_{k-1,j}^2 = e$ holds. Thus, characteristic equation (9) of $k-1$ level holds.

The conditions (8) and (9) for every $s^2, s \in G_k$ hold, so they hold for their product that is equivalent to conditions which hold for every commutator. □

Definition 3. *We define a subdirect product of group G_{k-1} with itself by equipping it with condition (8) and (9) of index parity on all of $k-1$ levels.*

Corollary 9. *The subdirect product $G_{k-1} \boxtimes G_{k-1}$ is defined by $k-2$ outer relations on level subgroups. The order of $G_{k-1} \boxtimes G_{k-1}$ is 2^{2^k-k-2}.*

Proof. We specify a subdirect product for the group $G_{k-1} \boxtimes G_{k-1}$ by using $(k-2)$ conditions for the subgroup levels. Each G_{k-1} has even index on $k-2$-th level; it implies that its relation for $l = k-1$ holds automatically. This occurs because of the conditions of parity for the index of the last level is characteristic of each of the multipliers G_{k-1}. Therefore, It is not an essential condition for determining a subdirect product.

Thus, to specify a subdirect product in the group $G_{k-1} \boxtimes G_{k-1}$, one need only $k-2$ outer conditions on subgroups of levels. Any of such conditions reduces the order of $G_{k-1} \times G_{k-1}$ by two times. Hence, taking into account that the order of G_{k-1} is $2^{2^{k-1}-2}$, we can conclude that the order of $G_{k-1} \boxtimes G_{k-1}$ as a subgroup of $G_{k-1} \times G_{k-1}$ is the following: $|G_{k-1} \boxtimes G_{k-1}| = \left(2^{2^{k-1}-2}\right)^2 : 2^{k-2} = 2^{2^k-4} : 2^{k-2} = 2^{2^k-k-2}$. Thus, we use $k-2$ additional conditions on level subgroup to define the subdirect product $G_{k-1} \boxtimes G_{k-1}$, which contain G'_k as a proper subgroup of G_k, because according to the conditions, which are realized in the commutator of G'_k, (9) and (8) indexes of levels are even. □

Corollary 10. *A commutator G'_k is embedded as a normal subgroup in $G_{k-1} \boxtimes G_{k-1}$.*

Proof. A proof of injective embedding G'_k into $G_{k-1} \boxtimes G_{k-1}$ immediately follows from last item of proof of Corollary 9. The minimality of G'_k as a normal subgroup of G_k and injective embedding G'_k into $G_{k-1} \boxtimes G_{k-1}$ immediately entails that $G'_k \triangleleft G_{k-1} \boxtimes G_{k-1}$. □

Theorem 5. *A commutator subgroup of G_k has form $G'_k = G_{k-1} \boxtimes G_{k-1}$, where the subdirect product is defined by relations (8) and (9). The order of G'_k (the commutator subgroup of $Syl_2 A_{2^k}$) is 2^{2^k-k-2}.*

Proof. Since according to Statement 2 (g_1, g_2) as elements of G'_k also satisfy relations (8) and (9), which define the subdirect product $G_{k-1} \boxtimes G_{k-1}$.

Also $g_1g_2 \in B'_{k-1}$ implies the parity of permutation defined by (g_1, g_2), because B'_{k-1} contains only an element with even index of level [15]. The group G'_k has two disjoint domains of transitivity so G'_k has the structure of a subdirect product of G_{k-1} which acts on this domains transitively. Thus, all elements of G'_k satisfy the conditions (8) and (9) which define subdirect product $G_{k-1} \boxtimes G_{k-1}$. Hence $G'_k < G_{k-1} \boxtimes G_{k-1}$ but G'_k can be equipped by some other relations; therefore, the presence of isomorphism has not yet been proven. For proving revers inclusion we have to show that every element from $G_{k-1} \boxtimes G_{k-1}$ can be expressed as some word $a^{-1}b^{-1}ab$, where $a, b \in G_k$. Therefore, it suffices to show the reverse inclusion. For this goal we use the fact that $G'_k < G_{k-1} \boxtimes G_{k-1}$. Recall that is known [15] that the order of G_k is 2^{2^k-2}.

As it was shown above, G'_k has k new conditions relatively to G_k. Each condition is valid in some level-subgroup. Each of condition reduces an order of the corresponding level subgroup 2 times, so the order of G'_k is 2^k times smaller. On every X^l, $l \leq k-1$, we have an even number of active v.p., by this reason there is the trivial permutation on X^0.

According to the Corollary 9, in the subdirect product $G_{k-1} \boxtimes G_{k-1}$ there are exactly $k-2$ conditions relative to $G_{k-1} \times G_{k-1}$, which are for the subgroups of levels. It has been shown that the relations (8) and (9) are fulfilled in G'_k.

Let α_{lm}, $0 \leq l \leq k-1$, $0 \leq m \leq 2^{l-1}$ be an automorphism from G_k having only one active v.p. in v_{lm}, and let α_{lm} have trivial permutations in rest of the vertices, so we can identify α_{lm} with a vertex permutation g_{lm}. Recall that partial case of notation of form α_{lm} is the generator $\alpha_l := \alpha_{l1}$ of G_k which was defined by us in [15] and denoted by us as α_l. Note that the order of α_{li}, $0 \leq l \leq k-1$ is 2. Thus, $\alpha_{ji} = \alpha_{ji}^{-1}$. We choose a generating set consisting of the following $2k-3$ elements: $(\alpha_{1,1;2}), \alpha_{2,1}, ..., \alpha_{k-1,1}, \alpha_{2,3}, ..., \alpha_{k-1,2^{k-2}+1}$, where $(\alpha_{1,1;2})$ is an automorphism having exactly two active v.p.s in v_{11} and v_{12}. Products of the form $(\alpha_{j1}\alpha_{l1}\alpha_{j1})\alpha_{l1}$ are denoted by P_{lj}. Using a conjugation by generator α_j, $0 \leq j < l$ we can express any v.p. on l-level, because $(\alpha_j\alpha_l\alpha_j) = \alpha_{l2^{l-j-1}+1}$. Defime the product $P_{lj} = (\alpha_j\alpha_l\alpha_j)\alpha_l$. Consider an algorithm of constructing any element of $G_{k-1} \boxtimes G_{k-1}$ as a product of commutators.

1. We need to show that every element of $G_{k-1} \boxtimes G_{k-1}$ satisfying the relations (8), (9) can be constructed as $\alpha^{-1}\beta^{-1}\alpha\beta$, $\alpha, \beta \in G_k$.

   This proves the absence of other relations in G'_k except those that in the subdirect product $G_{k-1} \boxtimes G_{k-1}$. Thereby we prove the embeddedness of G'_k in $G_{k-1} \boxtimes G_{k-1}$. We have to construct an element of form $P_{k-1}P_{k-2} \cdot ... \cdot P_1P_0$ as a product of elements of form $P_l = \prod_{t=1}^{m_l} P_{lj_t}$ satisfying relations (8) and (9). Where $P_{lj} = (\alpha_j\alpha_l\alpha_j)\alpha_l$ is commutator of α_l, α_j.
2. We have to construct an automorphism which has an arbitrary tuple of two active v.p.s satisfying the relations (8) and (9) on X^l as a product of P_{lj} and P_{li}. We use the generator α_l and conjugate by α_j, $j < l$. This corresponds to the tuple of v.p. of the form $(g_{l1}, e, ..., e, g_{lj}, e, ..., e)$, where g_{l1}, g_{lj} are non-trivial. Note that this tuple $(g_{l1}, e, ..., e, g_{li}, e, ..., e)$, which corresponds to P_{li}, is an element of direct product if we consider α_{lj} as an element of S_2 in vertices of X^l. To obtain a tuple of v.p. of form $(e, ..., e, g_{li}, e, ..., e, g_{lj}, e, ..., e) \in G_k(l)$ we simply multiply P_{lj} and $P_{li} \in G_k(l)$.
3. To obtain a tuple T of v.p. with $2m$ active v.p. satisfying the relations (8), (9) we construct $P_l = \prod_{t=1}^{m_l} P_{lj_t}$, $m < 2^l$ for varying $j_t \leq 2^l$, where the values of j_t correspond to the second coordinate of active v.p. from the tuple T, which we have to construct. To construct an arbitrary element h we form a corresponding product $h = \prod_{l=1}^{k} P_l$. On the $(k-1)$-th level, we choose the generator τ to be $\tau = \tau_{k-1,1}\tau_{k-1,\, 2^{k-1}}$, as defined in [15].

Since h satisfies the relations (8) and (9) for all $0 \leq l \leq k$ then $h \in G_{k-1} \boxtimes G_{k-1}$.

On the $(k-1)$-th level, we choose the generator τ which was defined in [15] as $\tau = \tau_{k-1,1}\tau_{k-1,\, 2^{k-1}}$. Recall that it was shown in [15] how to express any τ_{ij} using τ, $\tau_{i,2^{k-2}}$, $\tau_{j,2^{k-2}}$, where $i, j < 2^{k-2}$, in form of a product of commutators $\tau_{ij} = \tau_{i,2^{k-2}}\tau_{j,2^{k-2}} = (\alpha_i^{-1}\tau_{1,2^{k-2}}^{-1}\alpha_i\tau_{j,2^{k-2}})$.

Here $\tau_{i,2^{k-2}}$ was expressed as the commutator $\tau_{i,2^{k-2}} = \alpha_i^{-1}\tau_{1,2^{k-2}}^{-1}\alpha_i\tau_{1,2^{k-2}}$.

Thus, we express all tuples of elements satisfying to relations (8) and (9) by using only commutators of G_k. Thus, we get all tuples of each level subgroup elements satisfying the relations (8) and (9). This means we express every element of each level subgroup by commutators. In particular, to obtain a tuple of v.p. with $2m$ active v.p. on X^{k-2} of $v_{11}X^{[k-1]}$, we will construct the product for τ_{ij} for varying $i, j < 2^{k-2}$.

Thus, all vertex labelings of automorphisms, which appear in the representation of $G_{k-1} \boxtimes G_{k-1}$ by portraits as the subgroup of $AutX^{[k]}$, are also in the representation of G'_k.

Since there are faithful representations of $G_{k-1} \boxtimes G_{k-1}$ and G'_k by portraits of automorphisms from $AutX^{[k]}$, which coincide with each other, subgroup G'_k of $G_{k-1} \boxtimes G_{k-1} \simeq G'_k$ is equal to $G_{k-1} \boxtimes G_{k-1}$ (i.e., $G_{k-1} \boxtimes G_{k-1} = G'_k$). □

The archived results are confirmed by algebraic system GAP calculations. For instance, $|Syl_2 A_8| = 2^6 = 2^{2^3-2}$ and $|(SylA_{2^3})'| = 2^{2^3-3-2} = 8$. The order of G_2 is 4, the number of additional relations in the subdirect product is $k-2 = 3-2 = 1$. We have the same result $(4 \cdot 4) : 2^1 = 8$, which confirms Theorem 5.

Example 1. *Set $k = 4$ then $|(SylA_{16})'| = |(G_4)'| = 1024$, $|G_3| = 64$, since $k-2 = 2$, so according to our theorem above order of $Syl_2 A_{16} \boxtimes Syl_2 A_{16}$ is defined by $2^{k-2} = 2^2$ relations, and by this reason is equal to $(64 \cdot 64) : 4 = 1024$. Thus, orders are coincides.*

Example 2. *The true order of $(Syl_2 A_{32})'$ is $33554432 = 2^{25}$, $k = 5$. A number of additional relations which define the subdirect product is $k-2 = 3$. Thus, according to Theorem 5, $|\ (Syl_2 A_{16} \boxtimes Syl_2 A_{16})'\ | = 2^{14} 2^{14} : 2^{5-2} = 2^{28} : 2^{5-2} = 2^{25}$.*

According to calculations in GAP we have: $Syl_2 A_7 \simeq Syl_2 A_6 \simeq D_4$. Therefore, its derived subgroup $(Syl_2 A_7)' \simeq (Syl_2 A_6)' \simeq (D_4)' = C_2$.

The following structural law for Syllows 2-subgroups is typical. The structures of $Syl_2 A_n$ and $Syl_2 A_k$ are the same if all n and k have the same multiple of two as the multiplier in decomposition on $n!$ and $k!$ Thus, $Syl_2 A_{2k} \simeq Syl_2 A_{2k+1}$.

Example 3. *$Syl_2 A_7 \simeq Syl_2 A_6 \simeq D_4$, $Syl_2 A_{10} \simeq Syl_2 A_{11} \simeq Syl_2 S_8 \simeq (D_4 \times D_4) \rtimes C_2$. $Syl_2 A_{12} \simeq Syl_2 S_8 \boxtimes Syl_2 S_4$, by the same reasons that from the proof of Corollary 9 its commutator subgroup is decomposed as $(Syl_2 A_{12})' \simeq (Syl_2 S_8)' \times (Syl_2 S_4)'$.*

Lemma 7. *In G''_k the following equalities are true:*

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-2}+1}^{2^{l-1}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj}), \quad 2 < l < k. \tag{11}$$

In case $l = k-1$, the following conditions hold:

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{i-1}+1}^{2^{l-1}} \varphi(g_{lj}) = e, \quad \prod_{j=2^{l-1}}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}}^{2^{l}} \varphi(g_{lj}) = e. \tag{12}$$

In other terms, the subgroup G''_k has an even index of any level of $v_{11}X^{[k-2]}$ and of $v_{12}X^{[k-2]}$. The order of G''_k is equal to 2^{2^k-3k+1}.

Proof. As a result of derivation of G'_k, elements of $G''_k(1)$ are trivial. Due the fact that $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$, we can derivate G'_k by commponents. The commutator of G_{k-1} is already investigated in Theorem 5. As $G^2_{k-1} = G'_{k-1}$ by Corollary 7, it is more convenient to present a characteristic equalities in the second commutator $G''_k \simeq G'_{k-1} \boxtimes G'_{k-1}$ as equations in $G^2_{k-1} \boxtimes G^2_{k-1}$. As shown above, for $2 \leq l < k-1$, in G^2_{k-1} the following equalities are true:

$$\prod_{j=1}^{2^{l-1}} \varphi(g_{lj} g_{l\sigma(j)}) = \prod_{j=1}^{2^{l-1}} \varphi(g_{lj}) \prod_{j=1}^{2^{l-1}} \varphi(g_{l\sigma(j)}) = \prod_{j=1}^{2^{l-1}} \varphi(g_{lj}) \prod_{j=1}^{2^{l-1}} \varphi(g_{li}) = \prod_{j=1}^{2^{l-1}} \varphi(g_{lj}^2) = e \tag{13}$$

$$\prod_{j=1}^{2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-2}+1}^{2^{l-1}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj}). \tag{14}$$

The equality (14) holds since it is valid in the initial group $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$. The equalities

$$\prod_{j=2^{l-1}+1}^{2^{l-1}+2^{l-2}} \varphi(g_{lj}) = \prod_{j=2^{l-1}+2^{l-2}+1}^{2^{l}} \varphi(g_{lj})$$

hold for elements of second group G'_{k-1}, since the elements of the original group are endowed with these conditions.

In $(G'_k)^2$ any element g of $G'_k(l)$ satisfies the equality (14). Moreover, g satisfies the previous conditions (11) because of $(G_{k-1}(l))^2 = G'_{k-1}(l)$.

The similar conditions appear in $(G'_{k-1}(k-2))^2$ after squaring of G'_k. Thus, taking into account the characteristic equations of $G'_{k-1}(l)$, the subgroup $(G'_{k-1}(k-2))^2$ satisfies the equality:

$$\prod_{j=1}^{2^{k-3}} \varphi(g_{lj}) = \prod_{j=2^{k-3}+1}^{2^{k-2}} \varphi(g_{lj}) = e, \quad \prod_{j=2^{k-2}+1}^{2^{k-2}+2^{k-3}} \varphi(g_{lj}) = \prod_{j=2^{k-1}+2^{k-2}+1}^{2^{k-1}} \varphi(g_{lj}) = e. \tag{15}$$

Taking into account the structure $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$, we obtain after the derivation $G''_k \simeq (G_{k-2} \boxtimes G_{k-2}) \boxtimes (G_{k-2} \boxtimes G_{k-2})$. With respect to conditions (8) and (9) in the subdirect product, we have that the order of G''_k is $2^{2^k-k-2} : 2^{2k-3} = 2^{2^k-3k+1}$ because on each level $2 \leq l < k$, the order of level subgroup $G''_k(l)$ is 4 times smaller than the order of $G'_k(l)$. On the first level, one new condition arises that reduces the order of $G'_k(1)$ by 2 times. In total, we have $2(k-2)+1 = 2k-3$ new conditions for comparing with G'_k. □

Corollary 11. *Any minimal generating set of $Syl'_2 A_{2^k}$, $k > 2$ consists of $2k-3$ elements.*

Proof. The proof is based on two facts about $G'^2_k G''_k \simeq G'^2_k = G'_k$. More precisely it is based on Corollary 7 and on a calculating of the index $\left|G' : G'^2_k G''_k\right| = 2^{2k-3}$.

To justify that the index $\left|G' : G'^2_k G''_k\right| = 2^{2k-3}$, we take into consideration the orders of these subgroups from Theorem 5 and Lemma 7. Corollary 7 tell us that the subgroup G^2_k is equal to the subgroup G'_k, then the Frattiny subgroup $\Phi(G'_k) = G''_k = {G'_k}^2$. According to Corollary 7 the subgroup G^2_k is equal to the subgroup G'_k, then the Frattiny subgroup $\Phi(G'_k) = G''_k = {G'_k}^2$. Further, for finding the Frattiny factor, which is an elementary abelian 2-group, it is enough sufficient to calculate $\left|G' : G''_k\right|$ because of $\Phi(Syl'_2 A_{2^k}) = Syl''_2(A_{2^k})$. Due to Lemma 7, we have $G''_{k-1} \simeq G'_{k-2} \boxtimes G'_{k-2}$, hence the order of G''_{k-1} is equal to 2^{2^k-3k+1}. Taking into account that G''_k is normal subgroup of G'_k, we compute the order of Frattiny quotient is 2^{2k-3}. Thus, according to Frattiny theorem, a minimal generating set of $Syl'_2 A_{2^k}$ consists of $2k-3$ elements. It is well known [28], the orders of irreducible generating sets for p-group are equal to each other. □

In case $k = 2$ the $Syl_2 A_4 \simeq K_4$, therefore the commutator subgroup is trivial.

Example 4. *The size of (G''_4) is 32. The size of the direct product $(G'_3)^2$ is 64, but, due to relation on second level of G''_k, the direct product $(G'_3)^2$ transforms into the subdirect product $G'_3 \boxtimes G'_3$ that has two times less feasible combination on X^2. The number of additional relations in the subdirect product is $k-3 = 4-3 = 1$. Thus, the order of product is reduced by 2^1 times.*

Example 5. *The commutator subgroup of* $Syl_2'(A_8)$ *consists of elements:* $\{e, (13)(24)(57)(68), (12)(34), (14)(23)(57)(68), (56)(78), (13)(24)(58)(67), (12)(34)(56)(78), (14)(23)(58)(67)\}$. *The commutator* $Syl_2'(A_8) \simeq C_2^3$ *is an elementary abelian 2-group of order 8. This fact confirms our formula* $d(G_k) = 2k-3$, *because* $k = 3$ *and* $d(G_k) = 2k-3 = 3$. *A minimal generating set of* $Syl_2'(A_8)$ *consists of three generators:* $(1,3)(2,4)(5,7)(6,8), (1,2)(3,4), (1,3)(2,4)(5,8)(6,7)$.

Example 6. *The minimal generating set of* $Syl_2'(A_{16})$ *consists of five (that is* $2 \cdot 4 - 3$*) generators:* $(1,4,2,3)(5,6)(9,12)(10,11), (1,4)(2,3)(5,8)(6,7), (1,2)(5,6), (1,7,3,5)(2,8,4,6)(9,14,12,16) \times$
$\times\, (10,13,11,15), (1,7)(2,8)(3,6)(4,5)(9,16,10,15)(11,14,12,13)$.

Example 7. *A minimal generating set of* $Syl_2'(A_{32})$ *consists of seven (that is* $2 \cdot 5 - 3$*) generators:*
$(23,24)(31,32), (1,7)(2,8)(3,5,4,6)(11,12)(25,32)(26,31)(27,29)(28,30)$,
$(3,4)(5,8)(6,7)(13,14)(23,24)(27,28)(29,32)(30,31),\ (7,8)(15,16)(23,24)(31,32)$,
$(1,9,7,15)(2,10,8,16)(3,11,5,13)(4,12,6,14)(17,29,22,27,18,30,21,28)(19,32,23,26,20,31,24,25)$,
$(1,5,2,6)(3,7,4,8)(9,15)(10,16)(11,13)(12,14)(19,20)(21,24,22,23)(29,31)(30,32)$,
$(3,4)(5,8)(6,7)(9,11,10,12)(13,14)(15,16)(17,23,20,22,18,24,19,21)(25,29,27,32,26,30,28,31)$.

This confirms our formula of minimal generating set size $2 \cdot k - 3$.

The minimal generating set for G_4 can be presented in form of wreath recursion:

$$a_1 = (e,e)\sigma,\ b_2 = (a_1, e),\ a_3 = (b_2, e),\ b_4 = (b_3,\ b_3),$$

where $\sigma = (1,2)$. The minimal generating set for G'_4 can be presented in form of wreath recursion:

$$a_2 = (\sigma,\ \sigma),\ a_3 = (e, a_2), a_4 = (a_3, a_3),\ \ b_3 = (e, b_2), b_4 = (b_3, b_3).$$

where σ, a_3, a_4 are generators of the first multiplier G_3 and σ, b_3, b_4 are generators of the second.

5. Conclusions

The size of minimal generating set for commutator of Sylow 2-subgroup of alternating group A_{2^k} was proven to be equal to $2k-3$, where $k > 2$.

A new approach to presentation of Sylow 2-subgroups of alternating group A_{2^k} was applied. As a result, the short proof of a fact that commutator width of Sylow 2-subgroups of the alternating group A_{2^k} ($k > 2$), permutation group S_{2^k} and Sylow p-subgroups of $Syl_2 A_{p^k}$ ($Syl_2 S_{p^k}$) are equal to 1 was obtained. Commutator widths of permutational wreath products $B \wr C_n$ were investigated.

We constructed the minimal generating set of the commutator subgroup of the Sylow 2-subgroup of the alternating group. The inclusion problem [18] for $Syl_2 A_{2^k}$ and its subgroups as $(Syl_2 A_{2^k})'$ and $(Syl_2 A_{2^k})''$ was investigated by us. The relation between solving of the inclusion problem of and conjugacy search problem [19] in this group was established by us.

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