mathematics

Article

Green Functions of the First Boundary-Value Problem for a Fractional Diffusion—Wave Equation in Multidimensional Domains

Arsen Pskhu

Institute of Applied Mathematics and Automation, Kabardin-Balkar Scientific Center of Russian Academy of Sciences, 89-A Shortanov Street, 360000 Nalchik, Russia; pskhu@list.ru

Received: 12 February 2020; Accepted: 22 March 2020; Published: 26 March 2020

Abstract: We construct the Green function of the first boundary-value problem for a diffusion-wave equation with fractional derivative with respect to the time variable. The Green function is sought in terms of a double-layer potential of the equation under consideration. We prove a jump relation and solve an integral equation for an unknown density. Using the Green function, we give a solution of the first boundary-value problem in a multidimensional cylindrical domain. The fractional differentiation is given by the Dzhrbashyan–Nersesyan fractional differentiation operator. In particular, this covers the cases of equations with the Riemann–Liouville and Caputo derivatives.

Keywords: diffusion-wave equation; boundary-value problem; Green function; double-layer potential; fractional derivative; Dzhrbashyan–Nersesyan operator

MSC: 35R11, 34B27

1. Introduction

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \Delta_x\right) u(x,y) = f(x,y), \tag{1}$$

where $\frac{\partial^\alpha}{\partial y^\alpha}$ stands for a fractional derivative with respect to y of order $\alpha \in (0,2)$, $y > 0$, and

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the Laplace operator with respect to $x = (x_1, x_2, ..., x_n) \in S \subset \mathbb{R}^n$.

The fractional diffusion ($0 < \alpha \leq 1$) and diffusion-wave ($0 < \alpha < 2$) equations have attracted great attention in recent years. Active research in this direction began with works [1–4]. In the study of equations with fractional diffusion-wave operators in the main part, a wide range of methods and approaches were used, such as: integral transforms [5–9]; group analysis [10–13]; Fourier methods [14–18]; maximum principles [19,20]; functional methods [21]; asymptotic behavior [22]; Green function methods and method of potentials [23–32]; parametrix methods [33]; inverse problems and regularization methods [34]; reduction methods [35,36]; etc. We also mention the monographs [37–39], which reflect many of these approaches and contain vast bibliographies concerning the issue.

Considerable interest in the study of fractional differential equations, among other things, is also fueled by various applications in physics, mechanics, and simulation (see e.g., [40–46]). Of particular note are some recent applications of the fractional diffusion equation to economics and financial modeling (see e.g., [47–49]).

The novelty of the work is that we construct the Green function of the first boundary-value problem for an arbitrary bounded domain with a smooth boundary.

As it is adopted in the theory of boundary-value problems for parabolic and hyperbolic equations, the first boundary-value problem is the problem of finding solutions that take given initial values and values on the lateral boundary of the domain (see e.g., [50], p. 5, and [51], p. 39). For the equation under consideration, it is natural to adhere to this terminology (see Problem 1 below). Moreover, as is customary, by a Green function we mean a source function (elementary solution) that solves a specific problem in a particular domain (see e.g., [50], p. 84), i.e., the Green function depends on the domain and the form of boundary conditions, contrary to the fundamental solution. Having the Green function constructed and using the formula of general representation of solution (the Green formula) for a considered equation, one can find a solution to a boundary-value problem. In turn, the fundamental solution is a source function, in terms of which a solution of the Cauchy problems can be found. As a rule, the Green function is sought as a sum of the fundamental solution and a regular part, the form of which depends on the domain where the problem is considered and on the form of the boundary value (see Definition 4 in Section 6). Thus, the main problem in constructing a Green function is to find its regular part.

Here, we seek for the regular part of the Green function in the form of a double-layer potential. We prove a jump relation for it and solve the resulting integral equation for an unknown density. The concept of the double-layer potential, which we use here (see Definition 2 in Section 5), naturally generalizes the corresponding concepts from the theory of elliptic and parabolic equations (see e.g., [51], p. 382, and [50] p. 89).

Using the Green function, we give a representation of a solution of the first boundary-value problem in a multidimensional cylindrical domain.

Recently, the Green functions of boundary-value problems for the Equation (1) have been constructed for one-dimensional case ($n = 1$) in [23], and —for problems in multidimensional rectangular domains with $n > 1$—in [32]. We also note [31], in which the first boundary-value problem for the fractional diffusion equation ($0 < \alpha < 1$) with the Caputo derivative has been solved using the layer potential technique.

The fundamental solution for (1) is usually expressed in terms of the Wright function (for $n = 1$) and in terms of the Fox H-function (for $n \geq 2$). Here we use the representation obtained in [24] (see (14) below), and which is in complete agreement with the classical results (see e.g., [1–6]), as was shown in [24].

The fractional differentiation is given by the Dzhrbashyan–Nersesyan fractional differentiation operator. The operators of fractional differentiation in the sense of Riemann–Liouville and Caputo arespecial cases of the Dzhrbashyan–Nersesyan operator and can be expressed in terms of the latter. Therefore, the results obtained here also cover the cases of equations with the Riemann–Liouville and Caputo derivatives.

2. Fractional Differentiation

The fractional differentiation is given by the Dzhrbashyan–Nersesyan operator [52]. The Dzhrbashyan–Nersesyan operator of order α, with starting point at $y = \eta$, and associated with an ordered sequence

$$\{\gamma_0, \gamma_1, ..., \gamma_m\}, \tag{2}$$

is defined by
$$\frac{\partial^\alpha}{\partial(y-\eta)^\alpha} = D_{\eta y}^{\{\gamma_0,\gamma_1,...,\gamma_m\}} = D_{\eta y}^{\gamma_m-1} D_{\eta y}^{\gamma_{m-1}} \cdots D_{\eta y}^{\gamma_1} D_{\eta y}^{\gamma_0}, \tag{3}$$
where
$$\gamma_k \in (0,1] \qquad (k=0,1,...,m), \qquad \alpha = \sum_{k=0}^{m} \gamma_k - 1,$$
and $D_{\eta y}^{\gamma_m-1}$ and $D_{\eta y}^{\gamma_k}$ denote the Riemann–Liouville fractional integral and fractional derivative, respectively. The Riemann–Liouville operators of fractional integration and differentiation with respect to y, and with starting point at $y=\eta$, are defined by ([40], p. 11, [38], §2.1)
$$D_{\eta y}^{\zeta} g(y) = \operatorname{sign}(y-\eta) \int_\eta^y g(t) \frac{|y-t|^{-\zeta-1}}{\Gamma(-\zeta)} dt \qquad (\zeta<0), \qquad\qquad D_{\eta y}^{0} g(y) = g(y), \tag{4}$$
and
$$D_{\eta y}^{\zeta} g(y) = \operatorname{sign}^p(y-\eta) \frac{\partial^p}{\partial y^p} D_{\eta y}^{\zeta-p} g(y) \qquad (0<\zeta\le p, \qquad p\in\mathbb{N}). \tag{5}$$

The composition law for the Riemann–Liouville derivatives (see e.g., [38]) shows that
$$D_{\eta y}^{\{\gamma_0,\gamma_1,...,\gamma_m\}} g(y) = D_{\eta y}^{\alpha} \left[g(y) - \sum_{k=0}^{m-1} \frac{|y-\eta|^{\mu_k-1}}{\Gamma(\mu_k)} g_k \right], \tag{6}$$
where
$$g_k = \lim_{y\to\eta} D_{\eta y}^{\{\gamma_0,\gamma_1,...,\gamma_k\}} g(y) \qquad \text{and} \qquad \mu_k = \sum_{j=0}^{k} \gamma_j.$$

It should be noted that the operator (3) associated with the sequences $\{\zeta-m+1,1,1,...,1\}$ and $\{1,1,...,1,\zeta-m+1\}$ coincides with the Riemann–Liouville derivative and the Caputo derivative, respectively, i.e., we can write
$$D_{\eta y}^{\zeta} g(y) = D_{\eta y}^{\{\zeta-m+1,1,1,...,1\}} g(y) \qquad \text{and} \qquad \partial_{\eta y}^{\zeta} g(y) = D_{\eta y}^{\{1,...,1,\zeta-m+1\}} g(y).$$

Here, $\zeta\in(m-1,m]$ and $\partial_{\eta y}^{\zeta}$ is the Caputo fractional derivative, i.e. $\partial_{\eta y}^{\zeta} = \operatorname{sign}^m(y-\eta) D_{\eta y}^{\zeta-m} \frac{\partial^m}{\partial y^m}$.

3. Boundary-Value Problem

We consider the Equation (1) or, more precisely,
$$\left(D_{0y}^{\{\gamma_0,\gamma_1,...,\gamma_m\}} - \Delta_x \right) u(x,y) = f(x,y) \qquad (0<\alpha=\gamma_0+...+\gamma_m-1<2), \tag{7}$$
in the domain
$$\Omega = S\times(0,T) = \{(x,y):\, x\in S\subset\mathbb{R}^n,\, y\in(0,T)\},$$
where S is a bounded simply connected domain in $\mathbb{R}^n$. In what follows, ∂S and $\overline{S}$ denote, respectively, the boundary and the closure of S.

Definition 1. *By a regular solution of the Equation (7) we mean a function $u(x,y)$ such that: $y^{1-\mu}u(x,y)\in \overline{S}\times[0,T)$ for some $\mu>0$; in Ω, $u(x,y)$ has continuous derivatives with respect to x_j ($j=1,...,n$) up to the second order; the functions $D_{0y}^{\{\gamma_0,\gamma_1,...,\gamma_k\}} u(x,y)$ ($k=0,1,...,m-1$) have continuous derivatives with respect to $y\in(0,T)$*

for a fixed $x \in S$, and these functions are continuous in $S \times [0, T)$; and $u(x, y)$ satisfies the Equation (1) for all $(x, y) \in \Omega$.

The first boundary-value problem for the Equation (7) is stated as follows.

Problem 1. *Find a regular solution of the Equation (7) in Ω satisfying the conditions*

$$\lim_{\substack{x \to x^* \\ x \in S}} u(x, y) = \varphi(x^*, y) \qquad (x^* \in \partial S, \quad 0 < y < T) \tag{8}$$

and

$$\lim_{y \to 0} D_{0y}^{\{\gamma_0, \gamma_1, ..., \gamma_k\}} u(x, y) = \tau_k(x) \qquad (x \in S, \quad k = 0, 1, ..., m-1), \tag{9}$$

where $\varphi(x^*, y)$ and $\tau_k(x)$ are known (given) functions, $\varphi(x^*, y) \in C(\partial S \times (0, T))$ and $\tau_k(x) \in C(S)$.

Remark 1. *The function $\varphi(x^*, y)$ is defined on $\partial S \times (0, T)$, which is the lateral boundary of the cylindrical domain Ω, and specifies the boundary condition of Problem 1. The functions $\tau_k(x)$, $k = 0, 1, ..., m-1$, are defined on S (i.e., on the lower base of Ω) and should be considered to be the initial conditions of the problem (7)–(9). It should be noted that the number of necessary initial conditions to be imposed for the Equation (7), is equal to the number of elements in the sequence (2) and, in general, does not depend on the order α.*

Remark 2. *For the Equation (7) with the Riemann–Liouville fractional derivative (i.e., for $\{\gamma_0, \gamma_1, ..., \gamma_m\} = \{\alpha - m + 1, 1, ..., 1\}$), due to $\alpha \in (0, 2)$, m can only be equal to either 1 or 2. In this case the initial conditions (9) take one of the following forms*

$$\lim_{y \to 0} D_{0y}^{\alpha - 1} u(x, y) = \tau_0(x) \qquad \text{for} \quad 0 < \alpha \leq 1 \qquad (m = 1) \tag{10}$$

or

$$\lim_{y \to 0} D_{0y}^{\alpha - 2} u(x, y) = \tau_0(x), \qquad \lim_{y \to 0} D_{0y}^{\alpha - 1} u(x, y) = \tau_1(x) \qquad \text{for} \quad 1 < \alpha < 2 \qquad (m = 2). \tag{11}$$

Similarly, if fractional differentiation in (7) is given in the Caputo sense (i.e., $\{\gamma_0, \gamma_1, ..., \gamma_m\} = \{1, ..., 1, \alpha - m + 1\}$ with $m = 1$ or 2), then (9) may be written as

$$\lim_{y \to 0} u(x, y) = \tau_0(x) \qquad \text{for} \quad 0 < \alpha \leq 1 \qquad (m = 1) \tag{12}$$

or

$$\lim_{y \to 0} u(x, y) = \tau_0(x), \qquad \lim_{y \to 0} \frac{\partial}{\partial y} u(x, y) = \tau_1(x) \qquad \text{for} \quad 1 < \alpha < 2 \qquad (m = 2). \tag{13}$$

4. Fundamental Solution

Consider the function [24]

$$\Gamma_{\alpha,n}(x, y) = C_n y^{\beta(2-n)-1} f_\beta \left(|x| y^{-\beta}; n-1, \beta(2-n) \right). \tag{14}$$

Here and subsequently,

$$\beta = \frac{\alpha}{2}, \qquad C_n = 2^{-n} \pi^{\frac{1-n}{2}},$$

and

$$f_\beta(z;\mu,\delta)=\begin{cases}\dfrac{2}{\Gamma\left(\frac{\mu}{2}\right)}\displaystyle\int_1^\infty \phi\left(-\beta,\delta;-zt\right)\left(t^2-1\right)^{\frac{\mu}{2}-1}dt, & \mu>0,\\[2ex] \phi\left(-\beta,\delta;-z\right), & \mu=0,\end{cases}$$

where

$$\phi\left(-\beta,\delta;-z\right)=\sum_{k=0}^{\infty}\frac{(-z)^k}{k!\Gamma(\delta-\beta k)}$$

is the Wright function [53,54].

In [24], it was shown that the function $\Gamma_{\alpha,n}(x-s,y-t)$ is the fundamental solution of the Equation (7).

Let us recall some properties of the function (14).

Lemma 1. *Let $t<y$, $|x-s|>0$, and $\zeta\in\mathbb{R}$. Then $\Gamma_{\alpha,n}(x-s,y-t)$ is a solution of the equation*

$$\left(D_{ty}^{\alpha}-\Delta_x\right)D_{ty}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t)=0$$

as a function of x and y, and is a solution of

$$\left(D_{yt}^{\alpha}-\Delta_s\right)D_{yt}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t)=0 \tag{15}$$

as a function of s and t.

Lemma 2. *Let $\zeta\in\mathbb{R}$. Then*

$$\left|D_{\eta y}^{\zeta}\Gamma_{\alpha,n}(x,y)\right|\le Cy^{\beta(2-n)-\zeta-1}g_p\left(|x|y^{-\beta}\right)E\left(|x|y^{-\beta},\rho\right), \tag{16}$$

$$\left|\frac{\partial}{\partial x_k}D_{\eta y}^{\zeta}\Gamma_{\alpha,n}(x,y)\right|\le C|x_k|y^{-\beta n-\zeta-1}g_{p+2}\left(|x|y^{-\beta}\right)E\left(|x|y^{-\beta},\rho\right), \tag{17}$$

$$\left|\frac{\partial^2}{\partial x_k^2}D_{\eta y}^{\zeta}\Gamma_{\alpha,n}(x,y)\right|\le Cy^{-\beta n-\zeta-1}g_q\left(|x|y^{-\beta}\right)E\left(|x|y^{-\beta},\rho\right),$$

where

$$p=\begin{cases}n, & \textit{for}\quad \zeta\in\mathbb{N}\cup\{0\},\\ n+2, & \textit{for}\quad \zeta\notin\mathbb{N}\cup\{0\},\end{cases}\qquad q=\begin{cases}n+2, & \textit{for}\quad \zeta\in\mathbb{N}\cup\{0\}\quad \textit{or}\quad n=1,\\ n+4 & \textit{for}\quad \zeta\notin\mathbb{N}\cup\{0\}\quad \textit{and}\quad n\ge 2,\end{cases}$$

and

$$E(z,\rho)=\exp\left(-\rho z^{\frac{1}{1-\beta}}\right),\qquad g_n(z)=\begin{cases}1 & \textit{for}\quad n\le 3,\\ |\ln z|+1 & \textit{for}\quad n=4,\\ z^{4-n} & \textit{for}\quad n\ge 5,\end{cases}$$

$C=C(n,\alpha,\rho)$, $\rho<(1-\beta)\beta^{\frac{\beta}{1-\beta}}$, *and (by choosing C) ρ can be taken arbitrarily close to $(1-\beta)\beta^{\frac{\beta}{1-\beta}}$.*

Here and subsequently, the symbol C denotes different positive constants indicating in brackets the parameters on which they depend if necessary: $C=C(\alpha,\beta,...)$.

Proofs of Lemmas 1 and 2 are given in [24].

Corollary 1. *Let $t < y$, $|x-s| > 0$, and $\zeta \in \mathbb{R}$. Then $\Gamma_{\alpha,n}(x-s, y-t)$ satisfies the equations*

$$\left(D_{ty}^{\{\gamma_0,\gamma_1,\dots,\gamma_m\}} - \Delta_x\right) D_{ty}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t) = 0 \tag{18}$$

and

$$\left(D_{yt}^{\{\gamma_0,\gamma_1,\dots,\gamma_m\}} - \Delta_s\right) D_{yt}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t) = 0. \tag{19}$$

Proof. Combining (6) and (16) gives

$$D_{ty}^{\{\gamma_0,\gamma_1,\dots,\gamma_m\}} D_{ty}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t) = D_{ty}^{\alpha} D_{ty}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t)$$

and

$$D_{yt}^{\{\gamma_0,\gamma_1,\dots,\gamma_m\}} D_{yt}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t) = D_{yt}^{\alpha} D_{yt}^{\zeta}\Gamma_{\alpha,n}(x-s,y-t).$$

Lemmas 1 now leads to (18) and (19). □

5. Double-Layer Potential

In what follows, ν_s denotes the outer normal direction to ∂S at $s \in \partial S$, and $\frac{\partial}{\partial \nu_s}$ stands for the directional derivative in the direction ν_s.

Definition 2. *Let $w(s,t) \in L(\partial S \times (0,T))$. We call a function $V(x,y)$ defined by*

$$V(x,y) = \int_0^y \int_{\partial S} w(s,t) \frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt \qquad (x \in \mathbb{R}^n \setminus \partial S, \quad y \in (0,T)) \tag{20}$$

the double-layer potential with the density $w(s,t)$.

By Lemmas 1 and 2, it is obvious that $V(x,y)$ is a regular solution of (7) in each of the domains $S \times (0,T)$ and $\left(\mathbb{R}^n \setminus \overline{S}\right) \times (0,T)$. Our purpose now is to study the behavior of $V(x,y)$ on ∂S.

Let

$$B_\varepsilon^a = \{x \in \mathbb{R}^n : |x-a| < \varepsilon\}, \qquad \partial S_\varepsilon^a = B_\varepsilon^a \cap \partial S, \qquad \text{and} \qquad C_\varepsilon^a = B_\varepsilon^a \cap \left(\mathbb{R}^n \setminus \overline{S}\right). \tag{21}$$

Definition 3. *We say that a surface ∂S belongs to the Lyapunov class (see e.g., [55] (p. 64)) if: at each point of ∂S there is a well-defined tangent plane; any line parallel to ν_a intersects ∂S_ε^a at most once for some $\varepsilon > 0$; there are $C > 0$ and $\theta > 0$ such that $|\widehat{\nu_a \nu_b}| \le C|a-b|^\theta$ for any $a, b \in \partial S$ (where $\widehat{\nu_a \nu_b}$ stands for the angle between ν_a and ν_b).*

Lemma 3. *Let $n \ge 2$, $t^{1-\mu}w(s,t) \in C(\partial S \times [0,T])$ for some $\mu > 0$, ∂S be a surface of the Lyapunov class, $a \in \partial S$, and $y \in (0,T)$. Then*

$$\lim_{\substack{x \to a \\ x \in S}} \int_0^y \int_{\partial S} w(s,t) \frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt =$$
$$= -\frac{1}{2}w(a,y) + \int_0^y \int_{\partial S} w(s,t) \frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(a-s,y-t)d\sigma_s dt. \tag{22}$$

Proof. With the notation (21), we have

$$\lim_{\substack{x \to a \\ x \in S}} \int_0^y \int_{\partial S} w(s,t) \frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt = \int_0^y \int_{\partial S \setminus \partial S_\varepsilon^a} w(s,t) \frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(a-s,y-t)d\sigma_s dt +$$

$$+\lim_{\substack{x\to a\\ x\in S}}\left(\int_0^{y-\delta}+\int_{y-\delta}^{y}\right)\int_{\partial S_\varepsilon^a} w(s,t)\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt,$$

for any $\delta\in(0,y)$ and sufficiently small $\varepsilon>0$. By the formula (see [24])

$$\frac{\partial}{\partial x_j}\Gamma_{\alpha,n}(x,y)=-2\pi x_j\Gamma_{\alpha,n+2}(x,y),$$

we get

$$\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(a-s,y-t)=2\pi\langle a-s,\nu_s\rangle\Gamma_{\alpha,n+2}(a-s,y-t), \tag{23}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Since ∂S is a Lyapunov surface, we have

$$|\langle a-s,\nu_s\rangle|\le C|a-s|^{1+\theta} \tag{24}$$

for some $\theta>0$ in a sufficiently small neighborhood of the point a. Combining this with (16) and (17), we obtain

$$\lim_{\substack{x\to a\\ x\in S}}\int_0^y\int_{\partial S} w(s,t)\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt=\int_0^y\int_{\partial S} w(s,t)\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(a-s,y-t)d\sigma_s\, dt+$$

$$+\lim_{\varepsilon\to 0}\lim_{\substack{x\to a\\ x\in S}}\int_{y-\delta}^{y}\int_{\partial S_\varepsilon^a}[w(s,t)-w(a,y)]\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt+$$

$$+w(a,y)\lim_{\varepsilon\to 0}\lim_{\substack{x\to a\\ x\in S}}\int_{y-\delta}^{y}\int_{\partial S_\varepsilon^a}\frac{\partial}{\partial \nu_s}\Gamma_{\alpha,n}(x-s,y-t)d\sigma_s dt.$$

Let us denote by I_1 and I_2 the integrals in the second and the third summands, respectively, in the right-hand side of the last equality. By Lemma 2 and (23), we have

$$|I_1|\le C\omega_{\varepsilon,\delta}\int_{y-\delta}^{y}\int_{\partial S_\varepsilon^a}\frac{|\langle x-s,\nu_s\rangle|}{(y-t)^{\beta n+1}}g_{n+2}\left(\frac{|x-s|}{(y-t)^\beta}\right)E\left(\frac{|x-s|}{(y-t)^\beta},\rho\right)d\sigma_s dt,$$

where

$$\omega_{\varepsilon,\delta}=\sup\left\{|w(s,t)-w(a,y)|:\ |s-a|<\varepsilon,\ |y-t|<\delta\right\}.$$

A simple computation gives

$$|I_1|\le C\omega_{\varepsilon,\delta}\int_{\partial S_\varepsilon^a}\frac{|\langle x-s,\nu_s\rangle|}{|x-s|^n}d\sigma_s\int_0^\infty t^{-\beta n-1}g_{n+2}\left(t^{-\beta}\right)E\left(t^{-\beta},\rho\right)dt,$$

where

$$\int_{\partial S_\varepsilon^a}\frac{|\langle x-s,\nu_s\rangle|}{|x-s|^n}d\sigma_s\le C(S)\qquad\text{and}\qquad\int_0^\infty t^{-\beta n-1}g_{n+2}\left(t^{-\beta}\right)E\left(t^{-\beta},\rho\right)dt\le C(\beta,n).$$

Now

$$\lim_{\varepsilon\to 0}\lim_{\substack{x\to a\\ x\in S}}I_1=0, \tag{25}$$

which is due to the continuity of $w(s,t)$ in a neighborhood of (a,y) and arbitrary choice of δ.

Let us examine I_2. Using the Stokes formula gives

$$\int_{\partial S_\varepsilon^a} \frac{\partial}{\partial \nu_s} \Gamma_{\alpha,n}(x-s,y-t)d\sigma_s = \int_{\partial C_\varepsilon^a \setminus \partial S_\varepsilon^a} \frac{\partial}{\partial \nu_s} \Gamma_{\alpha,n}(x-s,y-t)d\sigma_s - \int_{C_\varepsilon^a} \Delta_s \Gamma_{\alpha,n}(x-s,y-t)ds.$$

By (15) and (16), we get

$$I_2 = \int_{\partial C_\varepsilon^a \setminus \partial S_\varepsilon^a} \frac{\partial}{\partial \nu_s} \left[D_{yt}^{-1} \Gamma_{\alpha,n}(x-s,y-t) \right]_{t=y-\delta} d\sigma_s - \int_{C_\varepsilon^a} \left[D_{yt}^{\alpha-1} \Gamma_{\alpha,n}(x-s,y-t) \right]_{t=y-\delta} ds$$

and

$$\lim_{\substack{x\to a \\ x\in S}} I_2 = \int_{\partial C_\varepsilon^a \setminus \partial S_\varepsilon^a} \frac{\partial}{\partial \nu_s} \left[D_{yt}^{-1} \Gamma_{\alpha,n}(a-s,y-t) \right]_{t=y-\delta} d\sigma_s - \\ - \int_{C_\varepsilon^a} \left[D_{yt}^{\alpha-1} \Gamma_{\alpha,n}(a-s,y-t) \right]_{t=y-\delta} ds = J_1 - J_2. \tag{26}$$

From (14) and (23), it follows that

$$J_1 = \frac{\pi^{\frac{1-n}{2}}}{2^{n+1}} \left[D_{yt}^{-1} y^{-\beta n-1} f_\beta \left(\varepsilon y^{-\beta}; n+1, -\beta n \right) \right]_{t=y-\delta} \int_{\partial C_\varepsilon^a \setminus \partial S_\varepsilon^a} \langle a-s, \nu_s \rangle d\sigma_s.$$

Using the formulas (see [24])

$$D_{0y}^{\zeta} y^{q-1} f_\beta \left(\varepsilon y^{-\beta}; p, q \right) = y^{q+\zeta-1} f_\beta \left(\varepsilon y^{-\beta}; p, q+\zeta \right)$$

and

$$\lim_{z\to 0} z^{p-1} f_\beta \left(z; p, q \right) = \frac{2\,\Gamma(p-1)}{\Gamma\left(\frac{p}{2}\right) \Gamma(\beta p + q - \beta)} \qquad (p>1),$$

we get

$$\lim_{\varepsilon\to 0} J_1 = \lim_{\varepsilon\to 0} \frac{\pi^{\frac{1-n}{2}}}{2^{n+1}} \lim_{\varepsilon\to 0} \left(\frac{\varepsilon}{\delta^\beta} \right)^n f_\beta \left(\varepsilon \delta^{-\beta}; n+1, 1-\beta n \right) \cdot \lim_{\varepsilon\to 0} \varepsilon^{-n} \int_{\partial C_\varepsilon^a \setminus \partial S_\varepsilon^a} \langle a-s, \nu_s \rangle d\sigma_s = \\ = \frac{\pi^{\frac{1-n}{2}}}{2^n} \frac{\Gamma(n)}{\Gamma\left(\frac{n+1}{2}\right)} \cdot \frac{1}{2} \lim_{\varepsilon\to 0} \varepsilon^{-n} \int_{\partial B_\varepsilon^a} \langle a-s, \nu_s \rangle d\sigma_s = \frac{\pi^{\frac{1-n}{2}}}{2^n} \frac{\Gamma(n)}{\Gamma\left(\frac{n+1}{2}\right)} \cdot \left(-\frac{n}{2} V_n \right).$$

Here V_n is the volume of the unit ball in $\mathbb{R}^n$. Thus

$$\lim_{\varepsilon\to 0} J_1 = -\frac{1}{2}. \tag{27}$$

Let us evaluate J_2. The inequality (16) gives

$$|J_2| \le C\delta^{-\beta n} \int_{C_\varepsilon^a} g_{n+2} \left(|a-s|\delta^{-\beta} \right) E \left(|a-s|\delta^{-\beta}, \rho \right) ds,$$

which means that

$$\lim_{\varepsilon\to 0} J_2 = 0.$$

The last equality, (26), and (27) yield

$$\lim_{\varepsilon\to 0}\lim_{\substack{x\to a\\ x\in S}} I_2 = -\frac{1}{2}.$$

Combining this with (25), we obtain (22). □

Formula (22) is called a jump relation for double-layer potential (20).

6. Green Function

Definition 4. *We call a function $G(x,y,s,t)$ the Green function of the problem (7)–(9) if $G(x,y,s,t)$ has the form $G(x,y,s,t) = \Gamma_{\alpha,n}(x-s,y-t) - W(x,y-t,s)$, where the function $W(x,y,s)$ satisfies the following conditions:*

1) $W(x,y,s)$ is a regular solution of the Equation (7) as a function of $(x,y)\in\Omega$ for any fixed $s\in S$;

2)
$$W(x,y,s)\in C(\overline{S}\times[0,T]\times S); \tag{28}$$

3)
$$W(x^*,y,s) = \Gamma_{\alpha,n}(x^*-s,y) \qquad \textit{for all} \quad x^*\in\partial S, \quad s\in S, \quad \textit{and} \quad y\in(0,T); \tag{29}$$

4)
$$\lim_{y\to 0} D_{0y}^{\{\gamma_0,\gamma_1,\ldots,\gamma_k\}} W(x,y,s) = 0 \qquad (x,s\in S, \quad k=0,1,\ldots,m-1). \tag{30}$$

Our goal now is to construct the function $G(x,y,s,t)$, or equivalently, its regular part $W(x,y,s)$. We will look for $W(x,y,s)$ in the form

$$W(x,y,s) = \int_0^y \int_{\partial S} w(\xi,\eta,s)\frac{\partial}{\partial\nu_\xi}\Gamma_{\alpha,n}(x-\xi,y-\eta)d\sigma_\xi\, d\eta, \tag{31}$$

where $w(\xi,\eta,s)$ is to be determined.

Let $x^*\in\partial S$, s be a fixed point of S, and ∂S be a Lyapunov surface. It follows from (22) that

$$\lim_{x\to x^*} W(x,y,s) = -\frac{1}{2}w(x^*,y,s) + \int_0^y \int_{\partial S} w(\xi,\eta,s)\frac{\partial}{\partial\nu_\xi}\Gamma_{\alpha,n}(x^*-\xi,y-\eta)d\sigma_\xi\, d\eta.$$

Assuming (29) to satisfy, we obtain that $w(x^*,y,s)$ is a solution of the integral equation

$$w(x^*,y,s) = -2\,\Gamma_{\alpha,n}(x^*-s,y) + 2\,(P\,w)\,(x^*,y,s), \tag{32}$$

where

$$(P\,w)\,(x^*,y,s) = \int_0^y \int_{\partial S} w(\xi,\eta,s)\frac{\partial}{\partial\nu_\xi}\Gamma_{\alpha,n}(x^*-\xi,y-\eta)d\sigma_\xi\, d\eta.$$

From now on, we assume that

$$w(x^*,y,s)\in C\,(\partial S\times(0,T)\times S) \qquad \text{and} \qquad |w|\le w_0(s)\frac{y^{\mu-1}}{\Gamma(\mu)} \tag{33}$$

for some $\mu>0$ and $w_0 = w_0(s) > 0$. By (16), (23), and (24), we get

$$|P\,w| \le \frac{w_0}{\Gamma(\mu)}\int_0^y \eta^{\mu-1}\int_{\partial S}\frac{|x^*-\xi|^{1+\theta}}{(y-\eta)^{\beta n+1}} g_{n+2}\left(\frac{|x^*-\xi|}{(y-\eta)^\beta}\right) E\left(\frac{|x^*-\xi|}{(y-\eta)^\beta},\rho\right) d\sigma_\xi\, d\eta \le$$

$$\leq \frac{w_0}{\Gamma(\mu)} \int_0^y \eta^{\mu-1}(y-\eta)^{\theta\beta-1} \int_{\partial S_0} |\lambda|^{1+\theta} g_{n+2}\left(|\lambda|\right) E\left(|\lambda|, \rho\right) d\sigma_\lambda\, d\eta,$$

where θ is a positive number (the choice of which depends only on ∂S), and

$$\partial S_0 = \partial S_0(x^*, y-\eta) = \{\lambda \in \mathbb{R}^n : \lambda = (x^* - \xi)(y-\eta)^\beta,\ \xi \in \partial S\}.$$

It is easy to check that

$$\int_{\partial S_0} |\lambda|^{1+\theta} g_{n+2}\left(|\lambda|\right) E\left(|\lambda|\right) d\sigma_\lambda \leq C \qquad \text{where} \quad C = C(S, T).$$

Thus, by

$$\int_0^y \eta^{\mu-1}(y-\eta)^{\beta\theta-1} d\eta = y^{\mu+\beta\theta-1} \frac{\Gamma(\mu)\Gamma(\beta\theta)}{\Gamma(\mu+\beta\theta)},$$

we have

$$|P\, w| \leq C\, \Gamma(\beta\theta) w_0 \frac{y^{\mu+\beta\theta-1}}{\Gamma(\mu+\beta\theta)}.$$

Repeated use of this inequality shows that

$$\left|P^k\, w\right| \leq \left[C\, \Gamma(\beta\theta)\right]^k w_0 \frac{y^{\mu+k\beta\theta-1}}{\Gamma(\mu+k\beta\theta)} \qquad (k \in \mathbb{N}).$$

This means that the integral Equation (32) has a unique solution of the class (33), and the solution of (32) is given by

$$w(x^*, y, s) = v(x^*, y, s) + \sum_{k=1}^{\infty} 2^k \left(P^k\, v\right)(x^*, y, s), \tag{34}$$

where

$$v(x^*, y, s) = -2\, \Gamma_{\alpha,n}(x^* - s, y).$$

Thus, by (31) and (34), $W(x, y, s)$ can be written as

$$W(x, y, s) = -\int_0^y \int_{\partial S} \left[\sum_{k=0}^{\infty} 2^{k+1} \Gamma^k_{\alpha,n}(\xi, \eta, s)\right] \frac{\partial}{\partial \nu_\xi} \Gamma_{\alpha,n}(x-\xi, y-\eta) d\sigma_\xi\, d\eta, \tag{35}$$

where $\Gamma^0_{\alpha,n}(\xi, \eta, s) = \Gamma_{\alpha,n}(\xi - s, \eta)$ and

$$\Gamma^{k+1}_{\alpha,n}(\xi, \eta, s) = \int_0^\eta \int_{\partial S} \Gamma^k_{\alpha,n}(\xi^*, \eta^*, s) \frac{\partial}{\partial \nu_{\xi^*}} \Gamma_{\alpha,n}(\xi - \xi^*, \eta - \eta^*) d\sigma_{\xi^*}\, d\eta^* \qquad (k \in \mathbb{N}).$$

Theorem 1. *Let $n \geq 2$ and ∂S be a surface of the Lyapunov class. Then the function $W(x, y, s)$ defined by (35) satisfies the properties 1)–4) of Definition 4, and the function*

$$G(x, y, s, t) = \Gamma_{\alpha,n}(x - s, y - t) - W(x, y - t, s) \tag{36}$$

is the Green function of the problem (7)–(9).

Proof. It follows from Corollary 1 and Lemma 2 that $W(x,y,s)$ is a regular solution of (7), and, moreover, (30) holds for any $x, s \in S$. By (22) and (31), we get

$$\lim_{\substack{x\to x^* \\ x\in S}} W(x^*,y,s) = -\frac{1}{2}w(x^*,y,s) + (P\,w)\,(x^*,y,s).$$

Combining this and (32) leads to (29). In addition, this means that $W(x,y,s)$ can be continued up to ∂S (with respect to x), and we get (28). □

Remark 3. *It is worth noting that the Green function $G(x,y,s,t)$ as well as the fundamental solution $\Gamma_{\alpha,n}(x,y)$ does not depend on the number and values of elements in $\{\gamma_0,...,\gamma_m\}$, but only on $\alpha = \gamma_0 + ... + \gamma_m - 1$. However, the sequence $\{\gamma_0,...,\gamma_m\}$ affects the form of solutions (see Remark 5 below).*

7. Representation of Solutions

Having constructed the Green function, we can find a representation for solutions to Problem 1.

Theorem 2. *Let $u(x,y)$ be a regular solution of the problem (7)–(9) such that $\frac{\partial}{\partial x_j}u(x,y) \in L\,(\partial S\times(0,T))$ ($j = 1,2,...,n$), and $y^{1-\mu}\varphi(x^*,y) \in C\,(\partial S\times[0,T])$ for some $\mu > 0$. Then*

$$u(x,y) = \sum_{k=0}^{m-1}\int_S \tau_k(s) D_{0y}^{\alpha-\mu_k} G(x,y,s,0)ds + \Phi(x,y) + F(x,y), \tag{37}$$

where $\mu_k = \gamma_0 + \gamma_1 + ... + \gamma_k$ ($k = 0,1,...,m-1$),

$$\Phi(x,y) = -\int_0^y \int_{\partial S} \varphi(s,t)\frac{\partial}{\partial \nu_s}G(x,y,s,t)d\sigma_s\,dt, \quad \textit{and} \quad F(x,y) = \int_0^y\int_S f(s,t)G(x,y,s,t)ds\,dt. \tag{38}$$

Proof. Using the general representation for solutions of (7) (see [24], Theorem 1), we can conclude that if $u(x,y)$ is a regular solution of the problem (7)–(9), then

$$\begin{aligned} u(x,y) = &\int_0^y\int_S f(s,t)G(x,y,s,t)ds\,dt + \sum_{k=0}^{m-1}\int_S \tau_k(s)\left[D_{yt}^{\alpha-\mu_k}G(x,y,s,t)\right]_{t=0} ds+ \\ &+\int_0^y\int_{\partial S}\left[G(x,y,s,t)\frac{\partial}{\partial\nu_s}u(s,t) - u(s,t)\frac{\partial}{\partial\nu_s}G(x,y,s,t)\right]d\sigma_s\,dt. \end{aligned} \tag{39}$$

It follows easily from the definitions (4) and (5) that $\left[D_{yt}^{\zeta}g(y-t)\right]_{t=0} = D_{0y}^{\zeta}g(y)$ for any $\zeta \in \mathbb{R}$. Taking into account (36), this means that

$$\left[D_{yt}^{\alpha-\mu_k}G(x,y,s,t)\right]_{t=0} = D_{0y}^{\alpha-\mu_k}G(x,y,s,0).$$

Combining the latter, (8), (29), and (39), we get (37). □

Remark 4. *Please note that Theorem 2 does not state that any function of the form (37) is a priori a solution to Problem 1. To guarantee this, it is necessary to impose additional smoothness conditions (such as Hölder continuity or differentiability) on the functions $f(x,y)$, $\varphi(x,y)$, and $\tau_k(x)$. These conditions can easily be adapted from [24].*

Let us reformulate the results of Theorem 2 for equations with the Riemann–Liouville and Caputo derivatives, i.e., for the equations

$$\left(D_{0y}^{\alpha} - \Delta_x\right) u(x,y) = f(x,y) \tag{40}$$

and

$$\left(\partial_{0y}^{\alpha} - \Delta_x\right) u(x,y) = f(x,y). \tag{41}$$

It follows from Theorem 2 that a regular solution of (40) satisfying the boundary condition (8) and one of the initial conditions (10) or (11) (depending on the segment in which α lies, see Remark 2) has the form

$$u(x,y) = \int_S \tau_0(s) G(x,y,s,0) ds + \Phi(x,y) + F(x,y) \tag{42}$$

for $\alpha \in (0,1]$ $(m = 1)$, or

$$u(x,y) = \int_S \tau_0(s) \frac{\partial}{\partial y} G(x,y,s,0) ds + \int_S \tau_1(s) G(x,y,s,0) ds + \Phi(x,y) + F(x,y) \tag{43}$$

for $\alpha \in (1,2)$ $(m = 2)$.

Likewise, a regular solution of (41) satisfying (8) and either (12) or (13) can be written as

$$u(x,y) = \int_S \tau_0(s) D_{0y}^{\alpha-1} G(x,y,s,0) ds + \Phi(x,y) + F(x,y) \tag{44}$$

for $\alpha \in (0,1]$ $(m = 1)$, or

$$u(x,y) = \int_S \tau_0(s) D_{0y}^{\alpha-2} G(x,y,s,0) ds + \int_S \tau_1(s) D_{0y}^{\alpha-1} G(x,y,s,0) ds + \Phi(x,y) + F(x,y) \tag{45}$$

for $\alpha \in (1,2)$ $(m = 2)$.

The functions $\Phi(x,y)$ and $F(x,y)$ in (42)–(45) are defined by (38).

Remark 5. *Comparing* (37), (42)–(45), *one can notice that the components of solutions introduced by the right side $f(x,y)$ and the boundary condition $\varphi(x^*,y)$ (i.e., The functions $\Phi(x,y)$ and $F(x,y)$) are the same for all cases, only dependent of α, and independent of the number and distribution of elements in $\{\gamma_0, \gamma_1, ..., \gamma_m\}$.*
Thus, the sequence $\{\gamma_0, \gamma_1, ..., \gamma_m\}$ or, in other words, the used form of fractional differentiation (among those that can be generated by the Dzhrbashyan–Nersesyan operator) affects only the part of the solution that corresponds to the initial conditions.

8. Conclusions

In this paper, we construct the Green function of the first boundary problem for the fractional diffusion-wave equation in a multidimensional cylindrical domains with a boundary of the Lyapunov class. We find the regular part of the Green function as a solution of an integral equation generated by a jump relation for the double-layer potential of the considered equation. We give the representation of solutions to the problem under study in terms of the Green function.

For fractional differentiation, we use the Dzhrbashyan–Nersesyan operator, which generates the Riemann–Liouville and Caputo derivatives as particular cases. This allow us to obtain the corresponding results for equations with these types of fractional derivatives as a consequence of the results obtained in the general case. Moreover, it is shown that the choice of the form of fractional differentiation (among the operators covered by the Dzhrbashyan–Nersesyan operator) affects only the part of solution corresponding to initial conditions, and the remaining components of solutions depend only of the order of fractional differentiation.

Funding: This research was funded by the Russian Foundation for Basic Research grant number 18-51-45005.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Wyss, W. The fractional diffusion equation. *J. Math. Phys.* **1986**, *27*, 2782–2785. [CrossRef]
2. Schneider, W.R.; Wyss, W. Fractional diffusion and wave equations. *J. Math. Phys.* **1989**, *30*, 134–144. [CrossRef]
3. Kochubei, A.N. Diffusion of fractional order. *Differ. Equ.* **1990**, *26*, 485–492.
4. Fujita, Y. Integrodifferential equation which interpolates the heat equation and the wave equation I, II. *Osaka J. Math.* **1990**, *27*, 309–321, 797–804.
5. Mainardi, F. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.* **1996**, *9*, 23–28. [CrossRef]
6. Mainardi, F.; Luchko, Y.; Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **2001**, *4*, 153–192.
7. Orsingher, E.; Beghin, L. Time-fractional telegraph equations and telegraph processes with brownian time. *Probab. Theory Relat. Fields* **2004**, *128*, 141–160.
8. Voroshilov, A.A.; Kilbas, A.A. The Cauchy problem for the diffusion-wave equation with the Caputo partial derivative. *Differ. Equ.* **2006**, *42*, 638–649. [CrossRef]
9. Atanackovic, T.M.; Pilipovic, S.; Zorica, D. A diffusion wave equation with two fractional derivatives of different order. *J. Phys. A Math. Theor.* **2007**, *40*, 5319–5333. [CrossRef]
10. Engler, H. Similiraty solutions for a class of hyperbolic integrodifferential equations. *Differ. Integr. Equ.* **1997**, *10*, 815–840.
11. Buckwar, E.; Luchko, Yu. Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl.* **1998**, *227*, 81–97. [CrossRef]
12. Gorenflo, R.; Luchko, Yu.; Mainardi, F. Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Comput. Appl. Math.* **2000**, *118*, 175–191. [CrossRef]
13. Gorenflo, R.; Iskenderov, A.; Luchko, Y. Mapping between solutions of fractional diffusion-wave equations. *Fract. Calcul. Appl. Anal.* **2000**, *3*, 75–86.
14. Shogenov, V.K.; Kumykova, S.K.; Shkhanukov-Lafishev, M.K. The generalized transport equation and fractional derivatives. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **1997**, *12*, 47–54.
15. Agrawal, O.P. Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam.* **2002**, *29*, 145–155. [CrossRef]
16. Chen, J.; Liu, F.; Anh, V. Analytical solution for the time-fractional telegraph equation by the method of separating variables. *J. Math. Anal. Appl.* **2008**, *338*, 1364–1377. [CrossRef]
17. Bazhlekova, E. On a nonlocal boundary value problem for the two-term time-fractional diffusion-wave equation. *AIP Conf. Proc.* **2013**, *1561*, 172–183.
18. Masaeva, O.K. Dirichlet problem for a nonlocal wave equation with Riemann- Liouville derivative. *Vestnik KRAUNC. Fiz. Mat. Nauki* **2019**, *27*, 6–11.
19. Luchko, Y. Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* **2009**, *351*, 218–223. [CrossRef]
20. Al-Refai, M.; Luchko, Y. Maximum principle for the fractional diffusion equations with the Riemann-Liouville fractional derivative and its applications. *Fract. Calcul. Appl. Anal.* **2014**, *17*, 483–498. [CrossRef]
21. Fedorov, V.E.; Streletskaya E.M. Initial-value problems for linear distributed-order differential equations in Banach spaces. *Electron. J. Differ. Equ.* **2018**, *176*, 1–17.
22. Kochubei, A.N. Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calc. Appl. Anal.* **2014**, *17*, 881–896. [CrossRef]

23. Pskhu, A.V. Solution of Boundary Value Problems for the Fractional Diffusion Equation by the Green Function Method. *Differ. Equ.* **2003**, *39*, 1509–1513. [CrossRef]
24. Pskhu, A.V. The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math.* **2009**, *73*, 351–392. [CrossRef]
25. Kemppainen, J. Properties of the single layer potential for the time fractional diffusion equation. *J. Integr. Equ. Appl.* **2011**, *23*, 541–563. [CrossRef]
26. Mamchuev, M.O. Necessary non-local conditions for a diffusion-wave equation. *Vestn. Samar. Gos. Univ. Estestvennonauchn. Ser.* **2014**, *7*, 45–59.
27. Pskhu, A.V. Fractional diffusion equation with discretely distributed differentiation operator. *Sib. Elektron. Mat. Izv.* **2016**, *13*, 1078–1098.
28. Mamchuev, M.O. Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients. *Differ. Equ.* **2016**, *52*, 789–797. [CrossRef]
29. Pskhu, A.V. The first boundary-value problem for a fractional diffusion-wave equation in a non-cylindrical domain. *Izv. Math.* **2017**, *81*, 1212–1233. [CrossRef]
30. Mamchuev, M.O. Solutions of the main boundary value problems for the time-fractional telegraph. Equation by the green function method *Fract. Calc. Appl. Anal.* **2017**, *20*, 190–211.
31. Kemppainen, J. Layer potentials for the time-fractional diffusion equation. In *Handbook of Fractional Calculus with Applications*; Volume 2: Fractional Differential Equations; Kochubei, A., Luchko, Y., Eds.; De Gruyter: Berlin, Germany, 2019; pp. 181–196.
32. Pskhu, A.V. Green function of the first boundary-value problem for the fractional diffusion wave equation in a multidimensional rectangular domain. *Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz. VINITI* **2019**, *167*, 52–61.
33. Eidelman, S.D.; Kochubei, A.N. Cauchy problem for fractional diffusion equations. *J. Differ. Equ.* **2004**, *199*, 211–255. [CrossRef]
34. Tuan, N.H.; Kirane, M.; Luu, V.C.H.; Bin-Mohsin, B. A regularization method for time-fractional linear inverse diffusion problems. *Electron. J. Differ. Equ.* **2016**, *290*, 1–18.
35. Pskhu, A.V. Solution of the First Boundary Value Problem for a Fractional-Order Diffusion Equation. *Differ. Equ.* **2003**, *39*, 1359–1363. [CrossRef]
36. Mamchuev, M.O. *Boundary Value Problems for Equations and Systems of Equations with the Partial Derivatives of Fractional Order*; Publishing house KBSC of RAS: Nalchik, Russia, 2013.
37. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka: Moscow, Russia, 2005.
38. Kilbas, A.A.; Srivastava, H.M.: Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
39. Kochubei, A.; Luchko, Y. (Eds.) *Handbook of Fractional Calculus with Applications*; Volume 2: Fractional Differential Equations; De Gruyter: Berlin, Germany, 2019.
40. Nakhushev, A.M. *Fractional Calculus and Its Applications*; Fizmatlit: Moscow, Russia, 2003.
41. Uchaikin, V.V. *Method of Fractional Derivatives*; Artishok: Ulyanovsk, Russia, 2008.
42. Tarasov, V.E. *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*; Higher Education Press: Beijing, China; Springer: Berlin/Heidelberg, 2010.
43. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*; World Scientific: Singapore, 2010.
44. Atanacković, T.M.; Pilipović, S.; Stanković B.; Zorica, D. *Fractional Calculus with Applications in Mechanics*, ISTE: London, UK; Wiley: Hoboken, NJ, USA, 2014.
45. Tarasov, V. (Ed.) *Handbook of Fractional Calculus with Applications*; Volume 4: Applications in Physics, Part A; De Gruyter: Berlin, Germany, 2019.
46. Tarasov, V. (Ed.) *Handbook of Fractional Calculus with Applications*; Volume 5: Applications in Physics, Part B; De Gruyter: Berlin, Germany, 2019.
47. Korbel, J.; Luchko , Y. Modeling of financial processes with a space-time fractional diffusion equation of varying order. *Fract. Calc. Appl. Anal.* **2016**, *19*, 1414–1433. [CrossRef]

48. Tarasov, V.E. On History of Mathematical Economics: Application of Fractional Calculus. *Mathematics* **2019**, *7*, 509. [CrossRef]
49. Aguilar, J.-P.; Korbel, J.; Luchko, Y. Applications of the Fractional Diffusion Equation to Option Pricing and Risk Calculations. *Mathematics* **2019**, *7*, 796. [CrossRef]
50. Il'in, A.M.; Kalashnikov, A.S.; Oleinik, O.A. Linear equations of the second order of parabolic type. *Russian Math. Surv.* **1962**, *17*, 1–143. [CrossRef]
51. Tikhonov, A.N.; Samarskii, A.A. *Equations of Mathematical Physics*; Dover Publications: New York, NY, USA, 2011.
52. Dzhrbashyan, M M.; Nersesyan, A.B. Fractional derivatives and the Cauchy problem for differential equations of fractional order. *Izv. Akad. Nauk Armenian SSR Matem.* **1968**, *3*, 3–29.
53. Wright, E.M. On the coefficients of power series having exponential singularities. *J. London Math. Soc.* **1933**, *8*, 71–79. [CrossRef]
54. Wright, E.M. The generalized Bessel function of order greater than one. *Quart. J. Math. Oxford Ser.* **1940**, *11*, 36–48. [CrossRef]
55. Hazewinkel, M., (Ed.). *Encyclopaedia of Mathematics*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK; Volume 6, 1990.