More Results on Italian Domination in $C_n \square C_m$

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Abstract: Italian domination can be described such that in an empire all cities/vertices should be stationed with at most two troops. Every city having no troops must be adjacent to at least two cities with one troop or at least one city with two troops. In such an assignment, the minimum number of troops is the Italian domination number of the empire/graph is denoted as γ_I . Determining the Italian domination number of a graph is a very popular topic. Li et al. obtained $\gamma_I(C_n \square C_3)$ and $\gamma_I(C_n \square C_4)$ (weak {2}-domination number of Cartesian products of cycles, J. Comb. Optim. 35 (2018): 75–85). Stepień et al. obtained $\gamma_I(C_n \square C_5) = 2n$ (2-Rainbow domination number of $C_n \square C_5$, Discret. Appl. Math. 170 (2014): 113–116). In this paper, we study the Italian domination number of the Cartesian products of cycles $C_n \square C_m$ for $m \geq 6$. For $n \equiv 0 \pmod{3}$, $m \equiv 0 \pmod{3}$, we obtain $\gamma_I(C_n \square C_m) = \frac{mn}{3}$. For other $C_n \square C_m$, we present a bound of $\gamma_I(C_n \square C_m)$. Since for $n = 6k$, $m = 3l$ or $n = 3k$, $m = 6l$ ($k, l \geq 1$), $\gamma_{r2}(C_n \square C_m) = \frac{mn}{3}$, (the Cartesian product of cycles with small 2-rainbow domination number, J. Comb. Optim. 30 (2015): 668–674), it follows in this case that $C_n \square C_m$ is an example of a graph class for which $\gamma_I = \gamma_{r2}$, which can partially answer the question presented by Brešar et al. on the 2-rainbow domination in graphs, Discret. Appl. Math. 155 (2007): 2394–2400.

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1. Introduction

Let $G = (V, E)$ be a finite connected simple graph. For every vertex $v \in V$, the open neighborhood of v , denoted by $N(v)$, is a set $\{u | (u, v) \in E(G)\}$. The degree of $v \in V$ is $\deg(v) = |N(v)|$, the maximum/minimum degree of G is denoted $\Delta(G)/\delta(G)$.

For two graphs G and H , the Cartesian product of them is the graph denoted $G \square H$, where $v_{i,j} \in V(G \square H)$ if and only if $v_i \in V(G)$ and $v_j \in V(H)$, and $(v_{i_1, j_1}, v_{i_2, j_2}) \in E(G \square H)$ if and only if $i_1 = i_2$ and $(j_1, j_2) \in E(H)$ or $j_1 = j_2$ and $(i_1, i_2) \in E(G)$.

A set $D \subseteq V(G)$ is a domination set if for each v of G either $v \in D$ or v is adjacent to some $u \in D$. The minimum cardinality of dominating sets of G is the domination number, denoted by $\gamma(G)$. Domination on graphs originates practical problems in Operations Research, so it has been extensively studied [1,2] and it has many variants, such as total domination [3], super domination [4], Roman domination [5], rainbow domination [6] and so on.

The problem of Roman domination can be described as a defense issue. It is said that in the Roman Empire all cities should be stationed at most two troops. Every city having no troop must be adjacent to at least one city in which two troops are stationed, so that one of the two troops could go to protect the city. The mathematical description is as the follows.

A Roman dominating function (RDF) on $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ if every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$.

Rainbow domination represents the situation in which there are k types of guards, it is required that each location/vertex that is not occupied with a guard has in its neighborhood all types of guards. The following is the definition of 2-rainbow domination.

A 2-rainbow dominating function (2RDF) on G is a function $f : V \rightarrow \mathcal{P}\{1, 2\}$ if each vertex $v \in V$ with $f(v) = \emptyset$ satisfies $\cup_{u \in N(v)} f(u) = \{1, 2\}$. The weight of f is $w(f) = \sum_{v \in V} |f(v)|$ and the minimum weight of 2RDFs on G is the 2-rainbow domination number of G , denoted by $\gamma_{r2}(G)$.

Brešar et al. [6] introduced a monochromatic version of 2-rainbow dominating functions, called weak 2-dominating function (W2DFs), when they studied 2-rainbow domination in trees. Chellali et al. [7] initiated the study of a variant of Roman dominating functions, call Roman $\{2\}$ -dominating functions, which is the same as W2DFs. Henning et al. [8] named the new concept Italian domination. The definition of the new domination is the following.

An Italian dominating function (IDF) on $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ if every vertex v with $f(v) = 0$ holds $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an IDF is the value $w(f) = \sum_{v \in V} f(v)$. The minimum weight of IDFs on G is called the Italian domination number, denoted by $\gamma_I(G)$.

Since Italian domination is a monochromatic version of 2-rainbow domination, the relationship between the two parameters receives more attentions. Chellali et al. [7] show $\gamma_I(G)$ is bounded above by $\gamma_{r2}(G)$ for a graph G . Brešar et al. [9] present a question “For which classes of graphs is $\gamma_{r2} = \gamma_I$ for every graph G of a class?”, and prove that for trees $\gamma_I(T) = \gamma_{r2}(T)$ [6]. Stępień et al. [10] show $\gamma_I(C_n \square C_5) = \gamma_{r2}(C_n \square C_5)$. To determine the Italian domination number of a graph also attracts more scholars. Li et al. [11] give the value of $\gamma_I(C_n \square C_3)$ and $\gamma_I(C_n \square C_4)$. Stępień et al. [10] obtain $\gamma_I(C_n \square C_5) = 2n$. Gao et al. [12] determine the exact values of $\gamma_I(C_n \square P_3)$ and $\gamma_I(C_3 \square P_m)$. They also present the bound of $\gamma_I(C_n \square P_m)$ for $n, m \geq 4$. There are many studies on some domination numbers related to Italian domination, such as global Italian domination number [13], independent Italian domination number [14], outer-independent Italian domination number [15] and perfect Italian domination number [16].

In this paper, we study the Italian domination number of the Cartesian products of cycles $C_n \square C_m$ for $m \geq 6$. First, we describe the algorithm based on the method of branch and bound. Then, with the algorithm we construct some good enough recursive Italian dominating functions. According to the characteristic of the neighborhood in $C_n \square C_m$, and in order to get the upper bound as small as possible, we set different mod numbers. Thus, we obtain some upper bounds of $\gamma_I(C_n \square C_m)$. Together with the lower bound of $\gamma_I(C_n \square C_m)$ provided by Chellali et al. [7], we obtain $\gamma_I(C_n \square C_m) = \frac{mn}{3}$ for $n \equiv 0 \pmod{3}$, $m \equiv 0 \pmod{3}$, and get a bound of $\gamma_I(C_n \square C_m)$ for other $C_n \square C_m$. Since for $n = 6k$, $m = 3l$ or $n = 3k$, $m = 6l$ ($k, l \geq 1$), $\gamma_{r2}(C_n \square C_m) = \frac{mn}{3}$ [17], it follows in this case $C_n \square C_m$ is an example of a graph class for which $\gamma_I = \gamma_{r2}$. Thus, we give another example of a graph class for which $\gamma_I = \gamma_{r2}$ besides $C_n \square C_5$ [10] and trees [6], partially answer the question of Brešar et al. [9].

Here are some existing results related to this paper.

For $n \geq 3$ [11],

$$\gamma_I(C_n \square C_3) = \begin{cases} n, & n \equiv 0 \pmod{3} \\ n + 1, & n \equiv 1, 2 \pmod{3} \end{cases}$$

For $n \geq 4$ [11],

$$\gamma_I(C_n \square C_4) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil, & n \equiv 0, 1, 3, 4, 5 \pmod{8} \\ \left\lceil \frac{3n}{2} \right\rceil + 1, & n \equiv 2, 6, 7 \pmod{8} \end{cases}$$

For $n \geq 5$ [10],

$$\gamma_I(C_n \square C_5) = 2n.$$

Theorem 1 ([17]). $\gamma_{r2}(C_n \square C_m) = \frac{mn}{3}$ if and only if $n = 6k, m = 3l$ or $n = 3k, m = 6l, k, l \geq 1$.

In this paper, we study the Italian domination number of $C_n \square C_m$ for $m \geq 6$. Before presenting our results, we need to give a lower bound on Italian domination numbers of graphs due to Chellali et al. [7], which is useful to our study.

Theorem 2 ([7]). If G is a connected graph with maximum degree $\Delta(G)$, then

$$\gamma_I(G) \geq \frac{2|V(G)|}{\Delta(G) + 2}.$$

In $G = C_n \square C_m$, $\Delta(G) = 4$, $|V(G)| = mn$, thus $\gamma_I(C_n \square C_m) \geq \lceil \frac{mn}{3} \rceil$.

2. An Algorithm for Italian Domination in $C_n \square C_m$

Let $G = C_n \square C_m$ with $V(G) = \{v_{i,j} | 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$ and $E(G) = \{e_{i,j} | e_{i,j} = (v_{i,j}, v_{i+1,j}), 0 \leq i \leq n-1, 0 \leq j \leq m-1\} \cup \{e'_{i,j} | e'_{i,j} = (v_{i,j}, v_{i,j+1}), 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$, where indices i and j are read modulo m and n respectively, then the graph is shown with Figure 1.

Figure 1. Graph $C_n \square C_m$.

To determine the Italian domination number is NP-complete [7], and the branch and bound method is the most well-known for solving NP-hard problems. However, branch and bound conditions are very different for different graphs. According to the characteristics of $C_n \square C_m$ and its neighborhoods, we develop valid branch and bound conditions and set the following rule.

Rule: In $C_n \square C_m$, $0 \leq i \leq n-1, 0 \leq j \leq m-1$, nn_1 and mm_1 are the number of rows and columns of the loop body, nn and mm are the number of rows and columns of the part which can be looped, then $(nn_1 \leq i \leq nn) \wedge (mm_1 \leq j \leq mm) \wedge (f(v_{i,j}) \neq f(v_{i-nn_1,j}) \vee f(v_{i,j}) \neq f(v_{i,j-mm_1}))$.

Based on above rule, we design an algorithm for Italian domination on $C_n \square C_m$.

3. Italian Domination Functions on $C_n \square C_m$

According to Algorithm 1, we construct some Italian dominating functions, and upon these functions we present upper bounds on the Italian domination number of $C_n \square C_m$.

Algorithm 1 Algorithm for finding IDFs of $C_n \square C_m$.

Set a threshold for Italian domination number T_{min} ;
 Initialization: current position $l = 0$, the consumed domination cost $p_1 = 0$;
 $l_0 : l++ ; jj = -1$;
 $l_1 : jj++ = 1 ; jj > 2$; return l_2 ;
 For current vertex, check whether it satisfies the following conditions:
if $(p_1 + s[l]) \geq T_{min}$, **then return** l_1 ; $s[l]$ is the value on the l^{th} vertex, $s[l] = 0, 1, 2$.
if current vertex cannot be dominated, **then return** l_1 ;
if it meets the Rule, **then return** l_1 ;
 $p_1 = p_1 + s[l]$; $jj_0[l] = jj$; the current consumed domination cost is stored in $jj_0[l]$.
if $l \leq n_1$, **then return** l_0 ; n_1 is the number of vertex.
if $p_1 < T_{min}$, **then** $T_{min} = p_1$;
for $(i = 1 ; i \leq n_1 ; i++)$
 $f[i] = jj_0[i]$; f is the dominating function.
Endfor
 $l++$;
 $l_2 : l--$;
if $l \geq 1$, **then** $jj = jj_0[l]$; $j = f[l]$; $p_1 = p_1 - j$; **return** l_1 ; **end if**
Output: T_{min} and the dominating function $f[1] \cdot \dots \cdot f[n_1]$.

Lemma 1. For $m \equiv 0(\text{mod } 3)$,

$$\gamma_1(C_n \square C_m) \leq \begin{cases} \frac{mn}{3}, & n \equiv 0(\text{mod } 3), \\ \frac{mn+m}{3}, & n \equiv 1, 2(\text{mod } 3). \end{cases}$$

Proof. We first construct an IDF g on $C_3 \square C_3$.

$$g(v_{i,j}) = \begin{cases} 1, & i = 0, j = 2 \vee i = 1, j = 1 \vee i = 2, j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $n \equiv 0(\text{mod } 3)$, we construct a recursive IDF f on $C_n \square C_m$: $f(v_{i,j}) = g(v_{i \bmod 3, j \bmod 3})$.

Figure 2 shows f on $C_6 \square C_6$, where red vertices stand for vertices of $f(v) = 1$, and blue vertices stand for vertices of $f(v) = 0$. From the construction of f , together with Figure 2, one can check f is an IDF and can calculate the weight $w(f) = 3 \times \frac{m}{3} \times \frac{n}{3} = \frac{mn}{3}$.

For $n \equiv 1, 2(\text{mod } 3)$, we construct IDF f on $C_n \square C_m$ as follows,

$$f(v_{i,j}) = \begin{cases} g(v_{i \bmod 3, j \bmod 3}), & 0 \leq i \leq n-2, \\ 1, & n \equiv 1(\text{mod } 3) \wedge i = n-1 \wedge j \equiv 0, 2(\text{mod } 3) \\ & \vee n \equiv 2(\text{mod } 3) \wedge i = n-1 \wedge j \equiv 0, 1(\text{mod } 3), \\ 0, & \text{otherwise.} \end{cases}$$

Figure 2 shows f on $C_7 \square C_6$ and $C_8 \square C_6$ and $w(f) = 3 \times \frac{m}{3} \times \frac{n}{3} + \frac{m}{3} = \frac{mn+m}{3}$. $\square$

Figure 2. f on $C_n \square C_m$ for $m \equiv 0(\text{mod } 3)$. Above the last horizontal dashed line, the block separated by the dashed lines is $C_3 \square C_3$ under g .

Lemma 2. For $m \equiv 1(\text{mod } 3)$, let $r = (n - 1) \text{ mod } 2m$, then

$$\gamma_I(C_n \square C_m) \leq \begin{cases} \frac{2mn+n}{6}, & n \equiv 0(\text{mod } 2m), \\ \frac{2mn+n+r+1}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 0(\text{mod } 2) \wedge \frac{r+1}{2} \equiv 0(\text{mod } 3), \\ \frac{2mn+n+2m-r-1}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 0(\text{mod } 2) \wedge \frac{r+1}{2} \equiv 1(\text{mod } 3), \\ \frac{2mn+n+2m-2}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 0(\text{mod } 2) \wedge \frac{r+1}{2} \equiv 2(\text{mod } 3), \\ \frac{2mn+n+2m+1}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 1(\text{mod } 2) \wedge \frac{r}{2} \equiv 0(\text{mod } 3), \\ \frac{2mn+n+r+1}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 1(\text{mod } 2) \wedge \frac{r}{2} \equiv 1(\text{mod } 3), \\ \frac{2mn+n+2m-r-1}{6}, & n \not\equiv 0(\text{mod } 2m) \wedge n \equiv 1(\text{mod } 2) \wedge \frac{r}{2} \equiv 2(\text{mod } 3), \end{cases}$$

i.e.,

$$\gamma_I(C_n \square C_m) \leq \frac{2mn + n + 2m + 1}{6}.$$

Proof. We first construct an IDF g on $C_{2m} \square C_m$,

$$g(v_{i,j}) = \begin{cases} 1, & i \text{ mod } 2 = 0 \wedge (j \leq \frac{i}{2} \wedge j \text{ mod } 3 = \frac{i}{2} \text{ mod } 3 \vee j \geq \frac{i+2}{2} \wedge j \text{ mod } 3 = \frac{i+2}{2} \text{ mod } 3) \\ \vee i \text{ mod } 2 = 1 \wedge (j \leq \frac{i+1}{2} \wedge j \text{ mod } 3 = \frac{i+3}{2} \text{ mod } 3 \vee j \geq \frac{i+3}{2} \wedge j \text{ mod } 3 = \frac{i-1}{2} \text{ mod } 3), \\ 0, & \text{otherwise.} \end{cases}$$

For $n \equiv 0(\text{mod } 2m)$, we construct an IDF f on $C_n \square C_m$: $f(v_{i,j}) = g(v_{i \text{ mod } 2m, j})$.

Figure 3 shows f on $C_{14} \square C_7$. The weight $w(f) = (\frac{m+2}{3} + \frac{m-1}{3}) \times \frac{n}{2} = \frac{2mn+n}{6}$.

For $n \not\equiv 0(\text{mod } 2m)$, we construct f on $C_n \square C_m$ as follows,

for $0 \leq i \leq n - 2$, $f(v_{i,j}) = g(v_{i \text{ mod } 2m, j})$;

for $i = n - 1$,

$$f(v_{i,j}) = \begin{cases} g(v_{i \text{ mod } 2m, j}), & \text{if } g(v_{i \text{ mod } 2m, j}) = 1, \\ 1, & \text{else if } \frac{r+1}{2} \equiv 0(\text{mod } 3) \wedge (j \leq \frac{r+1}{2} \wedge j \equiv 2(\text{mod } 3)) \\ & \vee \frac{r+1}{2} \equiv 1(\text{mod } 3) \wedge (j \geq \frac{r+1}{2} \wedge j \equiv 2(\text{mod } 3)) \\ & \vee \frac{r+1}{2} \equiv 2(\text{mod } 3) \wedge j \equiv 2(\text{mod } 3) \\ & \vee \frac{r}{2} \equiv 0(\text{mod } 3) \wedge j \equiv 2(\text{mod } 3) \\ & \vee \frac{r}{2} \equiv 1(\text{mod } 3) \wedge (j \leq \frac{r}{2} \wedge j \equiv 2(\text{mod } 3)) \\ & \vee \frac{r}{2} \equiv 2(\text{mod } 3) \wedge (j \geq \frac{r}{2} \wedge j \equiv 2(\text{mod } 3)), \\ 0, & \text{else.} \end{cases}$$

Figure 3 shows f on $C_{15} \square C_7$. In order to calculate the weight of f , let $V_i = \{v \in V(G) | f(v) = i\}$, $|V_i| = n_i$ for $i = 0, 1, 2$ and (V_{11}, V_{12}) be the order partition of V_1 induced by f , where

$$V_{11} = \{v_{i,j} | g(v_{i \bmod 2m,j}) = 1\}, V_{12} = V_1 - V_{11}.$$

Then we calculate the weight of f .

Case 1. $n \equiv 0 \pmod{2}$, $|V_{11}| = (\frac{m+2}{3} + \frac{m-1}{3}) \times \frac{n}{2} = \frac{2mn+n}{6}$, $w(f) = |V_{11}| + |V_{12}|$.

(1) $\frac{r+1}{2} \equiv 0 \pmod{3}$, $|V_{12}| = \frac{r+1}{2} \times \frac{1}{3} = \frac{r+1}{6}$, $w(f) = \frac{2mn+n+r+1}{6}$.

(2) $\frac{r+1}{2} \equiv 1 \pmod{3}$, $|V_{12}| = \frac{1}{3} \times (m - \frac{r+1}{2}) = \frac{2m-r-1}{6}$, $w(f) = \frac{2mn+n+2m-r-1}{6}$.

(3) $\frac{r+1}{2} \equiv 2 \pmod{3}$, $|V_{12}| = (\frac{r+1}{2} + 1) \times \frac{1}{3} + (m - \frac{r+1}{2} - 2) \times \frac{1}{3} = \frac{m-1}{3}$, $w(f) = \frac{2mn+n+2m-2}{6}$.

Figure 3. f on $C_n \square C_m$ for $m \equiv 1 \pmod{3}$.

Figure 4 shows four lines with f on $C_n \square C_{22}$ for $n \equiv 0 \pmod{2}$ in above three subcases, where blue vertices stand for $f(v) = 0$, red vertices stand for $f(v) = 1$ and $v \in V_{11}$ and pink vertices stand for $f(v) = 1$ and $v \in V_{12}$.

Figure 4. f on $C_n \square C_{22}$ for $n \equiv 0 \pmod{2}$.

Case 2. $n \equiv 1 \pmod{2}$, $|V_{11}| = (\frac{m+2}{3} + \frac{m-1}{3}) \times \frac{n-1}{2} + \frac{m+2}{3} = \frac{2mn+n+3}{6}$.

- (1) $\frac{r}{2} \equiv 0 \pmod{3}$, $|V_{12}| = \frac{r}{2} \times \frac{1}{3} + (m - \frac{r}{2} - 1) \times \frac{1}{3} = \frac{m-1}{3}$, $w(f) = \frac{2mn+n+2m+1}{6}$.
- (2) $\frac{r}{2} \equiv 1 \pmod{3}$, $|V_{12}| = (\frac{r}{2} - 1) \times \frac{1}{3} = \frac{r-2}{6}$, $w(f) = \frac{2mn+n+r+1}{6}$.
- (3) $\frac{r}{2} \equiv 2 \pmod{3}$, $|V_{12}| = (m - \frac{r}{2} - 2) \times \frac{1}{3} = \frac{2m-r-4}{6}$, $w(f) = \frac{2mn+n+2m-r-1}{6}$.

Figure 5 shows four lines with f on $C_n \square C_{22}$ for $n \equiv 1 \pmod{2}$ in the three subcases.

Figure 5. f on $C_n \square C_{22}$ for $n \equiv 1 \pmod{2}$.

□

Lemma 3. For $m \equiv 2 \pmod{3}$, $\gamma_I(C_n \square C_m) \leq \frac{mn+n}{3}$.

Proof. For $n \equiv 0 \pmod{3}$, we first construct an IDF g on $C_3 \square C_3$.

$$g(v_{i,j}) = \begin{cases} 1, & i = 0, j = 2 \vee i = 1, j = 1 \vee i = 2, j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we construct an IDF f on $C_n \square C_m$ as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i \bmod 3, j \bmod 3}), & 0 \leq j \leq m-3, \\ 1, & j = m-2 \wedge i \equiv 2 \pmod{3} \\ & \vee j = m-1 \wedge i \equiv 0, 1 \pmod{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 6 shows f on $C_9 \square C_8$. Then, $w(f) = 3 \times \frac{m-2}{3} \times \frac{n}{3} + 3 \times \frac{n}{3} = \frac{mn+n}{3}$. For $n \equiv 1 \pmod{3}$,

$$f(v_{i,j}) = \begin{cases} 1, & 0 \leq i \leq \lfloor \frac{m}{2} \rfloor \wedge (j \leq 2i \wedge j \bmod 3 = (2i+1) \bmod 3 \vee j \geq 2i \wedge j \bmod 3 = 2i \bmod 3) \\ & \vee \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq n-1 \wedge (j \leq m-2 \wedge j \bmod 3 = (2i-m) \bmod 3 \vee j = m-1 \wedge (2i-m) \bmod 3 > 0), \\ 0, & \text{otherwise.} \end{cases}$$

For $n \equiv 2(\text{mod } 3)$,

$$f(v_{i,j}) = \begin{cases} 1, & 0 \leq i \leq \lfloor \frac{m}{2} \rfloor \wedge (j \leq 2i \wedge j \text{ mod } 3 = (2i+2) \text{ mod } 3 \vee j \geq 2i \wedge j \text{ mod } 3 = (2i+1) \text{ mod } 3) \\ & \vee \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m-1 \wedge (j \leq 2i-m \wedge j \text{ mod } 3 = (2i-m-1) \text{ mod } 3 \vee (j \geq 2i-m \wedge j \text{ mod } 3 = (2i-m+1) \text{ mod } 3) \\ & \vee m \leq i \leq n-1 \wedge (j \leq m-2 \wedge (i+j) \text{ mod } 3 = 0) \vee (j = m-1 \wedge (2i-m-1) \text{ mod } 3 > 0)), \\ 0, & \text{otherwise.} \end{cases}$$

Figure 6 shows f on $C_{10} \square C_8$ and $C_{11} \square C_8$. Then, $w(f) = \frac{m+1}{3} \times n = \frac{mn+n}{3}$.

□

Figure 6. f on $C_n \square C_m$ for $m \equiv 2(\text{mod } 3)$.

4. The Italian Domination Number of $C_n \square C_m$

By Theorem 2 and Lemma 1, we can get

Theorem 3. For $m \equiv 0(\text{mod } 3)$, $n \equiv 0(\text{mod } 3)$, $\gamma_I(C_n \square C_m) = \frac{mn}{3}$.

By Theorem 2 and Lemmas 1–3, we can get

Theorem 4. For $m \not\equiv 0(\text{mod } 3)$ or $n \not\equiv 0(\text{mod } 3)$, $\lceil \frac{nm}{3} \rceil \leq \gamma_I(C_n \square C_m) \leq \lfloor \frac{2mn+n+2m+1}{6} \rfloor$.

Then for any integer n, m , the results of $\gamma_I(C_n \square C_m)$ are as shown in Table 1.

Table 1. Results of the Italian domination number of $C_n \square C_m$.

[11]	$m = 3$	$\gamma_I(C_n \square C_m) = \begin{cases} n, & n \equiv 0(\text{mod } 3), \\ n + 1, & n \equiv 1, 2(\text{mod } 3). \end{cases}$
	$m = 4$	$\gamma_I(C_n \square C_m) = \begin{cases} \lceil \frac{3n}{2} \rceil, & n \equiv 0, 1, 3, 4, 5(\text{mod } 8), \\ \lceil \frac{3n}{2} \rceil + 1, & n \equiv 2, 6, 7(\text{mod } 8). \end{cases}$
[10]	$m = 5$	$\gamma_I(C_n \square C_m) = 2n$.
this paper	$m \geq 6$	
	$m \equiv 0(\text{mod } 3), n \equiv 0(\text{mod } 3)$ $m \not\equiv 0(\text{mod } 3)$ or $n \not\equiv 0(\text{mod } 3)$	$\gamma_I(C_n \square C_m) = \frac{mn}{3}$. $\lceil \frac{nm}{3} \rceil \leq \gamma_I(C_n \square C_m) \leq \lfloor \frac{2mn+n+2m+1}{6} \rfloor$.

Italian domination is a monochromatic version of 2-rainbow domination, thus for a graph G the relationship between $\gamma_I(G)$ and $\gamma_{r2}(G)$ is attractive. Brešar et al. [9] presented a question “For which classes of graphs is $\gamma_{r2} = \gamma_I$ for every graph G of a class?”

As far as we know, trees and $C_n \square C_5$ are graphs for which $\gamma_I = \gamma_{r2}$.

By Theorems 1 and 3, for $n = 6k$, $m = 3l$ or $n = 3k$, $m = 6l$ ($k, l \geq 1$), $\gamma_I(C_n \square C_m) = \gamma_{r2}(C_n \square C_m)$. Thus, we give another example of a graph class for which $\gamma_I = \gamma_{r2}$.

5. Conclusions

In this paper, we investigate the Italian domination number of $C_n \square C_m$ for $m \geq 6$. We design an effective algorithm based on the characteristics of $C_n \square C_m$ using the branch and bound method. With the algorithm, we construct some good enough Italian dominating functions. Upon these functions, we can get upper bound on $\gamma_I(C_n \square C_m)$. Together with the lower bound, we obtain the exact value of $\gamma_I(C_n \square C_m) = \frac{mn}{3}$ for $m \equiv 0 \pmod{3}$, $n \equiv 0 \pmod{3}$.

This algorithm is promising and effective for other types of domination, such as double Roman domination, rainbow domination, and so on.

We will deepen our research in the following directions. (1) Italian domination is a generalization of Roman domination, and has a close relationship with 2-rainbow domination. We will study the relationship between these domination numbers for more graphs. (2) If an Italian dominating function on a graph G is looked as the function defining a cooperative game on a simplicial complex [18], there is a clear connection between Italian dominating function and cooperative games. We will investigate the application of Italian domination in cooperative games in our further study.

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