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Stochastic Comparisons of Parallel and Series Systems with Type II Half Logistic-Resilience Scale Components

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Abstract: This paper deals with stochastic comparisons of two parallel (series) systems with Type II half logistic-resilience scale (TIIHL-RS) distribution components with different baseline distribution functions. Under the conditions of interdependency and independency, the research shows that the system performance is better (worse) with the stronger component heterogeneity in the parallel (series) system under the usual stochastic order and the (reversed) hazard rate order.

Keywords: stochastic order; majorization order; Type II half logistic-resilience scale distribution; parallel system; series system

1. Introduction

Stochastic comparisons of system lifetimes is one of the topics in reliability theory and lifetime testing experiments. As a useful tool for measuring the relation of random variables, stochastic order has been widely used in the fields of reliability theory, actuarial science, applied probability statistics, and survival analysis.

Researchers have been dedicated greatly to the results of stochastic comparisons of parallel systems and series systems with exponential distribution components (see, for instance, the works of Balakrishnan and Zhao [1], Zhao and Li [2], Dykstra et al. [3], Joo and Mi [4], and Zhao and Balakrishnan [5], and the references therein for more details). However, the exponential distribution has the special feature of a constant failure rate, and it is not universally researched. Based on this, many scholars have generalized the exponential distribution to the Weibull distribution and gamma distribution (see the works of Fang and Zhang [6], Balakrishnan and Zhao [7], Zhao and Balakrishnan [8], Kochar and Torrado [9], Zhao et al. [10], Torrado and Kochar [11], Zhao et al. [12], and Zhang and Zhao [13], among others). In addition, there are some literature works that study the generalized exponential distribution, exponential Weibull distribution, and exponential generalized gamma distribution. The relevant research results can be found in the works of Balakrishnan [14], Fang and Zhang [15], Kundu et al. [16], Kundu and Chowdhury [17], and Haidari et al. [18], among others. Further, by introducing one or more parameters to a base distribution, the new distribution family has been favored by statisticians, for example adding a scale parameter λ to a base distribution $F(x)(\bar{F}(x))$, which gives the scale distribution family $F(\lambda x)(\bar{F}(\lambda x))$. For the case of the scale distribution family, Khaledi et al. [19] compared a series system with general scaled components. Li and Fang [20] investigated the ordering properties of the extremes of dependent scaled random variables. Further, Hazra et al. [21,22], adding a location parameter to a scale distribution, obtained a location-scaled distribution family and compared extreme order statistics from this distribution family. In addition, a new distribution family was obtained by adding a shape parameter α to the baseline distribution function. When the baseline distribution is a distribution function, the model is a proportional reversed hazard rate (PRHR) model. When the baseline distribution function is a survival function, the model is a PHR model. For the study of PRHR and PHR models, readers can refer to the works of Fang et al. [23] and Li and Fang [24], among others. Based on the scale distribution family, the PRHR distribution family, and the PHR distribution family, Fang et al. [23] constructed the scale-proportional (reversed) hazard rate distribution family, in the case of interdependency, where the dependence is characterized by the Archimedes copula, and they obtained the maximum order and minimum order results by means of the usual stochastic order, as well as the results of the minimum (maximum) order statistic in the sense of the (reversed) hazard rate order in the independent case. Similarly, Zhang et al. [25], by adding a resilience parameter to the scale model, introduced the resilience-scale (RS) distribution family, and they not only showed that the RS components' heterogeneity was directly proportional to the system performance in parallel and that the RS components' heterogeneity was directly inversely proportional to the system performance in the series system under the usual stochastic order, but also investigated two series (parallel) systems consisting of independent components in the sense of the (reversed) hazard rate ordering, the skewness, and the dispersiveness for the lifetimes of two parallel systems with independent heterogeneous and homogeneous components.

Alkaatreh et al. [26] proposed the $T-X$ distribution family, which uses the function $\omega(x)$ to associate the support of the random variable T with the range of the random variable X and $X \sim F(x)$. Let T be a continuous random variable with the density function $h(x)$ defined at $[a,b]$. Then, the distribution of $T-X$ is defined as follows

$$G(x) = \int_a^{\omega(F(x))} h(t)\mathrm{d}t, x \in R. \tag{1}$$

Based on $T-X$ distribution family, many scholars have done corresponding research, when the random variable T in Equation (1) follows the Weibull distribution, and $\omega(\cdot)$ is defined as

$$\omega(F(x)) = \frac{\bar{F}(\gamma x)}{1-\bar{F}(\gamma x)}, \tag{2}$$

which is called the Weibull-generated (Weibull-G) distribution family (see Cooray [27]). Chowdhury et al. [28] studied the property of the minimum of two heterogeneous samples each following the Weibull-G distribution. Similarly, when the random variable T in Equation (1) follows the Kumaraswamy distribution, $\omega(F(x)) = F(x)$, then the $T-X$ distribution family is called the Kumaraswamy-generated (Kumaraswamy-G) distribution family, and Kayakl [29] compared the lifetimes of two series (parallel) systems consisting Kumaraswamy-G components, under the assumption that one shape parameter is the same and the other shape parameter is heterogeneous; by obtaining sufficient conditions for the establishment of the usual stochastic order, the likelihood ratio order, and the dispersion order, when all shape parameters are heterogeneous, he showed by a counterexample that the hazard rate order was not established. Kundu and Chowdhury [30] further studied the maximums of two independent and heterogeneous samples each following the Kumaraswamy-G distribution under random shock. Recently, based on the $T-X$ distribution, Hassan et al. [31] gave the Type II half logistic generated (TIIHL-G) distribution family, the distribution function of which is:

$$H(x,\alpha,\lambda) = 1 - \int_0^{-\log G(x;\lambda)} \frac{2\alpha e^{-\alpha t}}{(1+e^{-\alpha t})^2}\mathrm{d}t = \frac{2G^{\alpha}(x;\lambda)}{1+G^{\alpha}(x;\lambda)},\ x>0, \lambda>0, \tag{3}$$

where the random variable T in Equation (1) follows the half logistic distribution, $\omega(G(x)) = -\log G(x;\lambda)$, $G(x;\lambda)$ is a baseline distribution, which is dependent on a parameter vector λ, and α is a resilience or shape parameter. Hereafter, a random variable X having the distribution function in Equation (3) is denoted by $X \sim$ TIIHL-G, and given the statistical properties of this distribution and some specific models, including the TIIHL-uniform distribution, the TIIHL BurrXII distribution, the TIIHL-Weibull distribution, and the TIIHL-quasi Lindley distribution. When $G(x;\lambda)$ follows the Weibull distribution, Hassan et al. [32] studied the TIIHL-Weibull (W) distribution with applications including mathematical properties, moments, quantile functions, order statistics, Renyi entropy, maximum likelihood estimation, and two data analyses, which showed that the distribution performed better than the beta Weibull distribution, the Mcdonald–Weibull distribution, and exponentiated Weibull distribution. Therefore, in reliability theory, this model is more widely used than E-W distribution and Weibull distribution. This paper aims to study the stochastic comparisons of two parallel or series systems consisting of n heterogeneous TIIHL-G components. Due to the complexity of the distribution, we study the special case of $G(x;\lambda) = G(\lambda x)$, which is a scale distribution family, and we construct the TIIHL-RS distribution family. The cumulative distribution function of TIIHL-RS is given by:

$$H(x,\alpha,\lambda) = \frac{2G^{\alpha}(\lambda x)}{1+G^{\alpha}(\lambda x)},\ x>0, \alpha>0, \lambda>0, \tag{4}$$

where α is a resilience parameter and λ is a scale parameter. Zhang et al. [26] considered the stochastic comparison of the resilience-scaled model $(F^{\alpha}(\lambda x))$ with the same baseline distribution $F(x)$. Let:

$$F(x) = \left[\frac{2G^{\alpha}(x)}{1+G^{\alpha}(x)}\right]^{1/\alpha}, x>0, \alpha>0. \tag{5}$$

Then, $F(x)$ is still a distribution function. Note that,

$$F^{\alpha}(\lambda x) = \frac{2G^{\alpha}(\lambda x)}{1+G^{\alpha}(\lambda x)}.$$

At first glance, it seems to that the TIIHL-RS model is just a simple case of the RS model by Zhang et al. (2018). However, the following examples show otherwise. Example 1 shows that the component with TIIHL-RS model performs better than the component with RS model and Example 2 shows a property of distribution may be not closed in the transformation Equation (5). Thus, the TIIHL-RS model is a different form of Zhang et al. (2018).

Example 1. *Owing to the arbitrariness of the baseline distribution $G(x)$, if $G(x) = F(x)$, let $X \sim F^{\alpha}(\lambda x)$ and $Y \sim \frac{2F^{\alpha}(\lambda x)}{1+F^{\alpha}(\lambda x)}$, then*

$$X \leq_{lr[hr,rh,st]} Y.$$

Example 2. *If baseline distributions $F(x) \neq G(x)$, let $\gamma(x)$ and $\gamma_1(x)$ be the hazard rate of $F(x)$ and $G(x)$, and $X_1 \sim F(x), Y_1 \sim G(x)$. Suppose $G(x) = x, 0 \leq x < 1$, which is Uniform distribution. Then, according to Equation (5), we have $F(x) = \left[\frac{2x^{\alpha}}{1+x^{\alpha}}\right]^{1/\alpha}, 0 \leq x < 1$. When $\alpha = 0.4$, as shown in Figure 1, we find Y_1 is increasing failure rate (IFR), but X_1's hazard rate curve is bathtub shaped.*

In particular, when $\alpha = 1$, $G(\lambda x) = \frac{1-e^{-\lambda x}}{1+3e^{-\lambda x}}$, this distribution tends to Half-logistic distribution. Dolati et al. [33] studied stochastic comparisons including usual stochastic ordering, hazard rate ordering, and the dispersive ordering between the smallest and largest order statistics with Half-logistic components. To the best of our knowledge, few scholars have conducted research on stochastic comparisons problems of TIIHL-RS model. As mentioned in Zhang et al. [26], in this paper, we investigate stochastic comparisons of lifetimes between parallel/series systems consisting of interdependent or independent heterogeneous

TIIHL-RS components with different baseline distributions in the sense of the usual stochastic order and the (reversed) hazard rate order, where the dependence is characterized by Archimedes copula. The TIIHL-RS model includes special cases such as the TIIHL index distribution, the TIIHL-Rayleigh distribution, the TIIHL-Weibull distribution, the TIIHL index Lomax distribution, and the Half-logistic distribution. Further, our results give extensions about Dolati et al. [33] work including usual stochastic ordering, hazard rate ordering.

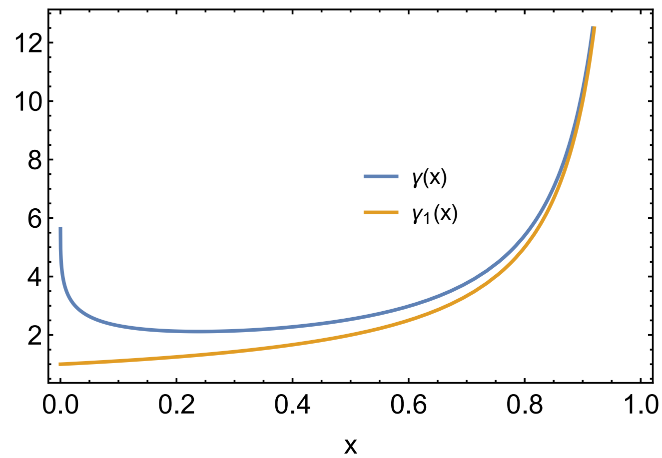

Figure 1. The hazard rate of X and Y.

The rest of the paper proceeds as following. In Table 1, we list a short list of the most important acronyms and notations. In Section 2, we briefly review some pertinent definitions, notations, and useful lemmas that are used throughout the paper. Section 3 presents the main result. In Section 3.1, we gave sufficient conditions to compare two parallel/series systems consisting of interdependent heterogeneous TIIHL-RS components in the sense of the usual stochastic order, and, in Section 3.2, we obtain the sufficient conditions of the (reversed) hazard rate order to compare two parallel (series) systems consisting of independent heterogeneous TIIHL-RS components. Finally, we conclude this paper in Section 4.

Table 1. Acronyms and notions.

cdf	Cumulative distribution function	$G_i(x)$	The baseline distribution, $i = 1,2$
pdf	Probability density function	$\bar{G}_i(x)$	The baseline survival distribution, $i = 1,2$
PHR	Proportional hazard rate model	$\gamma_i(x)$	The hazard rate of baseline distribution, $i = 1,2$
PRHR	Proportional reversed hazard rate model	$\tilde{\gamma}_i(x)$	The reversed hazard rate of baseline distribution, $i = 1,2$
SPHR	Scale-proportional hazard rate model	$h(x)$	$h(x) = \frac{\varphi}{\varphi'} \circ \phi(x) > 0$ is decreasing in x, for $x \geq 0$ and φ is log-convex
SPRHR	Scale-proportional reversed hazard rate model	$\eta_1(x,y)$	$\eta_1(x,y) = \frac{x^y}{1+x^y}$ is increasing in $x \in [0,1]$ and decreasing in $y \in (0,+\infty)$
TIIHL-RS	Type II half logistic resilience scale model		
RS	Resilience-scale model	$\xi_1(x,y)$	$\xi_1(x,y) = \frac{y}{1+x^y}$ is decreasing in $x \in [0,1]$ and increasing in$y \in (0,+\infty)$
TIIHL-G	Type II half logistic generated model		
TIIHL-W	Type II half logistic Weibull model	$\xi_2(x,y)$	$\xi_2(x,y) = \frac{\log x}{1+x^y}$ is increasing in $x \in [0,1]$ and $y \in (0,+\infty)$

2. Preliminaries

Throughout the manuscript, increasing and decreasing mean non-decreasing and non-increasing, respectively. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$. Denote the n-dimensional nonnegative vectors $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $\boldsymbol{y} = \{y_1, \ldots, y_n\}$, $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$, $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_n\}$, $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$, and $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_n\}$. The notion '$a \overset{sgn}{=} b$' means that a and b have the same sign, and $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$, for $i = 1, \ldots, n$. It is also assumed that all concerned random variables are nonnegative and absolutely continuous.

Let us first recall some notions of stochastic order, majorization order, Schur-convex (Schur-concave), and Archimedean copula.

Definition 1. *For two nonnegative random variables X and Y with density functions f and g and distribution functions $F(x)$ and $G(x)$, respectively, and let $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ be the corresponding survival functions, denote the hazard rate and the reversed hazard rate function of $X(Y)$ by $\gamma_X(x) = f(x)/\bar{F}(x)(\gamma_Y(x) = g(x)/\bar{G}(x))$ and $\tilde{\gamma}_X(x) = f(x)/F(x)(\tilde{\gamma}_Y(x) = g(x)/G(x))$, $x \geq 0$, respectively. Then, X is smaller than Y in the:*

(i) usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in [0, \infty)$;
(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in [0, \infty)$ or, equivalently, if $\gamma_X(x) \geq \gamma_Y(x)$; and
(iii) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in $x \in [0, \infty)$ or, equivalently, $\tilde{\gamma}_X(x) \leq \tilde{\gamma}_Y(x)$.

For more details basic properties and application of these orders, please refer to the work of Shake and Shanthikumar [34].

It is well known that the majorization is widely used to establish various stochastic inequalities. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two nonnegative vectors, write $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ and $y_{1:n} < y_{2:n} < \cdots < y_{n:n}$ as the increasing arrangement of the element of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 2. *The $\boldsymbol{x}$ is said to be larger than $\boldsymbol{y}$ in the:*

(i) majorized order (denoted by $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$), if $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n} \quad j = 1, \ldots, n-1$, and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$;
(ii) weakly supermajorized order (denoted by $\boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y}$), if $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n} \quad j = 1, \ldots, n$; and
(iii) weakly submajorized order (denoted by $\boldsymbol{x} \succeq_w \boldsymbol{y}$), if $\sum_{j=i}^{n} x_{j:n} \geq \sum_{j=i}^{n} y_{j:n} \quad i = 1, \ldots, n$.

It is well known that the following implications always hold:

$$\boldsymbol{x} \succeq_w \boldsymbol{y} \Leftarrow \boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y}.$$

Definition 3. *Let ϕ be a real valued function defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$. Then, ϕ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$, if*

$$\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \text{ implies } \phi(\boldsymbol{x}) \geq (\leq) \phi(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}.$$

For more details on the notion and basic properties of majorization related orders, one may refer to Marshall et al. [35]

Now, let us recall the concept of Archimedean copula.

Definition 4 ([36]). *A function $\psi : [0, \infty] \to (0, 1]$ is n-monotone, i.e., $(-1)^k \psi^{(k)}(x) \geq 0$, for $k = 1, 2 \ldots, n-2$, and $(-1)^{n-2}\psi^{(n-2)}(x)$ is decreasing and convex, where $\psi^{(k)}(x)$ is k^{th} derivative of $\psi(\cdot)$, satisfied $\psi(0) = 1$ and $\psi(\infty) = 0$. Then,*

$$C_\psi(u_1, \ldots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)), \quad u_i \in (0,1), i \in 1, 2, \ldots, n,$$

is called an Archimedean copula with the generator ψ, where $\phi = \psi^{-1}$ is the pseudo-inverse of ψ.

For convenience, from now on, we denote

$$\begin{aligned}
I_n^+ &= \{(x_1, \ldots, x_n) : \ 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}, \\
D_n^+ &= \{(x_1, \ldots, x_n) : \ x_1 \geq x_2 \geq \cdots \geq x_n > 0\}, \\
\mathcal{S}_n &= \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{pmatrix} : a_i, b_j > 0, \left(a_i - a_j\right)\left(b_i - b_j\right) \geq 0, i, j = 1, 2, \ldots, n \right\}, \\
\mathcal{U}_n &= \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{pmatrix} : a_i, b_j > 0, \left(a_i - a_j\right)\left(b_i - b_j\right) \leq 0, i, j = 1, 2, \ldots, n \right\}.
\end{aligned}$$

The following three lemmas are utilized to prove the main results.

Lemma 1 ([35]). *Let $\phi : D_n \to \mathbb{R}$ $(I_n \to \mathbb{R})$ is a symmetric and continuously differentiable function. Then, ϕ is Schur-convex (Schur-concave) on D_n (I_n), if and only if*

$$\phi_k(\mathbf{x}) \ \text{ is decreasing (increasing) in } k = 1, \ldots, n, \ \mathbf{x} \in \mathbb{D}_n(\mathbb{I}_n),$$

where $\phi_k(\mathbf{x}) = \partial\phi(\mathbf{x})/\partial x_k$.

Lemma 2 ([16]). *A real valued function ϕ on $\mathbb{R}^n$, satisfies*

$$x \overset{w}{\prec} y \Rightarrow \phi(x) \leq (\geq)\phi(y),$$

if and only if ϕ is decreasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$. Similarly, ϕ satisfies

$$x \prec_w y \Rightarrow \phi(x) \leq (\geq)\phi(y),$$

if and only if ϕ is increasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$.

Lemma 3. *Let the function $\eta_2 : (0,1] \times (0, \infty) \to (0, \infty)$ be defined as $\eta_2(x, y) = \frac{x^y}{1 - x^{2y}}$. Then,*

(i) $\eta_2(x, y)$ is increasing in x and is decreasing in y; and
(ii) $\eta_2(x, y)$ is convex in x, for all $y \geq 1$.

Proof. (i) Differentiating $\eta_2(x, y)$ with respect to x and y, we get

$$\begin{aligned}
\frac{\partial \eta_2(x,y)}{\partial x} &\overset{sgn}{=} yx^{y-1}(1 - x^{2y}) + 2x^{3y-1} = yx^{y-1}(1 + x^{2y}) > 0, \\
\frac{\partial \eta_2(x,y)}{\partial y} &\overset{sgn}{=} x^y(1 - x^{2y})\log x + 2x^{3y}\log x = x^y \log x < 0,
\end{aligned}$$

for all every $0 < x \leq 1,\ y > 0$.

(ii) When $y \geq 1$, we have

$$\frac{\partial \eta_2(x,y)}{\partial x} = \frac{yx^{y-1}(1+x^{2y})}{(1-x^{2y})^2} = yx^{y-1} \cdot f(x,y),$$

where $f(x,y) = \frac{1+x^{2y}}{(1-x^{2y})^2}$, it is easy to get $yx^{y-1} > 0$ is increasing in x, for all $y \geq 1$. Let $v = x^{2y}$; then,

$$f(v) = \frac{1+v}{(1-v)^2}, \ \ v > 0.$$

Differentiating $f(v)$ with respect to v,

$$f'(v) = (1-v)^2 + 2(1-v^2) = 3 - 2v - v^2 \geq 0,$$

since v is increasing in x, hence $\frac{\partial \eta_2(x,y)}{\partial x} > 0$ hold for all $0 < x \leq 1$ and $y \geq 1$. $\square$

3. Stochastic Comparison Results

In this section, we establish comparison results between two parallel (series) systems with TIIHL-RS distribution components with the different baseline distribution functions, respectively, in sense of the usual stochastic ordering and the (reversed) hazard rate ordering. Assume Z_1 and Z_2 are two nonnegative random variables with $Z_k \sim G_k(x)$ for $k = 1,2$. Further, for baseline distribution $G_1(x)$, let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a set of dependent TIIHL-RS distributed random vector with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda_i),\ i = 1,2,\ldots,n$, and an Archimedean copula having generator ψ_1. We denote $\boldsymbol{X} \sim$ TIIHL-RS$(G_1(x), \boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1)$. Similarly, for another baseline distribution $G_2(x)$, let $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be a set of dependent TIIHL-RS distributed random vector with $Y_i \sim$ TIIHL-RS$(G_2(x), \beta_i, \mu_i),\ i = 1,2,\ldots,n$, and an Archimedean copula having generator ψ_2. We denote $\boldsymbol{Y} \sim$ TIIHL-RS$(G_2(x), \boldsymbol{\beta}, \boldsymbol{\mu}, \psi_2)$. Suppose that $\bar{F}_{X_{n:n}}(x)$ and $\bar{F}_{Y_{n:n}}(x)$ be the reliability distributions of $X_{n:n}$ and $Y_{n:n}$, respectively, and $F_{X_{1:n}}(x)$ and $F_{Y_{1:n}}(x)$ be the distribution functions of $X_{1:n}$ and $Y_{1:n}$, respectively. Then,

$$\begin{aligned}
\bar{F}_{X_{n:n}}(x) &= \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x)) = 1 - \psi_1\left(\sum_{i=1}^{n} \phi_1(\frac{2{G_1}^{\alpha_i}(\lambda_i x)}{1+{G_1}^{\alpha_i}(\lambda_i x)})\right), \\
\bar{F}_{Y_{n:n}}(x) &= \Phi_2(\boldsymbol{\beta}, \boldsymbol{\mu}, \psi_2, G_2(x)) = 1 - \psi_2\left(\sum_{i=1}^{n} \phi_2(\frac{2{G_2}^{\beta_i}(\mu_i x)}{1+{G_2}^{\beta_i}(\mu_i x)})\right),
\end{aligned}$$

and

$$\begin{aligned}
F_{X_{1:n}}(x) &= \Psi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x)) = 1 - \psi_1\left(\sum_{i=1}^{n} \phi_1(\frac{1-{G_1}^{\alpha_i}(\lambda_i x)}{1+{G_1}^{\alpha_i}(\lambda_i x)})\right), \\
F_{Y_{1:n}}(x) &= \Psi_2(\boldsymbol{\beta}, \boldsymbol{\mu}, \psi_2, G_2(x)) = 1 - \psi_2\left(\sum_{i=1}^{n} \phi_2(\frac{1-{G_2}^{\beta_i}(\mu_i x)}{1+{G_2}^{\beta_i}(\mu_i x)})\right).
\end{aligned}$$

3.1. Heterogeneous Interdependent Case

In this subsection, we compare parallel and series systems each consisting of heterogeneous interdependent TIIHL-RS components, which have the different baseline distributions, respectively, with respect to usual stochastic order. Firstly, in the following theorem, we suppose the systems consisting of components with same resilience parameters as well as different scale parameters.

Theorem 1. *Let X_i and Y_i be two sets of dependent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda_i, \psi_1)$ and $Y_i \sim$ TIIHL-RS$(G_2(x), \alpha_i, \mu_i, \psi_2)$, respectively. Suppose $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{U}_n$, $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{S}_n$, $\phi_1 \circ \psi_2$ is super-additive and ψ_1 or ψ_2 is log-convex. Then,*

(1) $Z_1 \geq_{st} Z_2$, $\tilde{\gamma}(x)$ *is decreasing in* $x \in \mathbb{R}_+$, $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$*; and*
(2) $Z_1 \leq_{st} Z_2$, $\boldsymbol{\lambda} \geq \boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$.

Proof. (1) According to Lemma 1 of Fang et al. [23], $\phi_1 \circ \psi_2$ is super-additive, then

$$\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_1, G_2(x)) \geq \Phi_2(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_2, G_2(x)).$$

Since ψ_1 and ϕ_1 are decreasing, $\eta_1(G_1(\mu_i x), \alpha)$ is increasing in $G_1(\mu_i x)$ and $Z_1 \geq_{st} Z_2$; then, we have

$$\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_1, G_1(x)) \geq \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_1, G_2(x)).$$

Thus, we need only to prove

$$\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x)) \geq \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_1, G_1(x)).$$

In view of Lemma 1 and Lemma 2, we just need to prove that $\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))$ is decreasing and Schur-convex in λ_i, for all $\boldsymbol{\alpha} \in D_n^+$. Now,

$$\frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \lambda_k} = -\psi_1' \left(\sum_{i=1}^{n} \phi_1 \Big(\frac{2{G_1}^{\alpha_i}(\lambda_i x)}{1 + {G_1}^{\alpha_i}(\lambda_i x)} \Big) \right) \frac{\psi_1 \circ \phi_1}{\psi_1' \circ \phi_1} \left(\frac{2{G_1}^{\alpha_k}(\lambda_k x)}{1 + {G_1}^{\alpha_k}(\lambda_k x)} \right) \frac{x \alpha_k \tilde{\gamma}_1(\lambda_k x)}{1 + {G_1}^{\alpha_k}(\lambda_k x)} \leq 0.$$

We just need to show that $\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))$ is Schur-convex in $\boldsymbol{\lambda}$. After simplifications, we get

$$\begin{aligned} & \frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \lambda_k} - \frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \lambda_{k+1}} \\ = \; & -x\psi_1' \left(\sum_{i=1}^{n} \phi_1 \Big(\frac{2{G_1}^{\alpha_i}(\lambda_i x)}{1 + {G_1}^{\alpha_i}(\lambda_i x)} \Big) \right) \\ & \times \{ h[2\eta_1(G_1(\lambda_k x), \alpha_k)] \xi_1 [G_1(\lambda_k x), \alpha_k] \tilde{\gamma}(\lambda_k x) \\ & - h[2\eta_1(G_1(\lambda_{k+1} x), \alpha_{k+1})] \xi_1 [G_1(\lambda_{k+1} x), \alpha_{k+1}] \tilde{\gamma}(\lambda_{k+1} x) \}. \end{aligned}$$

By the monotonicity of the composite function, we find that $h[2\eta_1(G_1(\lambda x), \alpha)] \xi_1 [G_1(\lambda x), \alpha] \tilde{\gamma}_1(\lambda x)$ is decreasing in α and increasing in λ. Since $\boldsymbol{\lambda} \in I_n^+, \boldsymbol{\alpha} \in D_n^+$, i.e., $\lambda_k < \lambda_{k+1}, \alpha_k > \alpha_{k+1}$, we have

$$\frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \lambda_k} - \frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \lambda_{k+1}} \leq 0.$$

(2) Note that $\phi_1 \circ \psi_2$ is super-additive, ψ_1 and ϕ_1 are decreasing, $1 - \eta_1[G_1(\lambda_i x), \alpha]$ is decreasing in $G_1(\lambda_i x)$ and $Z_1 \leq_{st} Z_2$; then, we have

$$\Psi_1(\alpha, \boldsymbol{\lambda}, \psi_1, G_1(x)) \geq \Psi_1(\alpha, \boldsymbol{\lambda}, \psi_1, G_2(x)) \geq \Psi_2(\alpha, \boldsymbol{\lambda}, \psi_2, G_2(x)),$$

and

$$\Psi_2(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_2, G_2(x)) \geq \Psi_2(\boldsymbol{\alpha}, \boldsymbol{\mu}, \psi_2, G_2(x)).$$

Thus, we complete the proof. □

The following theorem compared two parallel systems consisting of heterogeneous interdependent components with different resilience parameters as well as same scale parameters in sense of usual stochastic order.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be a set of interdependent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda_i, \psi_1)$, and $Y_1, Y_2, \ldots, Y_n$ be another set of interdependent random variables with $Y_i \sim$ TIIHL-RS$(G_2(x), \beta_i, \lambda_i, \psi_2)$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{U}_n$, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{S}_n$, $\phi_1 \circ \psi_2$ is super-additive, and ψ_1 or ψ_2 is log-convex. Then,*

$$Z_1 \geq_{st} Z_2,\ \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

Proof. Assume that $\boldsymbol{\lambda} \in I_n^+, \boldsymbol{\alpha} \in D_n^+, \boldsymbol{\beta} \in D_n^+$, ψ_1 is log-convex. According Lemma 1 and Lemma 2, similar to the proof of Theorem 1(1), for all $\boldsymbol{\lambda} \in I_n^+$, we only prove $\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))$ is increasing and Schur-convex in α_i. The increasing property of $\Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))$ is obviously true. By using Lemma 1, Lemma 2, and $h[2\eta_1(G_1(\lambda x), \alpha)]\xi_1[G_1(\lambda x), \alpha]$ is increasing in α and decreasing λ, we obtain

$$\begin{aligned}
\frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \alpha_k} - \frac{\partial \Phi_1(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \psi_1, G_1(x))}{\partial \alpha_{k+1}} \quad = \quad & -\psi_1' \left(\sum_{i=1}^{n} \phi_1\Big(\frac{2{G_1}^{\alpha_i}(\lambda_i x)}{1 + {G_1}^{\alpha_i}(\lambda_i x)}\Big) \right) \\
& \times \{ h[2\eta_1(G_1(\lambda_k x), \alpha_k)]\xi_2[G_1(\lambda_k x), \alpha_k] \\
& - h[2\eta_1(G_1(\lambda_{k+1} x), \alpha_{k+1})]\xi_2[G_1(\lambda_{k+1} x), \alpha_{k+1}] \} \\
& \geq 0.
\end{aligned}$$

The prove of $\boldsymbol{\lambda} \in D_n^+, \boldsymbol{\alpha} \in I_n^+, \boldsymbol{\beta} \in I_n^+$ is similar to that of the $\boldsymbol{\lambda} \in I_n^+, \boldsymbol{\alpha} \in D_n^+, \boldsymbol{\beta} \in D_n^+$, and hence omitted for the sake of conciseness. □

The following theorem compared two series systems consisting of heterogeneous interdependent components with different resilience parameters as well as complete same scale parameters in the sense of usual stochastic order.

Theorem 3. *Let $X_1, X_2, \ldots, X_n$ be a set of interdependent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda, \psi_1)$, and $Y_1, Y_2, \ldots, Y_n$ be another set of interdependent random variables with $Y_i \sim$ TIIHL-RS$(G_2(x), \beta_i, \lambda, \psi_2)$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{S}_n$, $\phi_1 \circ \psi_2$ is super-additive, and ψ_1 or ψ_2 is log-convex. Then,*

$$Z_1 \leq_{st} Z_2,\ \boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}.$$

Proof. Assume that $\boldsymbol{\alpha} \in D_n^+$, according to Lemma 1 and Lemma 2, we just need to prove that, for all $k \in \{0, 1, \ldots, n\}$, $\Psi_1(\boldsymbol{\alpha}, \lambda, \psi_1, G_1(x))$ is decreasing and Schur-convex in α_k. From Lemma 3, it can be shown that $h[1 - \eta_1(G_1(\lambda x), \alpha)]\eta_2[G_1(\lambda x), \alpha]$ is increasing in α, we have

$$\begin{aligned}\frac{\partial \Psi_1(\boldsymbol{\alpha}, \lambda, \psi_1, G_1(x))}{\partial \alpha_k} - \frac{\partial \Psi_1(\boldsymbol{\alpha}, \lambda, \psi_1, G_1(x))}{\partial \alpha_{k+1}} &= 2\{\log G_1(\lambda x)\}\psi_1'\left(\sum_{i=1}^{n} \phi_1(\frac{1 - {G_1}^{\alpha_i}(\lambda x)}{1 + {G_1}^{\alpha_i}(\lambda x)})\right) \\ &\quad \times\{h[1 - \eta_1(G_1(\lambda x), \alpha_k)]\eta_2[G_1(\lambda x), \alpha_k] \\ &\quad -h[1 - \eta_1(G_1(\lambda x), \alpha_{k+1})]\eta_2[G_1(\lambda x), \alpha_{k+1}]\} \\ &\geq 0.\end{aligned}$$

By using $Z_1 \leq_{st} Z_2$, $\phi_1 \circ \psi_2$ is super-additive, and ψ_1 or ψ_2 is log-convex, it is easy to get

$$\Psi_1(\boldsymbol{\alpha}, \lambda, \psi_1, G_1(x)) \geq \Psi_1(\boldsymbol{\alpha}, \lambda, \psi_2, G_2(x)).$$

Then, we have

$$\Psi_1(\boldsymbol{\alpha}, \lambda, \psi_1, G_1(x)) \geq \Psi_2(\boldsymbol{\beta}, \lambda, \psi_2, G_2(x)).$$

The prove of $\boldsymbol{\alpha} \in I_n^+, \boldsymbol{\beta} \in I_n^+$ is similar to that of the $\boldsymbol{\lambda} \in I_n^+, \boldsymbol{\alpha} \in D_n^+, \boldsymbol{\beta} \in D_n^+$, and hence omitted for the sake of conciseness. □

Combining Theorem 1(1) and Theorem 2, we obtain more general result as following.

Theorem 4. *Let $X_1, X_2, \ldots, X_n$ be a set of interdependent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda_i, \psi_1)$, and $Y_1, Y_2, \ldots, Y_n$ be another set of interdependent random variables with $Y_i \sim$ TIIHL-RS$(G_2(x), \beta_i, \mu_i, \psi_2)$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{U}_n$, $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{S}_n$, $\tilde{\gamma}(x)$ is decreasing in $x \in \mathbb{R}_+$, $\phi_1 \circ \psi_2$ is super-additive, and ψ_1 or ψ_2 is log-convex. Then,*

$$Z_1 \geq_{st} Z_2,\ \boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu},\ \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

Combining Theorem 1(2) and Theorem 3, we obtain more general result as following.

Theorem 5. *Let $X_1, X_2, \ldots, X_n$ be a set of interdependent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha_i, \lambda, \psi_1)$, and $Y_1, Y_2, \ldots, Y_n$ be another set of interdependent random variables with $Y_i \sim$ TIIHL-RS$(G_2(x), \beta_i, \mu, \psi_2)$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{S}_n$, $\phi_1 \circ \psi_2$ is super-additive, and ψ_1 or ψ_2 is log-convex. Then,*

$$Z_1 \leq_{st} Z_2,\ \lambda \geq \mu, \boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}.$$

The following example shows that the condition of $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{U}_n$, $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{S}_n$, could not be relaxed in Theorem 4.

Example 3. *Consider two series systems consisting of two dependent components; we choose the Clayton–Oakes copula, i.e.,*

$$\psi(x) = (\theta x + 1)^{-1/\theta}, 0 < \theta < \infty.$$

Let $G_1(x) = 1 - e^{-x}$, $G_2(x) = 1 - e^{-2x}$, obvious $G_1(x) \leq G_2(x)$, suppose that $\theta_1 = 0.2$, $\theta_2 = 0.4$, $(\lambda_1, \lambda_2) = (0.8, 0.4)$, $(\mu_1, \mu_2) = (0.6, 0.8)$, $(\alpha_1, \alpha_2) = (0.8, 0.2)$, $(\beta_1, \beta_2) = (0.4, 0.5)$, it is easy to see $(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \in \mathcal{S}_n$, $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in$

$\mathcal{S}_n$. Figure 2 plots the reliability functions of $X_{2:2}$ and $Y_{2:2}$, which implies that the usual stochastic ordering does not hold between $X_{2:2}$ and $Y_{2:2}$.

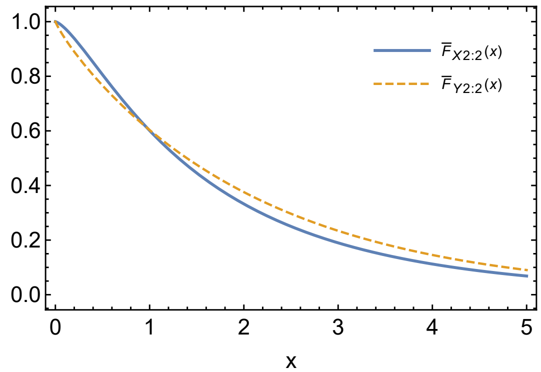

Figure 2. Plot of reliability functions of $X_{2:2}(x)$ and $Y_{2:2}(x)$.

Remark 1. *About the existence of the condition of φ or φ is log-convex and $\phi_1 \circ \psi_2$ is super-additive, one can refer to Zhang et al. [25].*

3.2. Heterogeneous independent case

In this subsection, we investigate the order properties of the lifetime of parallel and series systems with independent TIIHL-RS components.

Theorem 6. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are two sets independent random variables with $X_i \sim$ TIIHL-RS$(G_1(x), \alpha, \lambda_i, \psi_0)$, and $Y_i \sim$ TIIHL-RS$(G_2(x), \alpha, \mu_i, \psi_0)$, for all $i = 1, 2, \ldots, n$. Suppose $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{S}_n$, then*

(1) if $\alpha \le 1$, $x\tilde{\gamma}_1(x)$ or $x\tilde{\gamma}_2(x)$ is decreasing and convex in $x \in \mathbb{R}_+$, $Z_1 \ge_{rh} Z_2$, $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu} \Rightarrow X_{n:n} \ge_{rh} Y_{n:n}$.
(2) if $\alpha \ge 1$, $x\gamma_1(x)$ or $x\gamma_2(x)$ is increasing and convex in $x \in \mathbb{R}_+$, $Z_1 \le_{hr} Z_2$, $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow X_{1:n} \le_{hr} Y_{1:n}$.

Proof. (1) For $x > 0$, the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$ by:

$$\tilde{\gamma}_{X_{n:n}}(x) = \sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_1(\lambda_i x)}{1 + G_1{}^{\alpha}(\lambda_i x)}, \quad \tilde{\gamma}_{Y_{n:n}}(x) = \sum_{i=1}^{n} \frac{\alpha\mu_i\tilde{\gamma}_2(\mu_i x)}{1 + G_2{}^{\alpha}(\mu_i x)}.$$

Without loss of generality, we suppose that $\boldsymbol{\lambda} \in D_n^+$, according to stochastic order theory and Lemma 1 and Lemma 2, it only need to prove that

$$0 \ge \frac{\partial\tilde{\gamma}_{X_{n:n}}}{\partial\lambda_k} \ge \frac{\partial\tilde{\gamma}_{X_{n:n}}}{\partial\lambda_{k+1}}. \tag{6}$$

Let $\eta_3(x) = \frac{1}{1+x^a}$, $\eta_4(x) = x\tilde{\gamma}_1(x)$; note that, for all $a \le 1$, $\eta_3(x)$ is decreasing and convex in x. Thus, we have

$$\frac{\partial\tilde{\gamma}_{X_{n:n}}}{\partial\lambda_k} = \frac{\alpha}{x}\frac{\partial\eta_3(G_1(\lambda_k x))}{\partial\lambda_k x}\eta_4(\lambda_k x) + \frac{\alpha}{x}\eta_3(G_1(\lambda_k x))\eta_4'(\lambda_k x) \le 0.$$

Next, we need to show that $\tilde{\gamma}_{X_{n:n}}(x)$ is Schur-convex function, that is $\frac{\partial \tilde{\gamma}_{X_{n:n}}}{\partial \lambda_k} \geq \frac{\partial \tilde{\gamma}_{X_{n:n}}}{\partial \lambda_{k+1}}$.

$$\begin{aligned}\frac{\partial \tilde{\gamma}_{X_{n:n}}}{\partial \lambda_k} - \frac{\partial \tilde{\gamma}_{X_{n:n}}}{\partial \lambda_{k+1}} &= \frac{\alpha}{x}\{\frac{\partial \eta_3[G_1(\lambda_k x)]}{\partial \lambda_k x}\eta_4(\lambda_k x) + \eta_3[G_1(\lambda_k x)]\eta_4'(\lambda_k x) \\ &\quad - \frac{\partial \eta_3[G_1(\lambda_{k+1} x)]}{\partial \lambda_{k+1} x}\eta_4(\lambda_{k+1} x) + \eta_3[G_1(\lambda_{k+1} x)]\eta_4'(\lambda_{k+1} x)\} \\ &\geq 0.\end{aligned}$$

Thus, $\tilde{\gamma}_{X_{n:n}}(x)$ is a Schur-convex function, which shows that Equation (6) holds. Then,

$$\sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_1(\lambda_i x)}{1 + {G_1}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{\alpha\mu_i\tilde{\gamma}_1(\mu_i x)}{1 + {G_1}^{\alpha}(\mu_i x)}. \tag{7}$$

Now, $Z_1 \geq_{rh} Z_2$ implies $\tilde{\gamma}_1(x) \geq \tilde{\gamma}_2(x)$ and $G_1(x) \leq G_2(x)$, which yields

$$\sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_1(\lambda_i x)}{1 + {G_1}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_2(\lambda_i x)}{1 + {G_2}^{\alpha}(\lambda_i x)}.$$

Based on Equation (7), we have

$$\sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_1(\lambda_i x)}{1 + {G_1}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{\alpha\lambda_i\tilde{\gamma}_2(\lambda_i x)}{1 + {G_2}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{\alpha\mu_i\tilde{\gamma}_2(\mu_i x)}{1 + {G_2}^{\alpha}(\mu_i x)}.$$

(2) For $x > 0$, the hazard rate functions of $X_{1:n}$ and $Y_{1:n}$ by:

$$\gamma_{X_{1:n}}(x) = \sum_{i=1}^{n} \frac{2\alpha {G_1}^{\alpha}(\lambda_i x)\lambda_i\gamma_1(\lambda_i x)}{1 - 2{G_1}^{\alpha}(\lambda_i x)}, \quad \gamma_{Y_{1:n}}(x) = \sum_{i=1}^{n} \frac{2\alpha {G_2}^{\alpha}(\mu_i x)\mu_i\gamma_2(\mu_i x)}{1 - 2{G_2}^{\alpha}(\mu_i x)}.$$

Let

$$H(\alpha, \boldsymbol{\lambda}, x) = \frac{2\alpha}{x}\sum_{i=1}^{n} \frac{{G_1}^{\alpha}(\lambda_i x)\lambda_i x\gamma_1(\lambda_i x)}{1 - 2{G_1}^{\alpha}(\lambda_i x)}.$$

Without loss of generality, it is assumed that $\boldsymbol{\lambda} \in D_n^+$, then $\boldsymbol{\lambda} x \in D_n^+$. According to Lemma 1 and Lemma 2, it suffices to prove that $H(\alpha, \boldsymbol{\lambda}, x)$ is increasing and Schur-convex in $\boldsymbol{\lambda} x$. Taking the derivative of $H(\alpha, \boldsymbol{\lambda}, x)$ with respect to $\lambda_i x$, we have

$$\frac{\partial H(\alpha, \boldsymbol{\lambda}, x)}{\partial \lambda_k x} = \frac{2\alpha}{x}\left\{\frac{\partial \eta_2[G_1(\lambda_k x)]}{\partial \lambda_k x}\eta(\lambda_k x) + \eta_2[G_1(\lambda_k x)]\eta'(\lambda_k x)\right\},$$

where $\eta(x) = x\gamma_1(x)$; from Lemma 3, it is easy to see that, for all $\alpha \geq 1$, $\eta_2(x)$ is increasing and convex in x. Hence, $\frac{\partial H(\alpha,\boldsymbol{\lambda},x)}{\partial \lambda_k x} \geq 0$, and

$$\begin{aligned}\frac{\partial H(\alpha, \boldsymbol{\lambda}, x)}{\partial \lambda_k x} - \frac{\partial H(\alpha, \boldsymbol{\lambda}, x)}{\partial \lambda_{k+1} x} &= \frac{2\alpha}{x}\left\{\frac{\partial \eta_2[G_1(\lambda_k x)]}{\partial \lambda_k x}\eta(\lambda_k x) + \eta_2[G_1(\lambda_k x)]\eta'(\lambda_k x)\right\} \\ &\quad - \frac{2\alpha}{x}\left\{\frac{\partial \eta_2[G_1(\lambda_{k+1} x)]}{\partial \lambda_{k+1} x}\eta(\lambda_{k+1} x) + \eta_2[G_1(\lambda_{k+1} x)]\eta'(\lambda_{k+1} x)\right\}.\end{aligned}$$

Denote $\Lambda(x) = \frac{\partial \eta_2[G_1(x)]}{\partial x}\eta(x) + \eta_2[G_1(x)]\eta'(x)$, since $\eta_2(x),\ \eta(x)$ is increasing and convex functions, we have $\Lambda(x)$ is increasing in x, thus

$$\frac{\partial H(\alpha, \boldsymbol{\lambda}, x)}{\partial \lambda_k x} - \frac{\partial H(\alpha, \boldsymbol{\lambda}, x)}{\partial \lambda_{k+1} x} = \frac{2\alpha}{x}\left\{\Lambda(\lambda_k x) - \Lambda(\lambda_{k+1} x)\right\} \geq 0, \quad \text{for all } \boldsymbol{\lambda} x \in D_n^+.$$

Thus,

$$\sum_{i=1}^{n} \frac{2\alpha {G_1}^{\alpha}(\lambda_i x)\lambda_i \gamma_1(\lambda_i x)}{1 - 2{G_1}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{2\alpha {G_1}^{\alpha}(\mu_i x)\mu_i \gamma_1(\mu_i x)}{1 - 2{G_1}^{\alpha}(\mu_i x)}. \tag{8}$$

Similar to the first part proof, when $Z_1 \leq_{hr} Z_2$, according to Equation (8), we have

$$\sum_{i=1}^{n} \frac{2\alpha {G_1}^{\alpha}(\lambda_i x)\lambda_i \gamma_1(\lambda_i x)}{1 - 2{G_1}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{2\alpha {G_2}^{\alpha}(\lambda_i x)\lambda_i \gamma_2(\lambda_i x)}{1 - 2{G_2}^{\alpha}(\lambda_i x)} \geq \sum_{i=1}^{n} \frac{2\alpha {G_2}^{\alpha}(\mu_i x)\mu_i \gamma_2(\mu_i x)}{1 - 2{G_2}^{\alpha}(\mu_i x)}.$$

The proof is finished. □

A natural question is that whether the assumptions $\alpha \leq 1$ can be altered to $\alpha > 1$ in Theorem 6(1). The next example provides a negative answer.

Example 4. *Consider two series systems consisting of two independent components. Let* $G_1(x) = 1 - e^{-x}$, $G_2(x) = 1 - e^{-2x}$, *then* $Z_1 \geq_{rh} Z_2$. *Suppose that* $(\lambda_1, \lambda_2) = (0.4, 0.3)$, $(\mu_1, \mu_2) = (0.6, 0.5)$, $\alpha = 2$. *It is obvious that,* $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu}$; *then, according to Figure 3, the reversed hazard rate ordering of* $X_{2:2}$ *and* $Y_{2:2}$ *does not hold.*

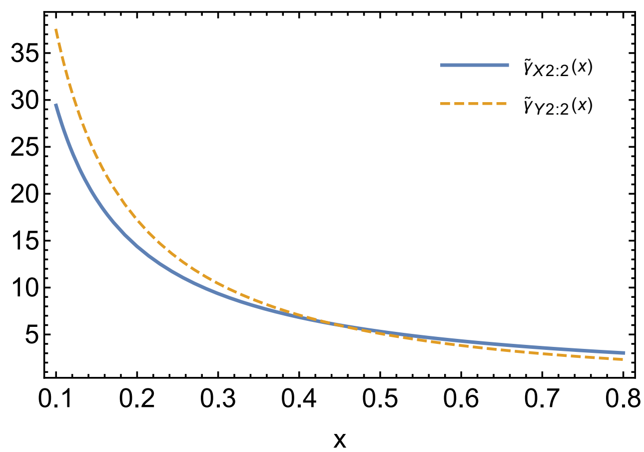

Figure 3. The reversed hazard rate functions of $X_{2:2}$ and $Y_{2:2}$.

Remark 2. *In particular, when* $\alpha = 1$, $G(\lambda x) = \frac{1-e^{-\lambda x}}{1+3e^{-\lambda x}}$, *the results in Theorem 4 and Theorem 5 generalize that of Proposition 1 in Dolati et al. [33] and Theorem 6(1) is a generalization of Proposition 4 (1) in Dolati et al. [33].*

Ordering results between order statistics are helpful in reliability. Here, we introduce numerical applications to illustrate the theoretical results. Selecting a row of subsystems with better performance from two sets of subsystems used to assemble a system is a key issue for reliability engineers.

Example 5. *Consider two systems* M *and* N, *composed of five types of subsystems, each of which has dependent (independent) heterogeneous TIIHL-W distribution. Assume system* M *is assembled with subsystems* $M_1, M_2, \ldots, M_5$, *and system* N *is assembled with subsystems* $N_1, N_2, \ldots, N_5$, *denote* X_i *and* Y_i *the lifetimes of* M_i *and* N_i *and* $X_i \sim$ *TIIHL-W*$(G_1, \alpha_i, \lambda_i, \psi)$ *and* $Y_i \sim$ *TIIHL-W*$(G_2, \alpha_i, \mu_i, \psi)$, $(i = 1, 2, \ldots, 5)$, *respectively. Setting*

(1) $\boldsymbol{\alpha} = (0.2, 0.3, 0.4, 0.5, 0.8)$, $\boldsymbol{\lambda} = (0.7, 0.6, 0.5, 0.2, 0.2)$, $\boldsymbol{\beta} = (0.1, 0.2, 0.4, 0.5, 0.6)$, $\boldsymbol{\mu} = (1.2, 0.8, 0.5, 0.4, 0.3)$. *Clearly,* $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu}$, $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$, *let* $G_1(x) = 1 - e^{-x}$, $G_2(x) = 1 - e^{-2x}$, $h(a, b) = 1/(1 - e^{-ax})^b$, *it is obvious that* $h(a, b)$ *is decreasing in a and increasing in b, and take* $\psi = (3x + 1)^{-1/3}$ *is the Clayton copula generator. We have,*

$$\begin{aligned} \bar{F}_{X_{5:5}} &= 1 - \psi\Big(\sum_{i=1}^{5} \phi\Big(\frac{2(1 - e^{-\lambda_i x})^{\alpha_i}}{1 + (1 - e^{-\lambda_i x})^{\alpha_i}}\Big)\Big) \\ &= 1 - \frac{2}{[h(0.7, 0.4) + 2h(0.7, 0.2) + h(0.6, 0.6) + 2h(0.6, 0.3) + h(0.5, 0.8)]^{1/2}} \\ &\quad - \frac{2}{[2h(0.5, 0.4) + h(0.2, 1.6) + h(0.2, 1) + 2h(0.2, 0.8) + 2h(0.2, 0.5) - 11]^{1/2}}, \end{aligned}$$

$$\begin{aligned} \bar{F}_{Y_{5:5}} &= 1 - \psi\Big(\sum_{i=1}^{5} \phi\Big(\frac{2(1 - e^{-2\mu_i x})^{\beta_i}}{1 + (1 - e^{-2\mu_i x})^{\beta_i}}\Big)\Big) \\ &= 1 - \frac{2}{[h(2.4, 0.2) + 2h(2.4, 0.1) + h(1.6, 0.4) + 2h(1.6, 0.2) + h(1, 0.8)]^{1/2}} \\ &\quad - \frac{2}{[2h(1, 0.4) + h(0.8, 1) + 2h(0.8, 0.5) + h(0.6, 1.2) + 2h(0.6, 0.6) - 11]^{1/2}}. \end{aligned}$$

Then,

$$\begin{aligned} \bar{F}_{X_{5:5}} - \bar{F}_{Y_{5:5}} &\overset{sgn}{=} [h(0.7, 0.4) - h(2.4, 0.2)] + [h(0.7, 0.2) - h(2.4, 0.1)] + [h(0.6, 0.6) - h(1.6, 0.4)] \\ &\quad + [h(0.6, 0.3) - h(1.6, 0.2)] + [h(0.5, 0.8) - h(1, 0.8)] + [h(0.5, 0.4) - h(1, 0.4)] \\ &\quad + [h(0.2, 1.6) - h(0.6, 1.2)] + [h(0.2, 1) - h(0.8, 1)] + [h(0.2, 0.8) - h(0.6, 0.6)] \\ &\quad + [h(0.2, 0.5) - h(0.8, 0.5)] \\ &\geq 0. \end{aligned}$$

where the inequality is derived from condition $h(a, b)$ *is decreasing in a and increasing in b. Thus, the conclusion of Theorem 4 holds, and we get* $X_{5:5} \geq_{st} Y_{5:5}$. *This means that M systems is better than N system in sense of usual stochastic order.*

(2) $\boldsymbol{\alpha} = (0.5, 0.6, 0.3, 0.4, 0.1)$, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda = 0.6$, $\boldsymbol{\beta} = (0.8, 0.7, 0.4, 0.3, 0.2)$, $\mu_1 = \mu_2 = \cdots = \mu_n = \mu = 0.4$. *Clearly,* $\lambda > \mu$, $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$, *let*$G_1(x) = 1 - e^{-2x}$, $G_2(x) = 1 - e^{-x}$, $p(a, b) = 4(1 - e^{-ax})^b/(1 - (1 - e^{-ax})^b)^2$, *it is obvious that* $p(a, b)$ *is increasing in a and decreasing in b, and take* $\psi = (3x + 1)^{-1/3}$ *is the Clayton copula generator. We have*

$$\begin{aligned} F_{X_{1:5}} &= 1 - \psi\Big(\sum_{i=1}^{5} \phi\Big(\frac{1 - (1 - e^{-\lambda x})^{\alpha_i}}{1 + (1 - e^{-\lambda x})^{\alpha_i}}\Big)\Big) \\ &= 1 - \frac{1}{[1 + p(1.2, 0.1) + p(1.2, 0.3) + p(1.2, 0.4) + p(1.2, 0.5) + p(1.2, 0.6)]^{1/2}}, \end{aligned}$$

$$\begin{aligned} F_{Y_{1:5}} &= 1 - \psi\Big(\sum_{i=1}^{5} \phi\Big(\frac{1 - (1 - e^{-2\mu x})^{\beta_i}}{1 + (1 - e^{-2\mu x})^{\beta_i}}\Big)\Big) \\ &= 1 - \frac{1}{[1 + p(0.8, 0.2) + p(0.8, 0.3) + p(0.8, 0.4) + p(0.8, 0.7) + p(0.8, 0.8)]^{1/2}}. \end{aligned}$$

Then,

$$\begin{aligned} F_{X_{1:5}} - F_{Y_{1:5}} &\overset{sgn}{=} [p(1.2, 0.1) - p(0.8, 0.2)] + [p(1.2, 0.3) - p(0.8, 0.3)] + [p(1.2, 0.4) - p(0.8, 0.4)] \\ &\quad + [p(1.2, 0.5) - p(0.8, 0.7)] + [p(1.2, 0.6) - p(0.8, 0.8)] \\ &\geq 0. \end{aligned}$$

where the inequality is derived from condition $p(a, b)$ is decreasing in a and increasing in b. Thus, the conclusion of Theorem 5 holds, and we get $X_{1:5} \leq_{st} Y_{1:5}$. This means that N systems is better than M system in sense of usual stochastic order.

(3) $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = 0.6$, $\boldsymbol{\lambda} = (0.7, 0.6, 0.5, 0.2, 0.2)$, $\boldsymbol{\mu} = (1.2, 0.8, 0.5, 0.4, 0.3)$. Clearly, $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu}$, let$G_1(x) = 1 - e^{-x}$, $G_2(x) = 1 - e^{-2x}$, $q_1(a) = a/(e^{ax} - 1)$, $q_2(a) = [1 + (1 - e^{-ax})^{0.6}]^{-1}$, and, $q(a) = q_1(a) \cdot q_2(a)$, it is obvious that $q(a, b)$ is decreasing in a. We have

$$\begin{aligned} \tilde{\gamma}_{X_{5:5}}(x) &= \sum_{i=1}^{n} \frac{\alpha \lambda_i \tilde{\gamma}_1(\lambda_i x)}{1 + {G_1}^{\alpha}(\lambda_i x)} \\ &= q(0.7) + q(0.6) + q(0.5) + q(0.2) + q(0.2), \end{aligned}$$

$$\begin{aligned} \tilde{\gamma}_{Y_{5:5}}(x) &= \sum_{i=1}^{n} \frac{\alpha \mu_i \tilde{\gamma}_2(\mu_i x)}{1 + {G_2}^{\alpha}(\mu_i x)} \\ &= q(2.4) + q(1.6) + q(1) + q(0.8) + q(0.6). \end{aligned}$$

Then,

$$\begin{aligned} \tilde{\gamma}_{X_{5:5}}(x) - \tilde{\gamma}_{Y_{5:5}}(x) &\overset{sgn}{=} [q(0.7) - q(2.4)] + [q(0.6) - q(1.6)] + [q(0.5) - q(1)] \\ &\quad + [q(0.2) - q(0.8)] + [q(0.2) - q(0.6)] \\ &\geq 0. \end{aligned}$$

where the inequality is derived from condition $q(a)$ is decreasing in a. Thus, the conclusion of Theorem 6(1) holds, and we get $X_{5:5} \geq_{rh} Y_{5:5}$. This means that M systems is better than N system in sense of reversed hazard rate order.

(4) $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = 1.6$, $\boldsymbol{\lambda} = (0.2, 0.3, 0.4, 0.5, 0.8)$, $\boldsymbol{\mu} = (0.7, 0.6, 0.5, 0.2, 0.2)$. Clearly, $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}$, let $G_1(x) = 1 - e^{-2x}$, $G_2(x) = 1 - e^{-x}$, $R(a) = [a(1 - e^{-ax})^{1.6}]/[1 - 2(1 - e^{-ax})^{1.6}]$, it is obvious that $R(a)$ is increasing in a. We have

$$\begin{aligned} \gamma_{X_{1:5}}(x) &= \sum_{i=1}^{n} \frac{2\alpha {G_1}^{\alpha}(\lambda_i x) \lambda_i \gamma_1(\lambda_i x)}{1 - 2{G_1}^{\alpha}(\lambda_i x)} \\ &= R(1.6) + R(1) + R(0.8) + R(0.6) + R(0.4), \end{aligned}$$

$$\begin{aligned} \gamma_{Y_{1:5}}(x) &= \sum_{i=1}^{n} \frac{2\alpha {G_2}^{\alpha}(\mu_i x) \mu_i \gamma_2(\mu_i x)}{1 - 2{G_2}^{\alpha}(\mu_i x)} \\ &= R(0.7) + R(0.6) + R(0.5) + R(0.2) + R(0.2). \end{aligned}$$

Then,

$$\begin{aligned}\gamma_{X_{1:5}}(x) - \gamma_{Y_{1:5}}(x) &\overset{sgn}{=} [R(1.6) - R(0.7)] + [R(1) - R(0.6)] + [R(0.8) - R(0.5)] \\ &\quad +[R(0.6) - R(0.2)] + [R(0.4) - R(0.2)] \\ &\geq 0.\end{aligned}$$

where the inequality is derived from condition $R(a)$ is increasing in a. Thus, the conclusion of Theorem 6(2) holds, and we get $X_{1:5} \leq_{hr} Y_{1:5}$. This means that N systems is better than M system in sense of hazard rate order.

4. Conclusions and Application

In this paper, we obtain the relevant results of the stochastic comparisons for the lifetimes of two series (parallel) systems consisting of heterogeneous TIIHL-RS components, in the sense of the usual stochastic order and the (reversed) hazard rate order, respectively. We show that the higher is the heterogeneity between the components of the system, the higher is the reliability of the parallel system, and the lower is the reliability of the series system. Our results provide a unified method for studying some special distributions, such as the TIIHL-exponential distribution, the TIIHL-Rayleigh distribution, the TIIHL-Weibull distribution, the TIIHL-exponential Lomax distribution, etc. In future research work, we will further obtain stronger order, such as likelihood ratio, and study the skewness and dispersion of system lifetimes, which is also of interest to the ordering properties of system lifetimes under random shock.

In Equation (4), let $G(x) = 1 - e^{-x^{\theta}}, \theta > 0, x > 0$, then

$$H(x, \alpha, \lambda) = \frac{2[1 - e^{-(\lambda x)^{\theta}}]^{\alpha}}{1 + [1 - e^{-(\lambda x)^{\theta}}]^{\alpha}},\ x > 0, \alpha > 0, \lambda > 0, \theta > 0.$$

The TIIHL-Weibull distribution was successfully applied to life analysis and reliability analysis by Hassan et al. [31] and Hassan et al. [32], and we believe that its application is not limited to this.

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