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A Strong Convergence Theorem under a New Shrinking Projection Method for Finite Families of Nonlinear Mappings in a Hilbert Space

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Abstract: In this paper, using a new shrinking projection method, we deal with the strong convergence for finding a common point of the sets of zero points of a maximal monotone mapping, common fixed points of a finite family of demimetric mappings and common zero points of a finite family of inverse strongly monotone mappings in a Hilbert space. Using this result, we get well-known and new strong convergence theorems in a Hilbert space.

Keywords: fixed point; demimetric mapping; inverse strongly monotone mapping; maximal monotone mapping; shrinking projection method; variational inequality problem

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to H$ be a mapping. Then we denote by $F(T)$ the set of fixed points of T. For a real number t with $0 \leq t < 1$, a mapping $U : C \to H$ is said to be a t-strict pseudo-contraction [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + t\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. In particular, if $t = 0$, then U is nonexpansive, i.e.,

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If U is a t-strict pseudo-contraction and $F(U) \neq \emptyset$, then we get that, for $x \in C$ and $p \in F(U)$,

$$\|Ux - p\|^2 \leq \|x - p\|^2 + t\|x - Ux\|^2.$$

From this inequality, we get that

$$\|Ux - x\|^2 + \|x - p\|^2 + 2\langle Ux - x, x - p\rangle \leq \|x - p\|^2 + t\|x - Ux\|^2.$$

Then we get that

$$2\langle x - Ux, x - p\rangle \geq (1 - t)\|x - Ux\|^2. \tag{1}$$

A mapping $U : C \to H$ is said to be generalized hybrid [2] if there exist real numbers α, β such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is said to be (α, β)-generalized hybrid. The class of generalized hybrid mappings covers several well-known mappings. A (1,0)-generalized hybrid mapping is nonexpansive . For $\alpha = 2$ and $\beta = 1$, it is nonspreading [3,4], i.e.,

$$2\|Ux - Uy\|^2 \le \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

For $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, it is also hybrid [5], i.e.,

$$3\|Ux - Uy\|^2 \le \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading mappings and hybrid mappings are not continuous; see [6]. If U is a generalized hybrid and $F(U) \neq \emptyset$, then we get that, for $x \in C$ and $p \in F(U)$,

$$\alpha\|p - Ux\|^2 + (1-\alpha)\|p - Ux\|^2 \le \beta\|p - x\|^2 + (1-\beta)\|p - x\|^2$$

and hence $\|Ux - p\|^2 \le \|x - p\|^2$. From this, we have that

$$2\langle x - p, x - Ux\rangle \ge \|x - Ux\|^2. \tag{2}$$

We also know that such a mapping exists in a Banach space. Let E be a smooth Banach space and let G be a maximal monotone mapping with $G^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of G for a positive number $\lambda > 0$, we obtain from [7,8] that, for $x \in E$ and $p \in G^{-1}0 = F(J_\lambda)$,

$$\langle J_\lambda x - p, J(x - J_\lambda x)\rangle \ge 0.$$

Then we get

$$\langle J_\lambda x - x + x - p, J(x - J_\lambda x)\rangle \ge 0$$

and hence

$$\langle x - p, J(x - J_\lambda x)\rangle \ge \|x - J_\lambda x\|^2, \tag{3}$$

where J is the duality mapping on E. Motivated by (1), (2) and (3), Takahashi [9] introduced a nonlinear mapping in a Banach space as follows: Let C be a nonempty, closed, and convex subset of a smooth Banach E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is said to be η-demimetric if, for $x \in C$ and $p \in F(U)$,

$$2\langle x - p, J(x - Ux)\rangle \ge (1-\eta)\|x - Ux\|^2.$$

According to this definition, we have that a t-strict pseudo-contraction U with $F(U) \neq \emptyset$ is t-demimetric, an (α, β)-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric and the metric resolvent J_λ with $G^{-1}0 \neq \emptyset$ is (-1)-demimetric. On the other hand, we know the shrinking projection method which was defined by Takahashi, Takeuchi, and Kubota [10] for finding fixed points of nonexpansive mappings in a Hilbert space. They proved the following strong convergence theorem [10].

Theorem 1 ([10]). *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $U : C \to C$ be a nonexpansive mapping. Assume that $F(U) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n U x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n = 1, 2, \ldots, \end{cases}$$

where a real number a and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following inequalities:

$$0 < a \leq \lambda_n \leq 1, \quad n = 1, 2, \ldots.$$

Then the sequence $\{x_n\}$ converges strongly to $u \in F(U)$, where $u = P_{F(U)} x_1$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

In this paper, using a new shrinking projection method, we prove a strong convergence theorem for finding a common point of the sets of zero points of a maximal monotone mapping, common fixed points for a finite family of demimetric mappings and common zero points of a finite family of inverse strongly monotone mappings in a Hilbert space. Using this result, we obtain well-known and new strong convergence theorems in a Hilbert space. In particular, using the shrinking projection method, we prove a strong convergence theorem for a finite family of generalized hybrid mappings with the variational inequalty problem in a Hilbert space.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$ and let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. When $\{x_n\}$ is a sequence in H, we denote by $x_n \to x$ the strong convergence of $\{x_n\}$ to $x \in H$ and by $x_n \rightharpoonup x$ the weak convergence. We have from [11,12] that, for $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{4}$$

Furthermore, we have that, for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{5}$$

Let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ with $F(U) \neq \emptyset$ is said to be quasi-nonexpansive if $\|Ux - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(U)$. If $U : C \to H$ is quasi- nonexpansive, then $F(U)$ is closed and convex; see [12,13]. For a nonempty, closed, and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is,

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall x \in H, \ y \in D. \tag{6}$$

A mapping P_D is said to be the metric projection of H onto D. The inequality (6) is equivalent to

$$\langle x - P_D x, y - P_D x\rangle \leq 0, \quad \forall x \in H, \ y \in D. \tag{7}$$

We obtain from (7) that P_D is firmly nonexpansive, that is,

$$\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle, \quad \forall x, y \in H.$$

In fact, from (7) we have that, for $x.y \in H$,

$$\langle x - P_D y + P_D y - P_D x, P_D y - P_D x\rangle \leq 0$$

and hence

$$\begin{aligned}\|P_D x - P_D y\|^2 &\leq \langle P_D x - P_D y, x - P_D y\rangle \\ &= \langle P_D x - P_D y, x - y + y - P_D y\rangle \\ &= \langle P_D x - P_D y, x - y\rangle + \langle P_D x - P_D y, y - P_D y\rangle \\ &\leq \langle P_D x - P_D y, x - y\rangle.\end{aligned}$$

Furthermore, using (7) and (5), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{8}$$

Let C be a nonempty, closed, and convex subset of H. A mapping $A : C \to H$ is said to be α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is an α-inverse-strongly monotone mapping and $0 < \mu \leq 2\alpha$, then we obtain from [12] that $I - \mu A : C \to H$ is nonexpansive, i.e.,

$$\|(I - \mu A)x - (I - \mu A)y\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{9}$$

For more results of inverse strongly monotone mappings, see also [12,14,15]. The variational inequalty problem for a nonlinear mapping $A : C \to H$ is to find an element $w \in C$ such that

$$\langle Aw, x - w\rangle \geq 0, \quad \forall x \in C. \tag{10}$$

The set of solutions of (10) is denoted by $VI(C, A)$. We also have that, for $\mu > 0$, $w = P_C(I - \mu A)w$ if and only if $w \in VI(C, A)$. In fact, let $\mu > 0$. Then, for $w \in C$,

$$\begin{aligned} w = P_C(I - \mu A)w &\Longleftrightarrow \langle (I - \mu A)w - w, w - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle -\mu Aw, w - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Aw, w - y\rangle \leq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Aw, y - w\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow w \in VI(C, A). \end{aligned} \tag{11}$$

Let G be a multi-valued mapping from H into H. The effective domain of G is denoted by $\mathrm{dom}(G)$, i.e., $\mathrm{dom}(G) = \{x \in H : Gx \neq \emptyset\}$. A multi-valued mapping $G \subset H \times H$ is called a monotone mapping on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(G)$, $u \in Gx$, and $v \in Gy$. A monotone mapping G on H is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on H. For a maximal monotone mapping G on H, we may define a single-valued mapping $J_r = (I + rG)^{-1} : H \to \mathrm{dom}(G)$, which is said to be the resolvent of G for $r > 0$. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of G for $r > 0$. We get from [8] that

$$A_r x \in G J_r x, \quad \forall x \in H,\ \ r > 0. \tag{12}$$

For a maximal monotone mapping G on H, let $G^{-1}0 = \{x \in H : 0 \in Gx\}$. It is known that $G^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive:

$$\|J_r x - J_r y\|^2 \leq \langle J_r x - J_r y, x - y\rangle, \quad \forall x, y \in H. \tag{13}$$

Takahashi, Takahashi, and Toyoda [16] proved the following result.

Lemma 1 ([16]). *Let G be a maximal monotone mapping on a Hilbert space H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following inequality holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 1, we get that, for $s, t > 0$ and $x \in H$,

$$\|J_s x - J_t x\|^2 \leq \frac{|s-t|}{s}\|J_s x - x\|\|J_s x - J_t x\|$$

and hence

$$\|J_s x - J_t x\| \leq \frac{|s-t|}{s}\|J_s x - J_t x\|. \tag{14}$$

Using the ideas of [17,18], Alsulami and Takahashi [19] proved the following lemma.

Lemma 2 ([19]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $G \subset H \times H$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $\kappa > 0$ and let $U : C \to H$ be a κ-inverse strongly monotone mapping. Suppose that $G^{-1}0 \cap U^{-1}0 \neq \emptyset$. Let $\lambda, r > 0$ and $z \in C$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rU)z$;
(ii) $0 \in Uz + Gz$;
(iii) $z \in G^{-1}0 \cap U^{-1}0$.

When a Banach space E is a Hilbert space, the definition of a demimetric mapping is as follows: Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\eta \in (-\infty, 1)$. A mapping $U : C \to H$ with $F(U) \neq \emptyset$ is said to be η-demimetric [9] if, for $x \in C$ and $q \in F(U)$,

$$\langle x - q, x - Ux\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2.$$

The following lemma which was essentially proved in [9] is important and crucial in the proof of the main result. For the sake of completeness, we give the proof.

Lemma 3 ([9]). *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let η be a real number with $\eta \in (-\infty, 1)$ and let U be an η-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

Proof. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$2\langle q - q_n, q - Uq\rangle \geq (1-\eta)\|q - Uq\|^2.$$

From $q_n \to q$, we have $0 \geq (1-\eta)\|q - Uq\|^2$. From $1 - \eta > 0$, we have $\|q - Uq\| = 0$ and hence $q = Uq$. This implies that $F(U)$ is closed.

Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $z = \alpha p + (1-\alpha)q$, where $\alpha \in [0, 1]$. Then we have that

$$2\langle z - p, z - Uz\rangle \geq (1-\eta)\|z - Uz\|^2 \text{ and } 2\langle z - q, z - Uz\rangle \geq (1-\eta)\|z - Uz\|^2.$$

From $\alpha \geq 0$ and $1 - \alpha \geq 0$, we also have that

$$2\langle \alpha z - \alpha p, z - Uz\rangle \geq \alpha(1-\eta)\|z - Uz\|^2$$

and $2\langle(1-\alpha)z-(1-\alpha)q, z-Uz\rangle \geq (1-\alpha)(1-\eta)\|z-Uz\|^2$. > From these inequalities, we get that

$$0 = 2\langle z-z, z-Uz\rangle \geq (1-\eta)\|z-Uz\|^2.$$

From $1-\eta > 0$ we get that $\|z-Uz\| = 0$ and hence $z = Uz$. This means that $F(U)$ is convex. □

Takahashi, Wen, and Yao [20] proved the following lemma which is also used in the proof of the main result.

Lemma 4 ([20]). *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\eta \in (-\infty, 1)$ and let a mapping $T : C \to H$ with $F(T) \neq \emptyset$ be η-demimetric. Let μ be a real number with $0 < \mu \leq 1-\eta$ and define $U = (1-\mu)I + \mu T$. Then U is a quasi-nonexpansive mapping of C into H.*

3. Main Result

In this section, using a new shrinking projection method, we obtain a strong convergence theorem for finding a common point of the sets of zero points of a maximal monotone mapping, common fixed points for a finite family of demimetric mappings and common zero points of a finite family of inverse strongly monotone mappings in a Hilbert space. Let C be a nonempty, closed and convex subset of a Hilbert space H. Then a mapping $T : C \to H$ is said to be demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Tx_n \to 0$, $w = Tw$ holds; see [21].

Theorem 2. *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\{k_1, \ldots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \ldots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j-demimetric and demiclosed mappings of C into itself and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Let A and G be maximal monotone mappings on H and let $J_r = (I + rA)^{-1}$ and $Q_\lambda = (I + \lambda G)^{-1}$ be the resolvents of A and G for $r > 0$ and $\lambda > 0$, respectively. Assume that*

$$\Omega = A^{-1}0 \cap (\cap_{j=1}^M F(T_j)) \cap (\cap_{i=1}^N (B_i + G)^{-1}0) \neq \emptyset.$$

For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ z_n = \sum_{i=1}^N \sigma_i Q_{\eta_n}(I - \eta_n B_i)y_n, \\ u_n = J_{r_n} z_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_n - z\| \leq \|x_n - z\|, \ \|z_n - z\| \leq \|y_n - z\| \\ \qquad\qquad\qquad \text{and } \ \langle z_n - z, z_n - u_n\rangle \geq \|z_n - u_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\}, \{r_n\} \subset (0, \infty)$, $\{\xi_1, \ldots, \xi_M\}, \{\sigma_1, \ldots, \sigma_N\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following:

(1) $0 < a \leq \lambda_n \leq \min\{1-k_1, \ldots, 1-k_M\}, \quad \forall n \in \mathbb{N}$;
(2) $0 < b \leq \eta_n \leq 2\min\{\mu_1, \ldots, \mu_N\}, \quad \forall n \in \mathbb{N}$;
(3) $0 < c \leq r_n, \quad \forall n \in \mathbb{N}$;
(4) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = P_\Omega x_1$.

Proof. Since a mapping B_i is μ_i-inverse strongly monotone for all $i \in \{1, \ldots, N\}$ and $0 < b \leq \eta_n \leq 2\mu_i$, we have that $Q_{\eta_n}(I - \eta_n B_i)$ is nonexpansive and

$$F(Q_{\eta_n}(I - \eta_n B_i)) = (B_i + G)^{-1}0$$

is closed and convex. Furthermore, we have from Lemma 3 that $F(T_j)$ is closed and convex. We also know that $A^{-1}0$ is closed and convex. Then,

$$\Omega = A^{-1}0 \cap (\cap_{j=1}^{M} F(T_j)) \cap (\cap_{i=1}^{N} (B_i + G)^{-1}0)$$

is nonempty, closed, and convex. Therefore, P_Ω is well defined.

We have that

$$\begin{aligned} \|y_n - z\| \le \|x_n - z\| &\iff \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\iff \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \le 0. \end{aligned}$$

Similarly, we have that

$$\|z_n - z\| \le \|y_n - z\| \iff \|z_n\|^2 - \|y_n\|^2 - 2\langle z_n - y_n, z\rangle \le 0.$$

Thus $\{z \in C : \|y_n - z\| \le \|x_n - z\| \text{ and } \|z_n - z\| \le \|y_n - z\|\}$ is closed and convex. We also have that $\{z \in C : \langle z_n - z, z_n - u_n\rangle \ge \|z_n - u_n\|^2\}$ is closed and convex. Then C_n is closed and convex for all $n \in \mathbb{N}$. Let us show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. We have that $\Omega \subset C_1 = C$. Assume that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. From Lemma 4 we have that, for $z \in \Omega$,

$$\begin{aligned} \|y_k - z\| &= \|\sum_{j=1}^{M} \xi_j((1-\lambda_k)I + \lambda_k T_j)x_k - z\| \\ &\le \sum_{j=1}^{M} \xi_j \|((1-\lambda_k)I + \lambda_k T_j)x_k - z\| \\ &\le \sum_{j=1}^{M} \xi_j \|x_k - z\| = \|x_k - z\|. \end{aligned} \tag{15}$$

Furthermore, since $Q_{\eta_k}(I - \eta_k B_i)$ is nonexpansive and hence quasi-nonexpansive, we have that, for $z \in \Omega$,

$$\begin{aligned} \|z_k - z\| &= \|\sum_{i=1}^{N} \sigma_i Q_{\eta_k}(I - \eta_k B_i)y_k - z\| \\ &\le \sum_{i=1}^{N} \sigma_i \|Q_{\eta_k}(I - \eta_k B_i)y_k - z\| \\ &\le \sum_{i=1}^{N} \sigma_i \|y_k - z\| = \|y_k - z\|. \end{aligned} \tag{16}$$

Since J_{r_k} is the resolvent of A and $u_k = J_{r_k} z_k$, we also have that

$$\langle z_k - J_{r_k} z_k, J_{r_k} z_k - z\rangle \ge 0, \quad \forall z \in \Omega.$$

From $\langle z_k - J_{r_k} z_k, J_{r_k} z_k - z_k + z_k - z\rangle \ge 0$, we have that

$$\langle z_k - J_{r_k} z_k, z_k - z\rangle \ge \|z_k - J_{r_k} z_k\|^2.$$

This implies that

$$\langle z_k - u_k, z_k - z\rangle \ge \|z_k - u_k\|^2.$$

From these, we have that $\Omega \subset C_{k+1}$. Therefore, we have by mathematical induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Thus $x_{n+1} = P_{C_{n+1}} x_1$ is well defined.

Since Ω is nonempty, closed, and convex, there exists $z_0 \in \Omega$ such that $z_0 = P_\Omega x_1$. By $x_{n+1} = P_{C_{n+1}} x_1$, we get that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z\|$$

for all $z \in C_{n+1}$. From $z_0 \in \Omega \subset C_{n+1}$ we obtain that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{17}$$

This implies that $\{x_n\}$ is bounded. Since $x_n = P_{C_n} x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get that

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|.$$

Thus $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then the limit of $\{\|x_1 - x_n\|\}$ exists. Put $\lim_{n\to\infty} \|x_n - x_1\| = c$. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $C_m \subset C_n$. >From $x_m = P_{C_m} x_1 \in C_m \subset C_n$ and (8), we have that

$$\|x_m - P_{C_n} x_1\|^2 + \|P_{C_n} x_1 - x_1\|^2 \leq \|x_1 - x_m\|^2.$$

This implies that

$$\|x_m - x_n\|^2 \leq \|x_1 - x_m\|^2 - \|x_n - x_1\|^2 \leq c^2 - \|x_n - x_1\|^2. \tag{18}$$

Since $c^2 - \|x_n - x_1\|^2 \to 0$ as $n \to \infty$, we have that $\{x_n\}$ is a Caushy sequence. Since H is complete and C is closed, there exists a point $u \in C$ such that $\lim_{n\to\infty} x_n = u$.

Using (18), we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By $x_{n+1} \in C_{n+1}$, we get that

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_n - x_{n+1}\|. \end{aligned} \tag{19}$$

This implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{20}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that $\|z_n - x_{n+1}\| \leq \|y_n - x_{n+1}\|$. We get from $\|y_n - x_{n+1}\| \to 0$ that $\|z_n - x_{n+1}\| \to 0$. From

$$\|y_n - z_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - z_n\|$$

we have that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{21}$$

Let us show $\|z_n - u_n\| \to 0$. We have from $x_{n+1} \in C_{n+1}$ that

$$\langle z_n - x_{n+1}, z_n - u_n \rangle \geq \|z_n - u_n\|^2.$$

Since $\|z_n - x_{n+1}\|\|z_n - u_n\| \geq \langle z_n - x_{n+1}, z_n - u_n \rangle \geq \|z_n - u_n\|^2$, we have that $\|z_n - x_{n+1}\| \geq \|z_n - u_n\|$. Then we get from $\|z_n - x_{n+1}\| \to 0$ that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{22}$$

Since T_j is k_j-demimetric for all $j \in \{1, \dots, M\}$, we get that, for $z \in \cap_{j=1}^M F(T_j)$,

$$\begin{aligned}
\langle x_n - z, x_n - y_n \rangle &= \langle x_n - z, x_n - \sum_{j=1}^{M} \xi_j ((1-\lambda_n)I + \lambda_n T_j) x_n \rangle \\
&= \sum_{j=1}^{M} \xi_j \langle x_n - z, x_n - ((1-\lambda_n)I + \lambda_n T_j) x_n \rangle \\
&= \sum_{j=1}^{M} \xi_j \lambda_n \langle x_n - z, x_n - T_j x_n \rangle \\
&\geq \sum_{j=1}^{M} \xi_j \lambda_n \frac{1-k_j}{2} \|x_n - T_j x_n\|^2 \\
&\geq \sum_{j=1}^{M} \xi_j a \frac{1-k_j}{2} \|x_n - T_j x_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty} \|y_n - x_n\| = 0$ that

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \quad \forall j \in \{1, \dots, M\}.$$

Since T_j are demiclosed for all $j \in \{1, \dots, M\}$ and $\lim_{n\to\infty} x_n = u$, we have that $u \in \cap_{j=1}^M F(T_j)$. Let us show that $u \in \cap_{i=1}^N (B_i + G)^{-1}0$. Since $Q_{\eta_n}(I - \eta_n B_i)$ is nonexpansive for all $i \in \{1, \dots, N\}$, we get that, for $z \in \cap_{i=1}^N (B_i + G)^{-1}0$,

$$\begin{aligned}
\langle y_n - z, y_n - z_n \rangle &= \langle y_n - z, y_n - \sum_{i=1}^{N} \sigma_i Q_{\eta_n}(I - \eta_n B_i) y_n \rangle \\
&= \sum_{i=1}^{N} \sigma_i \langle y_n - z, y_n - Q_{\eta_n}(I - \eta_n B_i) y_n \rangle \\
&\geq \sum_{i=1}^{N} \sigma_i \frac{1}{2} \|y_n - Q_{\eta_n}(I - \eta_n B_i) y_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty} \|y_n - z_n\| = 0$ that

$$\lim_{n\to\infty} \|y_n - Q_{\eta_n}(I - \eta_n B_i) y_n\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Since $\{\eta_n\}$ is bounded, we get that there exists a subsequence $\{\eta_{n_l}\}$ of $\{\eta_n\}$ such that $\lim_{l\to\infty} \eta_{n_l} = \eta$ and $0 < b \leq \eta \leq 2\min\{\mu_1, \dots, \mu_N\}$. For such η, we get that, for $i \in \{1, \dots, N\}$ and a subsequence $\{y_{n_l}\}$ of $\{y_n\}$ corresponding to the sequence $\{\eta_{n_l}\}$,

$$\begin{aligned}
\|y_{n_l} - Q_\eta(I - \eta B_i) y_{n_l}\| &\leq \|y_{n_l} - Q_{\eta_{n_l}}(I - \eta_{n_l} B_i) y_{n_l}\| \\
&\quad + \|Q_{\eta_{n_l}}(I - \eta_{n_l} B_i) y_{n_l} - Q_{\eta_{n_l}}(I - \eta B_i) y_{n_l}\| \\
&\quad + \|Q_{\eta_{n_l}}(I - \eta B_i) y_{n_l} - Q_\eta(I - \eta B_i) y_{n_l}\| \\
&\leq \|y_{n_l} - Q_{\eta_{n_l}}(I - \eta_{n_l} B_i) y_{n_l}\| \\
&\quad + \|(I - \eta_{n_l} B_i) y_{n_l} - (I - \eta B_i) y_{n_l}\| \\
&\quad + \|Q_{\eta_{n_l}}(I - \eta B_i) y_{n_l} - Q_\eta(I - \eta B_i) y_{n_l}\| \\
&\leq \|y_{n_l} - Q_{\eta_{n_l}}(I - \eta_{n_l} B_i) y_{n_l}\| + |\eta_{n_l} - \eta| \|B_i y_{n_l}\| \\
&\quad + \frac{|\eta_{n_l} - \eta|}{\eta} \|Q_\eta(I - \eta B_i) y_{n_l} - (I - \eta B_i) y_{n_l}\|.
\end{aligned}$$

On the other hand, we get that, for a fixed $y \in C$ and $i \in \{1, \dots, N\}$,

$$\begin{aligned} b\|B_i y_n\| \le \eta_n \|B_i y_n\| &= \|\eta_n B_i y_n\| \\ &= \|y_n - (y - \eta_n B_i y) + y - \eta_n B_i y - (y_n - \eta_n B_i y_n)\| \\ &\le \|y_n - y\| + \eta_n \|B_i y\| + \|(I - \eta_n B_i) y - (I - \eta_n B_i) y_n\| \\ &\le \|y_n - y\| + 2\min\{\mu_1, \dots, \mu_N\}\|B_i y\| + \|y - y_n\|. \end{aligned}$$

Since $\{y_n\}$ is bounded, we have that $\{B_i y_n\}$ is bounded for all $i \in \{1, \dots, N\}$. Thus we get that

$$\lim_{l \to \infty} \|x_{n_l} - Q_\eta (I - \eta B_i) x_{n_l}\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Since $\lim_{l \to \infty} x_{n_l} = u$ and $Q_\eta (I - \eta B_i)$ are demiclosed for all $i \in \{1, \dots, N\}$, we get $u \in \cap_{i=1}^N (B_i + G)^{-1} 0$. Let us show $u \in A^{-1}0$. We have from (22) that

$$\lim_{n \to \infty} \|z_n - u_n\| = 0.$$

Using $r_n \ge c$, we get

$$\lim_{n \to \infty} \frac{1}{r_n} \|z_n - u_n\| = 0.$$

Therefore, we have

$$\lim_{n \to \infty} \|A_{r_n} z_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|z_n - u_n\| = 0.$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - u_n, p^* - A_{r_n} z_n \rangle \ge 0$ for all $n \in \mathbb{N}$. Since $z_n \to u$ and hence $u_n \to u$, we get $\langle p - u, p^* \rangle \ge 0$. From the maximallity of A, we have $u \in A^{-1}0$. Therefore, we have $u \in \Omega$.

Since $z_0 = P_\Omega x_1$, $u \in \Omega$ and $x_n \to u$, we have from (17) that

$$\|x_1 - z_0\| \le \|x_1 - u\| = \lim_{n \to \infty} \|x_1 - x_n\| \le \|x_1 - z_0\|.$$

Then $u = z_0$. Therefore, we have $x_n \to u = z_0$. This completes the proof. □

4. Applications

In this section, using Theorem 2, we obtain well-known and new strong convergence theorems in Hilbert spaces. We know the following lemma proved by Marino and Xu [22]; see also [23]. For the sake of completeness, we give the proof.

Lemma 5 ([22,23])**.** *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup u$ and $x_n - Ux_n \to 0$, then $u \in F(U)$.*

Proof. Let us show that a nonexpansive mapping $T : C \to H$ is demiclosed. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. We have that

$$\begin{aligned} \|u - Tu\|^2 &= \|u - x_n + x_n - Tu\|^2 \\ &= \|u - x_n\|^2 + \|x_n - Tu\|^2 + 2\langle u - x_n, x_n - Tu \rangle \\ &= \|u - x_n\|^2 + \|x_n - Tx_n + Tx_n - Tu\|^2 + 2\langle u - x_n, x_n - u + u - Tu \rangle \\ &= \|u - x_n\|^2 + \|x_n - Tx_n\|^2 + \|Tx_n - Tu\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu \rangle \\ &\quad - 2\|u - x_n\|^2 + 2\langle u - x_n, u - Tu \rangle \\ &\le \|u - x_n\|^2 + \|x_n - Tx_n\|^2 + \|x_n - u\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu \rangle \end{aligned}$$

$$
\begin{aligned}
&- 2\|u - x_n\|^2 + 2\langle u - x_n, u - Tu\rangle \\
&= \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, Tx_n - Tu\rangle + 2\langle u - x_n, u - Tu\rangle \to 0.
\end{aligned}
$$

Then, $u = Tu$. It is obvious that a mapping $B = I - U : C \to H$ is $\frac{1-k}{2}$-inverse strongly monotone. Put $\alpha = \frac{1-k}{2}$. We have that

$$
\alpha\|Bx - By\|^2 \leq \langle x - y, Bx - By\rangle, \quad \forall x, y \in C. \tag{23}
$$

From $U = I - B$ and (9), we have that

$$
I - 2\alpha B = I - 2\alpha(I - U) = (1 - 2\alpha)I + 2\alpha U
$$

is nonexpansive. If $x_n \rightharpoonup u$ and $x_n - Ux_n \to 0$, then

$$
x_n - ((1 - 2\alpha)I + 2\alpha U)x_n = 2\alpha(I - U)x_n \to 0.
$$

Since $(1 - 2\alpha)I + 2\alpha U$ is nonexpansive, we have $u \in F((1 - 2\alpha)I + 2\alpha U) = F(U)$. This implies that U is demiclosed. $\square$

Furthermore, we know the following lemma from Kocourek, Takahashi, and Yao [2]; see also [24].

Lemma 6 ([2,24])**.** *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup u$ and $x_n - Ux_n \to 0$, then $u \in F(U)$.*

We prove a strong convergence theorem for a finite family of strict pseudo-contractions in a Hilbert space.

Theorem 3. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{k_1, \ldots, k_M\} \subset [0, 1)$ and let $\{T_j\}_{j=1}^M$ be a finite family of k_j-strict pseudo-contractions of C into itself. Assume that $\cap_{j=1}^M F(T_j) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by*

$$
\begin{cases}
y_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $a \in \mathbb{R}$, $\{\lambda_n\} \subset (0, \infty)$ and $\{\xi_1, \ldots, \xi_M\} \subset (0, 1)$ satisfy the following:

(1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \ldots, 1 - k_M\}, \quad \forall n \in \mathbb{N}$;
(2) $\sum_{j=1}^M \xi_j = 1$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j)$, where $z_0 = P_{\cap_{j=1}^M F(T_j)} x_1$.

Proof. Since T_j is a k_j-strict pseudo-contraction of C into itself with $F(T_j) \neq \emptyset$, from (1), T_j is a k_j-demimetric mapping. Furthermore, we have from Lemma 5 that T_j is demiclosed. We also have that if $B_i = 0$ for all $i \in \{1, \ldots, N\}$ in Theorem 2, then B_i is a 1-inverse strongly monotone mapping. Putting $\eta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 2, we have that $z_n = y_n$ for all $n \in \mathbb{N}$. Furthermore, putting $A = G = 0$ and $\eta_n = r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 2, we have that

$$
Q_{v_n} = J_{r_n} = I, \quad \forall v_n > 0, \ r_n > 0.
$$

Then we have that $u_n = z_n = y_n$ for all $n \in \mathbb{N}$. Thus, we get the desired result from Theorem 2. $\square$

As a direct result of Theorem 3, we have Theorem 1 in Introduction. We can also prove the following strong convergence theorem for a finite family of inverse strongly monotone mappings in a

Hilbert space. Let g be a proper, lower semicontinuous and convex function of a Hilbert space H into $(-\infty, \infty]$. The subdifferential ∂g of g is defined as follows:

$$\partial g(x) = \{z \in H : g(x) + \langle z, y - x\rangle \leq g(y),\ \forall y \in H\}$$

for all $x \in H$. We have from Rockafellar [25] that ∂g is a maximal monotone mapping. Let D be a nonempty, closed, and convex subset of a Hilbert space H and let i_D be the indicator function of D, i.e.,

$$i_D(x) = \begin{cases} 0, & x \in D, \\ \infty, & x \notin D. \end{cases}$$

Then i_D is a proper, lower semicontinuous and convex function on H and then the subdifferential ∂i_D of i_D is a maximal monotone mapping. Thus we define the resolvent J_λ of ∂i_D for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_D)^{-1} x$$

for all $x \in H$. We get that, for $x \in H$ and $u \in D$,

$$\begin{aligned} u = & J_\lambda x \Longleftrightarrow x \in u + \lambda \partial i_D u \Longleftrightarrow x \in u + \lambda N_D\, u \\ & \Longleftrightarrow x - u \in \lambda N_D\, u \\ & \Longleftrightarrow \frac{1}{\lambda}\langle x - u, v - u\rangle \leq 0,\ \forall v \in\ D \\ & \Longleftrightarrow \langle x - u, v - u\rangle \leq 0,\ \forall v \in\ D \\ & \Longleftrightarrow u = P_D x, \end{aligned}$$

where $N_D u$ is the normal cone to D at u, i.e.,

$$N_D u = \{z \in H : \langle z, v - u\rangle \leq 0,\ \forall v \in D\}.$$

Theorem 4. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{\mu_1, \ldots, \mu_N\} \subset (0, \infty)$. Let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that $\cap_{i=1}^N VI(C, B_i) \neq \emptyset$. Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $b \in \mathbb{R}$, $\{\eta_n\} \subset (0, \infty)$ and $\{\sigma_1, \ldots, \sigma_N\} \subset (0, 1)$ satisfy the following:

(1) $0 < b \leq \eta_n \leq 2\min\{\mu_1, \ldots, \mu_N\}, \quad \forall n \in \mathbb{N};$
(2) $\sum_{i=1}^N \sigma_i = 1.$

Then $\{x_n\}$ converges strongly to $z_0 \in \cap_{i=1}^N VI(C, B_i)$, where $z_0 = P_{\cap_{i=1}^N VI(C,B_i)} x_1$.

Proof. Putting $G = \partial i_C$ in Theorem 2, we get that for $\eta_n > 0$, $J_{\eta_n} = P_C$. Furthermore, we have $(\partial i_C)^{-1}0 = C$ and $(B_i + \partial i_C)^{-1}0 = VI(C, B_i)$. In fact, we get that, for $z \in C$,

$$\begin{aligned} z \in (B_i + \partial i_C)^{-1}0 & \Longleftrightarrow 0 \in B_i z + \partial\, i_C z \\ & \Longleftrightarrow 0 \in B_i z + N_C z \Longleftrightarrow -B_i z \in\ N_C z \\ & \Longleftrightarrow \langle -B_i z, v - z\rangle \leq 0,\ \forall v \in\ C \\ & \Longleftrightarrow \langle B_i z, v - z\rangle \geq 0,\ \forall v \in\ C \end{aligned}$$

$$\Longleftrightarrow z \in VI(C, B_i).$$

The identity mapping I is a $\frac{1}{2}$-demimetric mapping of C into H. Put $T_j = I$ for all $j \in \{1, \dots, M\}$ and $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 2. Then we get that $y_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, putting $A = 0$, we have $u_n = z_n$. Thus, we get the desired result from Theorem 2. □

We prove a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 5. *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into itself and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Suppose that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ z_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)y_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\| \text{ and } \|z_n - z\| \leq \|y_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq 1, \quad \forall n \in \mathbb{N};$
(2) $0 < b \leq \eta_n \leq 2\min\{\mu_1, \dots, \mu_N\}, \quad \forall n \in \mathbb{N};$
(3) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1.$

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C,B_i))} x_1$.

Proof. Since T_j is a generalized hybrid mapping of C into itself such that $F(T_j) \neq \emptyset$, from (2), T_j is 0-demimetric. Furthermore, from Lemma 6, T_j is demiclosed. Furtheremore, put $G = \partial i_C$ as in the proof of Theorem 4. Then we have that $Q_{\eta_n}(I - \eta_n B_i) = P_C(I - \eta_n B_i)$ in Theorem 2. We also have that if $A = 0$, then $J_{r_n} = I$ and $u_n = z_n$. Therefore, we get the desired result from Theorem 2. □

We prove a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of nonexpansive mappings in a Hilbert space.

Theorem 6. *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into itself and let $\{U_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H. Suppose that $\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i)) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ z_n = \sum_{i=1}^N \sigma_i((1-\eta_n)I + \eta_n U_i)y_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\| \text{ and } \|z_n - z\| \leq \|y_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$ and $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq 1, \quad \forall n \in \mathbb{N};$
(2) $0 < b \leq \eta_n \leq 1, \quad \forall n \in \mathbb{N};$
(3) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1.$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} F(U_i))$*, where* $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} F(U_i))} x_1$.

Proof. As in the proof of Theorem 5, T_j is 0-demimetric and demiclosed. Since U_i is nonexpansive, $B_i = I - U_i$ is a $\frac{1}{2}$-inverse strongly monotone mapping. Furthermore, we get that

$$I - \eta_n B_i = I - \eta_n (I - U_i) = (1 - \eta_n) I + \eta_n U_i.$$

Putting $A = G = 0$, we get the desired result from Theorem 2. □

We finally prove a strong convergence theorem for resolvents of a maximal monotone mapping in a Hilbert space.

Theorem 7. *Let H be a Hilbert space. Let A be a maximal monotone mapping on H and let $J_r = (I + rA)^{-1}$ be the resolvents of A for $r > 0$. Suppose that $A^{-1}0 \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} u_n = J_{r_n} x_n, \\ C_{n+1} = \{z \in C_n : \langle x_n - z, x_n - u_n \rangle \geq \|x_n - u_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $c \in \mathbb{R}$ and $\{r_n\} \subset (0, \infty)$ satisfy the following:

$$0 < c \leq r_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0$, where $z_0 = P_{A^{-1}0} x_1$.

Proof. Put $T_j = I$ and $B_i = 0$ for all $j \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$ in Theorem 2. Furthermore, put $G = 0$. Then we have that $x_n = y_n = z_n$. Thus we get the desired result from Theorem 2. □

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