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Some Fractional Dynamic Inequalities of Hardy's Type Via Conformable Calculus

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Abstract: In this article, we prove some new fractional dynamic inequalities on time scales via conformable calculus. By using chain rule and Hölder's inequality on timescales we establish the main results. When $\alpha = 1$ we obtain some well-known time-scale inequalities due to Hardy, Copson, Bennett and Leindler inequalities.

Keywords: fractional hardy's inequality; fractional bennett's inequality; fractional copson's inequality; fractional leindler's inequality; timescales; conformable fractional calculus; fractional hölder inequality

MSC: 26A15; 26D10; 26D15; 39A13; 34A40; 34N05

In 1920, Hardy [1] established the inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} w(i)\right)^{k} \leq \left(\frac{k}{k-1}\right)^{k}\sum_{n=1}^{\infty} w^{k}(n), \quad k>1. \tag{1}$$

where $w(n)$ is a positive sequence defined for all $n \geq 1$. After that, Hardy [2], by using the calculus of variations, proved the continuous inequality of (1) which has the form

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x} g(s)ds\right)^{k} dx \leq \left(\frac{k}{k-1}\right)^{k}\int_0^{\infty} g^{k}(x)dx, \tag{2}$$

for a given positive function g, which is integrable over $(0, x)$, and g^k is convergent and integrable over $(0, \infty)$ and $k > 1$. In (1) and (2), $(k/\,(k-1))^k$ is a sharp constant. As a generalization of (2), Hardy [3] showed that when $k > 1$, then

$$\int_0^{\infty} x^{-h}\left(\int_0^{x} g(s)ds\right)^{k} dx \leq \left(\frac{k}{h-1}\right)^{k}\int_0^{\infty} x^{k-h} g^{k}(x)dx, \text{ for } h>1, \tag{3}$$

and

$$\int_0^{\infty} x^{-h}\left(\int_x^{\infty} g(s)ds\right)^{k} dx \leq \left(\frac{k}{1-h}\right)^{k}\int_0^{\infty} x^{k-h} g^{k}(x)dx, \text{for } h<1. \tag{4}$$

The constants $(k/(h-1))^k$ and $(k/(1-h))^k$ in (3) and (4) are the best possible. Copson [4] demonstrated that if $g(x) > 0$, $k > 1$ and $g^k(x)$ is integrable on the interval $(0, \infty)$, then

$$\int_x^\infty \left(\frac{g(s)}{s}\right) ds,$$

converges for $x > 0$ and

$$\int_0^\infty \left(\int_x^\infty \frac{g(s)}{s} ds\right)^k dx \le k^k \int_0^\infty g^k(x) dx, \tag{5}$$

where k^k is the best possible constant. Some of the generalizations of the discrete Hardy inequality (1) and the discrete version of (5) and its extensions are due to Leindler, we refer to the papers the papers [5–8]. For example, Leindler in [5] proved that if $p > 1$, $\lambda(n)$, $g(n) > 0$, then

$$\sum_{n=1}^\infty \lambda(n) \left(\sum_{s=1}^n g(s)\right)^p \le p^p \sum_{n=1}^\infty \lambda^{1-p}(n) \left(\sum_{s=n}^\infty \lambda(s)\right)^p g^p(n), \tag{6}$$

and

$$\sum_{n=1}^\infty \lambda(n) \left(\sum_{k=n}^\infty g(k)\right)^p \le p^p \sum_{n=1}^\infty \lambda^{1-p}(n) \left(\sum_{k=1}^n \lambda(k)\right)^p g^p(n). \tag{7}$$

The converses of (6) and (7) are proved by Leindler in [6]. He proved that if $0 < p \le 1$, then

$$\sum_{n=1}^\infty \lambda(n) \left(\sum_{k=1}^n g(k)\right)^p \ge p^p \sum_{n=1}^\infty \lambda^{1-p}(n) \left(\sum_{k=n}^\infty \lambda(k)\right)^p g^p(n), \tag{8}$$

and

$$\sum_{n=1}^\infty \lambda(n) \left(\sum_{k=n}^\infty g(k)\right)^p \ge p^p \sum_{n=1}^\infty \lambda^{1-p}(n) \left(\sum_{k=1}^n \lambda(p)\right)^p g^p(n). \tag{9}$$

For more generalization Copson in [9] showed that if $k > 1$, $\lambda(j) \ge 0$, $w(j) \ge 0$, $\forall j \ge 1$, $\Omega(m) = \sum_{j=1}^m \lambda(j)$, and $h > 1$, then

$$\sum_{m=1}^\infty \frac{\lambda(m)}{\Omega^h(m)} \left(\sum_{j=1}^m w(j)\lambda(j)\right)^k \le \left(\frac{k}{h-1}\right)^k \sum_{m=1}^\infty \lambda(m)\Omega^{k-h}(m) w^k(m), \tag{10}$$

and if $0 \le h < 1$and $k > 1$, then

$$\sum_{m=1}^\infty \frac{\lambda(m)}{\Omega^h(m)} \left(\sum_{j=m}^\infty w(j)\lambda(j)\right)^k \le \left(\frac{k}{1-h}\right)^k \sum_{m=1}^\infty \lambda(m)\Omega^{k-h}(m) w^k(m). \tag{11}$$

The integral versions of the inequalities (10) and (11) was proved by Copson in [10] (Theorems 1 and 3). In particular, he proved that if $k \ge 1$, $h > 1$, and $\Omega(s) = \int_0^s \lambda(t)dt$, then

$$\int_0^\infty \frac{\lambda(s)}{\Omega^h(s)} \Phi^k(s) ds \le \left(\frac{k}{h-1}\right)^k \int_0^\infty \frac{\lambda(s)}{\Omega^{h-k}(s)} g^k(s) ds, \tag{12}$$

where $\Phi(s) = \int_0^s \lambda(t)g(t)dt$, and if $k > 1$, $0 \le h < 1$, then

$$\int_0^\infty \frac{\lambda(s)}{\Omega^h(s)} \Phi^k(s) ds \le \left(\frac{k}{1-h}\right)^k \int_0^\infty \frac{\lambda(s)}{\Omega^{h-k}(s)} g^k(s) ds, \tag{13}$$

where $\Phi(s) = \int_s^\infty \lambda(t)g(t)dt$. Leindler in [5] and Bennett in [11] presented interesting different inequalities. Leindler established that if $k > 1$, $\Omega^*(m) = \sum_{j=m}^{\infty} \lambda(j) < \infty$, and $0 \leq h < 1$, then

$$\sum_{m=1}^{\infty} \frac{\lambda(m)}{(\Omega^*(m))^h} \left(\sum_{j=1}^{m} w(j)\lambda(j) \right)^k \leq \left(\frac{k}{1-h} \right)^k \sum_{m=1}^{\infty} \lambda(m)(\Omega^*(m))^{k-h} w^k(m), \tag{14}$$

and Bennett in [11] showed that if $1 < h \leq k$, then

$$\sum_{m=1}^{\infty} \frac{\lambda(m)}{(\Omega^*(m))^h} \left(\sum_{j=m}^{\infty} w(j)\lambda(j) \right)^k \leq \left(\frac{k}{h-1} \right)^k \sum_{m=1}^{\infty} \lambda(m)(\Omega^*(m))^{k-h} w^k(m). \tag{15}$$

In last decades, studying the dynamic equations and inequalities on time scales become a main field in applied and pure mathematics, we refer to [12–14] and the references they are cited. In fact, the book [13] includes forms of the above inequalities on time-scale and their extensions. The timescales idea is returned to Stefan Hilger [15], who investigated the research of dynamic equations on timescales. The books by Bohner and Peterson in [16,17] summarized and organized most timescales calculus. The three most common timescales calculuses are difference, differential, and quantum calculus (see [18]), i.e., at $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^s : s \in \mathbb{N}_0\}$ where $q > 1$.

In recent years, a lot of work has been published for fractional inequalities and the subject becomes an active field of research and several authors were interested in proving inequalities of fractional type by using the Riemann-Liouville and Caputo derivative (see [19–21]).

On the other hand, the authors in [22,23] introduced a new fractional calculus called the conformable calculus and gave a new definition of the derivative with the base properties of the calculus based on the new definition of derivative and integrals. By using conformable calculus, some authors have studied classical inequalities like Chebyshev's inequality [24], Hermite-Hadamard's inequality [25–27], Opial's inequality [28,29] and Steffensen's inequality [30].

The main question that arises now is: Is it possible to prove new fractional inequalities on timescales and give a unified approach of such studies? This in fact needs a new fractional calculus on timescales. Very recently Torres and others, in [31,32], combined a time scale calculus and conformable calculus and obtained the new fractional calculus on timescales. So, it is natural to look on new fractional inequalities on timescales and give an affirmative answer to the above question.

In particular, in this paper, we will prove the fractional forms of the classical Hardy, Bennett, Copson and Leindler inequalities. The paper is divided into two sections. Section 2 is an introduction of basics of fractional calculus on timescales and Section 3 contains the main results.

1. Preliminaries and Basic Lemmas

We present the fundamental results about the fractional timescales calculus. The results are adapted from [16,17,31,32]. A time-scale $\mathbb{T}$ is non-empty closed subset of $\mathbb{R}$ ($\mathbb{R}$ is the real numbers). The operators of backward jump and forward jump express of the closest point $t \in \mathbb{T}$ on the right and left of t is defined by, respectively:

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{16}$$

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{17}$$

where $\sup \phi = \inf \mathbb{T}$ and $\inf \phi = \sup \mathbb{T}$ (ϕ denotes the empty set), for any $t \in \mathbb{T}$ the notation $f^\sigma(t)$ refer to $f(\sigma(t))$, i.e., $f^\sigma = f \circ \sigma$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$.

Definition 1. *The number $T_\alpha^\Delta(f)(t)$ (provided it exists) of the function $f:\mathbb{T}\to\mathbb{R}$, for $t>0$ and $\alpha\in(0,1]$ is the number which has the property that for any $\epsilon>0$, there exists a neighborhood U of t S. T.*

$$\left|[f^\sigma(t)-f(s)]t^{1-\alpha}-T_\alpha^\Delta(f(t))(\sigma(t)-s)\right|\le\epsilon\,|\sigma(t)-s)|\,,\quad \textit{for all } t\in U$$

$T_\alpha^\Delta(f(t))$ is called the conformable $\alpha-$fractional derivative of function f of order α at t , for conformable fractional derivative on $\mathbb{T}$ at 0, we define it with $T_\alpha^\Delta(f(0)=\lim_{t\to 0^+}T_\alpha^\Delta(f(t))$.

The conformable fractional derivative has the following properties

Theorem 1. *Let v, $u:\mathbb{T}\to\mathbb{R}$ are conformable fractional derivative from order $\alpha\in(0,1]$, then the following properties are hold:*

(i) *The $v+u:\mathbb{T}\to\mathbb{R}$ is conformable fractional derivative and*

$$T_\alpha^\Delta(v+u)=T_\alpha^\Delta(v)+T_\alpha^\Delta(u).$$

(ii) *Fora all $k\in\mathbb{R}$, then $kv:\mathbb{T}\to\mathbb{R}$ is $\alpha-$fractional differentiable and*

$$T_\alpha^\Delta(kv)=kT_\alpha^\Delta(v).$$

(iii) *If v and u are $\alpha-$fractional differentiable, we have $vu:\mathbb{T}\to\mathbb{R}$ is $\alpha-$fractional differentiable and*

$$T_\alpha^\Delta(vu)=T_\alpha^\Delta(v)\;u+(v\circ\sigma)\;T_\alpha^\Delta(u)=T_\alpha^\Delta(v)\;(u\circ\sigma)+v\;T_\alpha^\Delta(u).$$

(iv) *If v is $\alpha-$fractional differentiable, then $1/v$ is $\alpha-$fractional differentiable with*

$$T\left(\frac{1}{v}\right)=-\frac{T_\alpha^\Delta(v)}{v\;(v\circ\sigma)}.$$

(v) *If v and u are $\alpha-$fractional differentiable, then v/u is $\alpha-$fractional differentiable with*

$$T_\alpha^\Delta(v/u)=\frac{T_\alpha^\Delta(v)u-vT_\alpha^\Delta(u)}{u(u\circ\sigma)},$$

valid $\forall\ t\in\mathbb{T}^k$,where $u(t)(u(\sigma(t))\neq 0$.

Lemma 1. *Let $v:\mathbb{T}\to\mathbb{R}$ is continuous and $\alpha-$fractional differentiable at $t\in\mathbb{T}$ for $\alpha\in(0,1]$, and $u:\mathbb{R}\to\mathbb{R}$ is continuous and differentiable. Then there exists $d\in[t,\sigma(t)]$ with*

$$T_\alpha^\Delta(u\circ v)(t)=u^{'}(v(d))T_\alpha^\Delta(v(t)).\tag{18}$$

Lemma 2. *Let $u:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $\alpha\in(0,1]$, and $v:\mathbb{T}\to\mathbb{R}$ be $\alpha-$fractional differentiable. Then $(u\circ v):\mathbb{T}\to\mathbb{R}$ is $\alpha-$fractional differentiable and we have*

$$T_\alpha^\Delta(u\circ v)(s)=\left(\int_0^1 u^{'}\left(v(s)+h\mu(s)s^{\alpha-1}T_\alpha^\Delta(v(s))\right)dh\right)T_\alpha^\Delta(v(s)).\tag{19}$$

Definition 2. *Let $0<\alpha\le 1$, the $\alpha-$fractional integral of f, is defined as*

$$\int f(s)\Delta_\alpha s=\int f(s)s^{\alpha-1}\Delta s.$$

The conformable fractional integral satisfying the next properties

Theorem 2. *Assume $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$. Let $u, v : \mathbb{T} \to \mathbb{R}$. Then*

(i) $\int_a^b [v(s) + u(s)]\Delta_\alpha s = \int_a^b v(s)\Delta_\alpha s + \int_a^b u(s)\Delta_\alpha s.$
(ii) $\int_a^b \lambda v(s)\Delta_\alpha s = \lambda \int_a^b v(s)\Delta_\alpha s.$
(iii) $\int_a^b v(s)\Delta_\alpha s = -\int_b^a v(s)\Delta_\alpha s.$
(iv) $\int_a^b v(s)\Delta_\alpha s = \int_a^c v(s)\Delta_\alpha s + \int_c^b v(s)\Delta_\alpha s.$
(v) $\int_a^a v(s)\Delta_\alpha s = 0.$

Lemma 3. *Assume $\mathbb{T}$ be a time-scale, $a, b \in \mathbb{T}$ where $b > a$. Let u, v are conformable $\alpha-$ fractional differentiable, $\alpha \in (0, 1]$. Then the formula of integration by parts is given by*

$$\int_a^b v(s) T_\alpha^\Delta u(s)\Delta_\alpha s = [v(s)u(s)]_a^b - \int_a^b u^\sigma(s) T_\alpha^\Delta v(s)\Delta_\alpha s. \tag{20}$$

Lemma 4. *Assume $\mathbb{T}$ be a time-scale, $a, b \in \mathbb{T}$ and $\alpha \in (0, 1]$. Let $u, v : \mathbb{T} \to \mathbb{R}$. Then*

$$\int_a^b |v(s)u(s)|\Delta_\alpha s \leq \left[\int_a^b |v(s)|^k \Delta_\alpha s\right]^{\frac{1}{k}} \left[\int_a^b |u(s)|^l \Delta_\alpha s\right]^{\frac{1}{l}}, \tag{21}$$

where $k > 1$ and $1/k + 1/l = 1$.

2. Main Results

Throughout the paper, we will assume that the functions are nonnegative on $[a, \infty)_{\mathbb{T}}$ and its integrals exist and are finite. We start with the fractional time-scale inequality of Copson's type.

Theorem 3. *Assume $1 < c < k$, define*

$$\Phi(x) := \int_a^x \lambda(s)\Delta_\alpha s \text{ and } \Omega(x) := \int_a^x \lambda(s)g(s)\Delta_\alpha s.$$

If

$$\Omega(\infty) < \infty, \text{ and } \int_a^\infty \frac{\lambda(s)}{(\Phi^\sigma(s))^{c-\alpha+1}}\Delta_\alpha s < \infty,$$

then

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty \frac{\lambda(x)\Phi^{k(\alpha-c)}(x)}{(\Phi^\sigma(x))^{(c-\alpha+1)(1-k)}} g^k(x)\Delta_\alpha x. \tag{22}$$

Proof. By employing the formula of integration by parts (20) on the term

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x,$$

with $u^\sigma(x) = (\Omega^\sigma(x))^k$ and $x_\alpha^\Delta v(x) = \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}$, we have that

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x = v(x)\Omega^k(x)\Big|_a^\infty + \int_a^\infty -v(x) x_\alpha^\Delta \left(\Omega^k(x)\right) \Delta_\alpha x, \tag{23}$$

where

$$-v(x) = \int_x^\infty \frac{\lambda(s)}{(\Phi^\sigma(s))^{c-\alpha+1}}\Delta_\alpha s = \int_x^\infty x_\alpha^\Delta \Phi(s) \left(\Phi^\sigma(s)\right)^{\alpha-c-1} \Delta_\alpha s.$$

By using the chain rule (18), we obtain that

$$\begin{aligned}
-x_{\alpha}^{\Delta}\left(\Phi^{\alpha-c}(x)\right) &= -(\alpha-c)\Phi^{\alpha-c-1}(d)x_{\alpha}^{\Delta}\Phi(x), \text{ where } d\in[x,\sigma(x)] \\
&= \frac{(c-\alpha)\, x_{\alpha}^{\Delta}\Phi(x)}{\Phi^{c-\alpha+1}(d)} \\
&\geq \frac{(c-\alpha)\, x_{\alpha}^{\Delta}\left(\Phi(x)\right)}{\left(\Phi^{\sigma}(x)\right)^{c-\alpha+1}}.
\end{aligned}$$

Then we have

$$x_{\alpha}^{\Delta}\left(\Phi(x)\right)\left(\Phi^{\sigma}(x)\right)^{\alpha-c-1} \leq \frac{-1}{c-\alpha}x_{\alpha}^{\Delta}\Phi^{\alpha-c}(x),$$

and thus

$$-v(x) = \int_x^{\infty}\frac{\lambda(s)}{\left(\Phi^{\sigma}(s)\right)^{c-\alpha+1}}\Delta_{\alpha}s \leq \frac{-1}{c-\alpha}\int_x^{\infty}x_{\alpha}^{\Delta}\Phi^{\alpha-c-1}(s)\Delta_{\alpha}s \leq \frac{\Phi^{\alpha-c}(x)}{c-\alpha}. \tag{24}$$

Again, by using the chain rule (18) to calculate

$$x_{\alpha}^{\Delta}\left(\Omega^{k}(x)\right) = k\Omega^{k-1}(d)x_{\alpha}^{\Delta}\left(\Omega(x)\right), \text{ where } d\in[x,\sigma(x)],$$

at $x_{\alpha}^{\Delta}\left(\Omega(x)\right) = \lambda(x)g(x) \geq 0$ and $d \leq \sigma(x)$, we get that

$$x_{\alpha}^{\Delta}\left(\Omega^{k}(x)\right) \leq k\lambda(x)g(x)\left(\Omega^{\sigma}(x)\right)^{k-1}. \tag{25}$$

Since $\Omega(a) = 0$, $v(\infty) = 0$ and from (24), (25) and (23) we have

$$\int_a^{\infty}\frac{\lambda(x)}{\left(\Phi^{\sigma}(x)\right)^{c-\alpha+1}}\left(\Omega^{\sigma}(x)\right)^{k}\Delta_{\alpha}x \leq \frac{k}{c-\alpha}\int_a^{\infty}\Phi^{\alpha-c}(x)\lambda(x)g(x)\left(\Omega^{\sigma}(x)\right)^{k-1}\Delta_{\alpha}x,$$

which reformulated as

$$\begin{aligned}
&\int_a^{\infty}\frac{\lambda(x)}{\left(\Phi^{\sigma}(x)\right)^{c-\alpha+1}}\left(\Omega^{\sigma}(x)\right)^{k}\Delta_{\alpha}x \\
&= \frac{k}{c-\alpha}\int_a^{\infty}\frac{\lambda(x)\Phi^{\alpha-c}(x)g(x)}{\left(\lambda(x)\left(\Phi^{\sigma}(x)\right)^{\alpha-c-1}\right)^{\frac{k-1}{k}}}\left(\frac{\lambda(x)\left(\Omega^{\sigma}(x)\right)^{k}}{\left(\Phi^{\sigma}(x)\right)^{c-\alpha+1}}\right)^{\frac{k-1}{k}}\Delta_{\alpha}x.
\end{aligned}$$

Using Hölder's inequality (21) on

$$\int_a^{\infty}\frac{\lambda(x)\Phi^{\alpha-c}(x)g(x)}{\left(\lambda(x)\left(\Phi^{\sigma}(x)\right)^{\alpha-c-1}\right)^{\frac{k-1}{k}}}\left(\frac{\lambda(x)\left(\Omega^{\sigma}(x)\right)^{k}}{\left(\Phi^{\sigma}(x)\right)^{c-\alpha+1}}\right)^{\frac{k-1}{k}}\Delta_{\alpha}x$$

with indices k and $k/(k-1)$, we have

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} (\Omega^\sigma(x))^k \Delta_\alpha x \leq$$

$$\frac{k}{c-\alpha}\left[\int_a^\infty \left[\frac{\lambda(x)\Phi^{\alpha-c}(x)g(x)}{\left[\lambda(x)\left(\Phi^\sigma(x)\right)^{\alpha-c-1}\right]^{\frac{k-1}{k}}}\right]^k \Delta_\alpha x\right]^{\frac{1}{k}}$$

$$\times\left[\int_a^\infty \left[\left[\frac{\lambda(x)\left(\Omega^\sigma(x)\right)^k}{(\Phi^\sigma(x))^{c-\alpha+1}}\right]^{\frac{k-1}{k}}\right]^{\frac{k}{k-1}} \Delta_\alpha x\right]^{\frac{k-1}{k}},$$

then

$$\left[\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} (\Omega^\sigma(x))^k \Delta_\alpha x\right]^{\frac{1}{k}} \leq \frac{k}{c-\alpha}\left[\int_a^\infty \frac{\lambda(x)\Phi^{k(\alpha-c)}(x)g^k(x)}{(\Phi^\sigma(x))^{(c-\alpha+1)(1-k)}}\Delta_\alpha x\right]^{\frac{1}{k}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} (\Omega^\sigma(x))^k \Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty \frac{\lambda(x)\Phi^{k(\alpha-c)}(x)g^k(x)}{(\Phi^\sigma(x))^{(c-\alpha+1)(1-k)}}\Delta_\alpha x,$$

that is the desired inequality (22). The proof is complete. □

Corollary 1. *At $\alpha = 1$ in Theorem 3 , we obtain the inequality*

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c}} (\Omega^\sigma(x))^k \Delta x \leq \left(\frac{k}{c-1}\right)^k \int_a^\infty \frac{\Phi^{k(1-c)}(x)}{(\Phi^\sigma(x))^{c(1-k)}}\lambda(x)g^k(x)\Delta x.$$

that is the timescales version of inequality (2.8) in [33].

Corollary 2. *At $\alpha = 1$, and $\mathbb{T} = \mathbb{R}$ $(\Phi^\sigma(x) = \Phi(x))$ in Theorem 3, we obtain the integral inequality*

$$\int_a^\infty \frac{\lambda(x)}{\Phi^c(x)} \left(\int_a^x \lambda(s)g(s)ds\right)^k dx \leq \left(\frac{k}{c-1}\right)^k \int_a^\infty \Phi^{k-c}(x)\lambda(x)g^k(x)dx,$$

which is of Copson type.

Corollary 3. *At $\alpha = 1$, $T = \mathbb{R}$, $\lambda(x) = 1$ and $a = 0$, $\left(\Phi(x) = \int_0^x \lambda(s)ds = x\right)$ in Theorem 3, we have Hardy-Littlewood integral inequality (3)*

$$\int_0^\infty \frac{1}{x^c}\left(\int_0^x g(s)ds\right)^k dx \leq \left(\frac{k}{c-1}\right)^k \int_0^\infty \frac{1}{x^{c-k}} g^k(x)dx.$$

Also, if $c = k$, we obtain the standard Hardy inequality (2)

$$\int_0^\infty \frac{1}{x^k}\left(\int_0^x g(s)ds\right)^k dx \leq \left(\frac{k}{k-1}\right)^k \int_0^\infty g^k(x)dx.$$

Theorem 4. *Let $0 \leq c < 1$ and $k > 1$. Define*

$$\Phi(x) := \int_a^x \lambda(s)\Delta_\alpha s \text{ and } \Omega(x) := \int_x^\infty \lambda(s)g(s)\Delta_\alpha s.$$

If

$$\Omega(a) < \infty, \textit{ and } \int_a^\infty \frac{\lambda(s)}{(\Phi^\sigma(s))^{c-\alpha+1}} \Delta_\alpha s < \infty,$$

then

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \Omega^k(x) \Delta_\alpha x \leq \left(\frac{k}{\alpha - c}\right)^k \int_a^\infty (\Phi^\sigma(x))^{k-c+\alpha-1} \lambda(x) g^k(x) \Delta_\alpha x. \tag{26}$$

Proof. By using the formula of integration by parts (20) on

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \Omega^k(x) \Delta_\alpha x,$$

with $v(x) = \Omega^k(x)$ and $x_\alpha^\Delta u(x) = \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}$, we have

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \Omega^k(x) \Delta_\alpha x = u(x)\Omega^k(x)\Big|_a^\infty + \int_a^\infty u^\sigma(x) x_\alpha^\Delta \left(-\Omega^k(x)\right) \Delta_\alpha x, \tag{27}$$

where

$$u(x) = \int_a^x \frac{\lambda(s)}{(\Phi^\sigma(s))^{c-\alpha+1}} \Delta_\alpha s = \int_a^x x_\alpha^\Delta \Phi(s) \left(\Phi^\sigma(s)\right)^{\alpha-c-1} \Delta_\alpha s.$$

By using chain rule (18), then for $d \in [x, \sigma(x)]$, we get that

$$\begin{aligned} x_\alpha^\Delta \left(\Phi^{\alpha-c}(x)\right) &= (\alpha - c)\Phi^{\alpha-c-1}(d) x_\alpha^\Delta \left(\Phi(x)\right) = \frac{(\alpha - c)\, x_\alpha^\Delta \left(\Phi(x)\right)}{\Phi^{c-\alpha+1}(d)} \\ &\geq \frac{(\alpha - c)\, x_\alpha^\Delta \left(\Phi(x)\right)}{(\Phi^\sigma(x))^{c-\alpha+1}}. \end{aligned}$$

So

$$x_\alpha^\Delta \left(\Phi(x)\right) \left(\Phi^\sigma(x)\right)^{\alpha-c-1} \leq \frac{1}{\alpha - c} x_\alpha^\Delta \left(\Phi^{\alpha-c}(x)\right),$$

and then,

$$\begin{aligned} u^\sigma(x) &= \int_a^{\sigma(x)} x_\alpha^\Delta \left(\Phi(s)\right) \left(\Phi^\sigma(s)\right)^{\alpha-c-1} \Delta_\alpha s \\ &\leq \frac{1}{\alpha - c} \int_a^{\sigma(x)} x_\alpha^\Delta \left(\Phi^{\alpha-c}(s)\right) \Delta_\alpha s \leq \frac{(\Phi^\sigma(x))^{\alpha-c}}{\alpha - c}. \end{aligned} \tag{28}$$

Again, by using chain rule (18), we obtain

$$-x_\alpha^\Delta \left(\Omega^k(x)\right) = -k\Omega^{k-1}(d) x_\alpha^\Delta \left(\Omega(x)\right), \text{where } d \in [x, \sigma(x)],$$

since $x_\alpha^\Delta \left(\Omega(x)\right) = -\lambda(x) g(x) \geq 0$ and $d \geq x$, then

$$-x_\alpha^\Delta \left(\Omega^k(x)\right) \leq k\lambda(x) g(x) \Omega^{k-1}(x). \tag{29}$$

Using $\Phi(a) = 0, \Omega(\infty) = 0$ and (28), (29) and (27), we have that

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}} \Omega^k(x) \Delta_\alpha x \leq \frac{k}{\alpha - c} \int_a^\infty \left(\Phi^\sigma(x)\right)^{\alpha-c} \lambda(x) g(x) \Omega^{k-1}(x) \Delta_\alpha x,$$

which reformulated as

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\Omega^k(x)\Delta_\alpha x =$$

$$\frac{k}{\alpha-c}\int_a^\infty \frac{(\Phi^\sigma(x))^{\alpha-c}\lambda(x)g(x)}{\left(\lambda(x)\left(\Phi^\sigma(x)\right)^{\alpha-c-1}\right)^{\frac{k-1}{k}}}\left(\frac{\lambda(x)\Omega^k(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\right)^{\frac{k-1}{k}}\Delta_\alpha x.$$

By employing Hölder's inequality (21) on

$$\int_a^\infty \frac{(\Phi^\sigma(x))^{\alpha-c}\lambda(x)g(x)}{\left(\lambda(x)\left(\Phi^\sigma(x)\right)^{\alpha-c-1}\right)^{\frac{k-1}{k}}}\left(\frac{\lambda(x)\Omega^k(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\right)^{\frac{k-1}{k}}\Delta_\alpha x$$

with indices k and $k/(k-1)$, we have

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\Omega^k(x)\Delta_\alpha x \leq$$

$$\frac{k}{\alpha-c}\left[\int_a^\infty\left[\frac{(\Phi^\sigma(x))^{1-c}\lambda(x)g(x)}{\left[\lambda(x)\left(\Phi^\sigma(x)\right)^{\alpha-c-1}\right]^{\frac{k-1}{k}}}\right]^k\Delta_\alpha x\right]^{\frac{1}{k}}$$

$$\times\left[\int_a^\infty\left[\left[\frac{\lambda(x)\Omega^k(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\right]^{\frac{k-1}{k}}\right]^{\frac{k}{k-1}}\Delta_\alpha x\right]^{\frac{k-1}{k}},$$

then we have

$$\left[\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c-\alpha+1}}\Omega^k(x)\Delta_\alpha x\right]^{\frac{1}{k}} \leq \frac{k}{\alpha-c}\left[\int_a^\infty \frac{\lambda(x)g^k(x)}{(\Phi^\sigma(x))^{c-k-\alpha+1}}\Delta_\alpha x\right]^{\frac{1}{k}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c}}\Omega^k(x)\Delta_\alpha x \leq \left(\frac{k}{\alpha-c}\right)^k\int_a^\infty (\Phi^\sigma(x))^{k-c+\alpha-1}\lambda(x)g^k(x)\Delta_\alpha x,$$

that is the desired inequality (26). The proof is complete. □

Corollary 4. *At $\alpha=1$ in Theorem 4, then*

$$\int_a^\infty \frac{\lambda(x)}{(\Phi^\sigma(x))^{c}}\Omega^k(x)\Delta x \leq \left(\frac{k}{1-c}\right)^k\int_a^\infty (\Phi^\sigma(x))^{k-c}\lambda(x)g^k(x)\Delta x.$$

which is the timescales version inequality (2.22) in [33].

Corollary 5. *At $\alpha=1$, and $T=\mathbb{R}$ in Theorem 4, we obtain the next integral inequality*

$$\int_a^\infty \frac{\lambda(x)}{\Phi^c(x)}\left(\int_x^\infty \lambda(s)g(s)ds\right)^k dx \leq \left(\frac{k}{1-c}\right)^k\int_a^\infty \Phi^{k-c}(x)\lambda(x)g^k(x)dx,$$

which considered an extension of Hardy's inequality (4) as in the following corollary.

Corollary 6. *At $\alpha = 1$, $T = \mathbb{R}$, $\lambda(x) = 1$ and $a = 0$ in Theorem 4, we have Hardy-Littlewood integral inequality (4)*

$$\int_0^\infty \frac{1}{x^c}\left(\int_x^\infty g(s)ds\right)^k dx \le \left(\frac{k}{1-c}\right)^k \int_0^\infty \frac{1}{x^{c-k}} g^k(x)dx.$$

A generalization of Leindler's inequality (14) on fractional time scales will be proved in the next theorem.

Theorem 5. *Assume $0 \le c < 1 < k$, define*

$$\Phi(x) := \int_x^\infty \lambda(s)\Delta_\alpha s \text{ and } \Omega(x) := \int_a^x \lambda(s)g(s)\Delta_\alpha s.$$

If

$$\Omega(\infty) < \infty, \text{ and } \int_a^\infty \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)}\Delta_\alpha s < \infty,$$

then

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \le \left(\frac{k}{\alpha-c}\right)^k \int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x)g^k(x)\Delta_\alpha x. \tag{30}$$

Proof. Using the formula of integration by parts (20) on

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\Omega^k(x)\Delta_\alpha x,$$

with $u^\sigma(x) = (\Omega^\sigma(x))^k$ and $x_\alpha^\Delta v(x) = \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}$, we have

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\left(\Omega^\sigma(x)\right)^k \Delta_\alpha x = v(x)\Omega^k(x)\Big|_a^\infty + \int_a^\infty -v(x)x_\alpha^\Delta\left(\Omega^k(x)\right)\Delta_\alpha x, \tag{31}$$

where

$$v(x) = -\int_x^\infty \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)}\Delta_\alpha s.$$

By chain rule (18), we see for $d \in [x, \sigma(x)]$ that

$$\begin{aligned} -x_\alpha^\Delta \Phi^{\alpha-c}(x) &= -(\alpha-c)\Phi^{\alpha-c-1}(d)x_\alpha^\Delta\left(\Phi(x)\right) = \frac{-(\alpha-c)(-\lambda(x))}{\Phi^{c-\alpha+1}(d)} \\ &\ge \frac{(\alpha-c)\,\lambda(x)}{\Phi^{c-\alpha+1}(x)}. \end{aligned}$$

Hence

$$-v(x) = \int_x^\infty \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)}\Delta_\alpha s \le \frac{-1}{\alpha-c}\int_x^\infty x_\alpha^\Delta \Phi^{\alpha-c}(s)\Delta_\alpha s \le \frac{\Phi^{\alpha-c}(x)}{\alpha-c}, \tag{32}$$

from chain rule (18), we obtain

$$x_\alpha^\Delta\left(\Omega^k(x)\right) = k\Omega^{k-1}(d)x_\alpha^\Delta\left(\Omega(x)\right), \text{ where } d \in [x, \sigma(x)],$$

since

$$x_\alpha^\Delta\left(\Omega(x)\right) = \lambda(x)g(x) \ge 0 \text{ and } d \le \sigma(x),$$

we get

$$x_\alpha^\Delta\left(\Omega^k(x)\right) \le k\lambda(x)g(x)\left(\Omega^\sigma(x)\right)^{k-1}. \tag{33}$$

Using $\Omega(a) = 0$, $v(\infty) = 0$ and (32), (33) and (31), we get that

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq \frac{k}{\alpha - c} \int_a^\infty \Phi^{\alpha-c}(x)\lambda(x)g(x) \left(\Omega^\sigma(x)\right)^{k-1} \Delta_\alpha x,$$

which reformulated as

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq$$
$$\frac{k}{\alpha - c} \int_a^\infty \frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\lambda(x)g(x)\frac{\lambda^{\frac{k-1}{k}}(x)\left(\Omega^\sigma(x)\right)^{k-1}}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\Delta_\alpha x.$$

Using Hölder's inequality (21) on

$$\int_a^\infty \frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\lambda(x)g(x)\frac{\lambda^{\frac{k-1}{k}}(x)\left(\Omega^\sigma(x)\right)^{k-1}}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\Delta_\alpha x$$

with indices k and $k/(k-1)$, we have

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq$$
$$\frac{k}{\alpha - c}\left[\int_a^\infty \left[\frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\lambda(x)g(x)\right]^k \Delta_\alpha x\right]^{\frac{1}{k}} \times$$
$$\left[\int_a^\infty \left[\lambda^{\frac{k-1}{k}}(x)\frac{\left(\Omega^\sigma(x)\right)^{k-1}}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\right]^{\frac{k}{k-1}} \Delta_\alpha x\right]^{\frac{k-1}{k}},$$

then

$$\left[\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x\right]^{\frac{1}{k}} \leq \frac{k}{\alpha - c}\left[\int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x)g^k(x)\Delta_\alpha x\right]^{\frac{1}{k}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq \left(\frac{k}{\alpha - c}\right)^k \int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x)g^k(x)\Delta_\alpha x,$$

that is the desired inequality (30). The proof is complete. □

Corollary 7. *At $\alpha = 1$ in Theorem 5, we get*

$$\int_a^\infty \frac{\lambda(x)}{\Phi^c(x)} \left(\Omega^\sigma(x)\right)^k \Delta x \leq \left(\frac{k}{1 - c}\right)^k \int_a^\infty \Phi^{k-c}(x)\lambda(x)g^k(x)\Delta x,$$

which inequality (2.36) in [33].

A generalization of Bennett's inequality (15) on fractional timescales will be proved in the next theorem.

Theorem 6. *Assume $0 < \alpha \leq 1$, $1 < c \leq k$, and define*

$$\Phi(x) := \int_x^\infty \lambda(s)\Delta_\alpha s \text{ and } \Omega(x) := \int_x^\infty \lambda(s)g(s)\Delta_\alpha s.$$

If

$$\Omega(a) < \infty \text{ and } \int_a^\infty \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)} \Delta_\alpha s < \infty,$$

then

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \Omega^k(x) \Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x) g^k(x) \Delta_\alpha x. \tag{34}$$

Proof. Using the formula of integration by parts (20) on

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \Omega^k(x) \Delta_\alpha x,$$

with $v(x) = \Omega^k(x)$ and $x_\alpha^\Delta u(x) = \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}$, then

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \Omega^k(x) \Delta_\alpha x = u(x)\Omega^k(x)\Big|_a^\infty + \int_a^\infty u^\sigma(x) x_\alpha^\Delta \left(-\Omega^k(x)\right) \Delta_\alpha x, \tag{35}$$

where

$$u(x) = \int_a^x \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)} \Delta_\alpha s.$$

By using chain rule (18), we see for $d \in [x, \sigma(x)]$ that

$$\begin{aligned} x_\alpha^\Delta \left(\Phi^{\alpha-c}(x)\right) &= (\alpha - c)\Phi^{\alpha-c-1}(d) x_\alpha^\Delta \left(\Phi(x)\right) = \frac{(\alpha-c)(-\lambda(x))}{\Phi^{c-\alpha+1}(d)} \\ &\geq \frac{(c-\alpha)\,\lambda(x)}{\Phi^{c-\alpha+1}(x)}. \end{aligned}$$

we get,

$$\begin{aligned} u^\sigma(x) &= \int_a^{\sigma(x)} \frac{\lambda(s)}{\Phi^{c-\alpha+1}(s)} \Delta_\alpha s \leq \frac{1}{c-\alpha} \int_a^{\sigma(x)} x_\alpha^\Delta \Phi^{\alpha-c}(s) \Delta_\alpha s \\ &= \frac{(\Phi^\sigma(x))^{\alpha-c}}{c-\alpha} - \frac{\Phi^{\alpha-c}(a)}{c-\alpha} \leq \frac{\Phi^{\alpha-c}(x)}{c-\alpha}, \end{aligned} \tag{36}$$

from chain rule (18), we find that

$$x_\alpha^\Delta \left(\Omega^k(x)\right) = k\Omega^{k-1}(d) x_\alpha^\Delta \Omega(x), \text{ where } d \in [x, \sigma(x)],$$

since

$$x_\alpha^\Delta \left(\Omega(x)\right) = -\lambda(x) g(x) \leq 0 \text{ and } x \leq d,$$

then

$$-x_\alpha^\Delta \left(\Omega^k(x)\right) \leq k\lambda(x) g(x) \Omega^{k-1}(x). \tag{37}$$

Using $v(a) = 0$, $\Omega(\infty) = 0$ and (36), (37) and (35), we get that

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)} \left(\Omega^\sigma(x)\right)^k \Delta_\alpha x \leq \frac{k}{c-\alpha} \int_a^\infty \Phi^{\alpha-c}(x)\lambda(x) g(x) \Omega^{k-1}(x) \Delta_\alpha x,$$

which reformulated as

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\Omega^k(x)\Delta_\alpha x \leq$$

$$\frac{k}{c-\alpha}\int_a^\infty \frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\lambda(x)g(x)\lambda^{\frac{k-1}{k}}(x)\frac{\Omega^{k-1}(x)}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\Delta_\alpha x,$$

Using Hölder's inequality (21) on

$$\int_a^\infty \lambda(x)g(x)\frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\frac{\lambda^{\frac{k-1}{k}}(x)\Omega^{k-1}(x)}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\Delta_\alpha x$$

with indices k and $k/(k-1)$, we have

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\Omega^k(x)\Delta_\alpha x \leq$$

$$\frac{k}{c-\alpha}\left[\int_a^\infty\left[\lambda(x)g(x)\frac{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}{\lambda^{\frac{k-1}{k}}(x)\Phi^{c-\alpha}(x)}\right]^k\Delta_\alpha x\right]^{\frac{1}{k}}\times$$

$$\left[\int_a^\infty\left[\lambda^{\frac{k-1}{k}}(x)\frac{\Omega^{k-1}(x)}{\Phi^{(c-\alpha+1)\left(\frac{k-1}{k}\right)}(x)}\right]^{\frac{k}{k-1}}\Delta_\alpha x\right]^{\frac{k-1}{k}},$$

then

$$\left[\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\left(\Omega(x)\right)^k\Delta_\alpha x\right]^{\frac{1}{k}} \leq \frac{k}{c-\alpha}\left[\int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x)g^k(x)\Delta_\alpha x\right]^{\frac{1}{k}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(x)}{\Phi^{c-\alpha+1}(x)}\left(\Omega^\sigma(x)\right)^k\Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k\int_a^\infty \Phi^{k-c+\alpha-1}(x)\lambda(x)g^k(x)\Delta_\alpha x,$$

that is the desired inequality (34). The proof is complete. □

Corollary 8. *At $\alpha = 1$ in Theorem 5, we have the inequality*

$$\int_a^\infty \frac{\lambda(x)}{\Phi^c(x)}\left(\Omega^\sigma(x)\right)^k\Delta x \leq \left(\frac{k}{c-1}\right)^k\int_a^\infty \Phi^{k-c}(x)\lambda(x)g^k(x)\Delta x,$$

which the inequality (2.49) in [33].

3. Conclusions

The new fractional calculus on timescales is presented with applications to some new fractional inequalities on timescales like Hardy, Bennett, Copson and Leindler types. Inequalities are considered in rather general forms and contain several special integral and discrete inequalities. The technique is based on the applications of well-known inequalities and new tools from fractional calculus.

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