mathematics

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A Topological Coincidence Theory for Multifunctions via Homotopy

Donal O'Regan

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway H91 TK33, Ireland; donal.oregan@nuigalway.ie

Received: 21 February 2020; Accepted: 12 March 2020; Published: 16 March 2020

Abstract: A new simple result is presented which immediately yields the topological transversality theorem for coincidences.

Keywords: essential maps; homotopy

1. Introduction

The topological transversality theorem of Granas [1] states that if F and G are continuous compact single valued maps and $F \cong G$ then F is essential if and only if G is essential. These concepts were generalized to multimaps (compact and noncompact) and for Φ–essential maps in a general setting (see [2,3] and the references therein). In this paper we approach this differently and we present a very general topological transversality theorem for coincidences.

For convenience we desribe now a class of maps one could consider in this setting. Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is called acyclic if F has acyclic values.

2. Topological Transversality Theorem

In this paper we will consider two classes $\mathbf{A}$ and $\mathbf{B}$ of maps. These are abstract classes which include many types of maps in the literature (see Remark 1). Let E be a completely regular space (i.e., a Tychonoff space) and U an open subset of E. We let $\overline{U}$ (respectively, ∂U) denote the closure (respectively, the boundary) of U in E.

Definition 1. *We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is a upper semicontinuous (u.s.c.) compact map.*

Remark 1. *Examples of $F \in \mathbf{A}(\overline{U}, E)$ might be that $F : \overline{U} \to K(E)$ has convex values or $F : \overline{U} \to K(E)$ has acyclic values.*

In this paper we fix a $\Phi \in B(\overline{U}, E)$ (i.e., $\Phi \in \mathbf{B}(\overline{U}, E)$ and $\Phi : \overline{U} \to K(E)$ is a u.s.c. map).

Definition 2. *We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ and $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$.*

Next we consider homotopy for maps in $A_{\partial U}(\overline{U}, E)$. We present two interpretations.

Definition 3. *Two maps $F, G \in A_{\partial U}(\overline{U}, E)$ are said to be homotopic in $A_{\partial U}(\overline{U}, E)$, written $F \cong G$ in $A_{\partial U}(\overline{U}, E)$, if there exists a u.s.c. compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,\cdot\,, \eta(\,\cdot\,)) \in \mathbf{A}(\overline{U}, E)$ for any*

continuous function $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $\Psi_t(x) = \Psi(x,t)$*),* $\Psi_0 = F$ *and* $\Psi_1 = G$.

Remark 2. *Alternatively we could use the following definition for* $\cong$ *in* $A_{\partial U}(\overline{U}, E)$*:* $F \cong G$ *in* $A_{\partial U}(\overline{U}, E)$ *if there exists a u.s.c. compact map* $\Psi : \overline{U} \times [0,1] \to K(E)$ *with* $\Psi \in \mathbf{A}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $\Psi_t(x) = \Psi(x,t)$*),* $\Psi_0 = F$ *and* $\Psi_1 = G$*. If we use this definition then we always assume for any map* $\Theta \in \mathbf{A}(\overline{U} \times [0,1], E)$ *and any map* $f \in \mathbf{C}(\overline{U}, \overline{U} \times [0,1])$ *then* $\Theta \circ f \in \mathbf{A}(\overline{U}, E)$*; here* $\mathbf{C}$ *denotes the class of single valued continuous functions.*

Definition 4. *Let* $F \in A_{\partial U}(\overline{U}, E)$*. We say* F *is* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$ *if for every map* $J \in A_{\partial U}(\overline{U}, E)$ *with* $J|_{\partial U} = F|_{\partial U}$ *there exists a* $x \in U$ *with* $\Phi(x) \cap J(x) \neq \emptyset$.

We now present a simple result. From this result the topological transversality theorem will be immediate. In our next theorem E will be a completely regular topological space and U will be an open subset of E.

Theorem 1. *Let* $F \in A_{\partial U}(\overline{U}, E)$ *and let* $G \in A_{\partial U}(\overline{U}, E)$ *be* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$*. Also suppose*

$$\begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \\ \text{we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E). \end{cases} \tag{1}$$

Then F *is* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$.

Proof. In the proof below we assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 3. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. From (1) there exists a u.s.c. compact map $H^J : \overline{U} \times [0,1] \to K(E)$ with $H^J(\,\cdot\,, \eta(\,\cdot\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$ and $H^J_1 = J$. Let

$$K = \left\{ x \in \overline{U} : \ \Phi(x) \cap H^J(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1] \right\}$$

and

$$D = \left\{ (x,t) \in \overline{U} \times [0,1] : \ \Phi(x) \cap H^J(x,t) \neq \emptyset \right\}.$$

Now $D \neq \emptyset$ (note G is Φ–essential in $A_{\partial U}(\overline{U}, E)$) and D is closed (note Φ and H^J are u.s.c.) and so D is compact (note H^J is a compact map). Let $\pi : \overline{U} \times [0,1] \to \overline{U}$ be the projection. Now $K = \pi(D)$ is closed (see Kuratowski's theorem ([4], p. 126) and so in fact compact (recall projections are continuous). Also note $K \cap \partial U = \emptyset$ (since $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$) so since E is Tychonoff there exists a continuous map (called the Urysohn map) $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x, \mu(x))$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x,0) = G(x)$ and $R(x) \cap \Phi(x) = G(x) \cap \Phi(x)$). Now since G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $\Phi(x) \cap R(x) \neq \emptyset$ (i.e., $\Phi(x) \cap H^J_{\mu(x)}(x) \neq \emptyset$). Thus $x \in K$ so $\mu(x) = 1$ and $\Phi(x) \cap H^J_1(x) \neq \emptyset$ that is, $\Phi(x) \cap J(x) \neq \emptyset$. $\square$

Remark 3. *(i). In the proof of Theorem 1 it is simple to adjust the proof if we use* $\cong$ *in* $A_{\partial U}(\overline{U}, E)$ *from Remark 2 if we note* $H^J(x, \mu(x)) = H^J \circ g(x)$ *where* $g : \overline{U} \to \overline{U} \times [0,1]$ *is given by* $g(x) = (x, \mu(x))$.

(ii). One could replace u.s.c. in the Definition of $A(\overline{U}, E)$, $B(\overline{U}, E)$*, Definition 3 and Remark 2 with any condition that guarantees that* K *in the proof of Theorem 1 is closed; this is all that is needed if* E *is normal. If* E *is Tychonoff and not normal the one can also replace the compactness of the map in* $A(\overline{U}, E)$*, Definition 3 and Remark 2 with any condition that guarantees that* K *in the proof of Theorem 1 is compact.*

(iii). Theorem 1 immediately yields a general Leray–Schauder type alternative for coincidences. Let E *be a completely metrizable locally convex space,* U *an open subset of* E, $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ *is*

Φ–essential in $A_{\partial U}(\overline{U},E)$, $\Phi(x)\cap[t\,F(x)+(1-t)\,G(x)]=\emptyset$ for $x\in\partial U$ and $t\in(0,1)$, and $\eta(\,\cdot\,)\,J(\,\cdot\,)+(1-\eta(\,\cdot\,))\,G(\,\cdot\,)\in\mathbf{A}(\overline{U},E)$ for any continuous function $\eta:\overline{U}\to[0,1]$ with $\eta(\partial U)=0$ and any map $J\in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U}=F|_{\partial U}$. Then F is Φ–essential in $A_{\partial U}(\overline{U},E)$.

The proof is immediate from Theorem 1 since topological vector spaces are completely regular and note if $J\in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U}=F|_{\partial U}$ then with $H^J(x,t)=t\,J(x)+(1-t)\,G(x)$ note $H^J_0=G$, $H^J_1=J$, $H^J:\overline{U}\times[0,1]\to K(E)$ is a u.s.c. compact (see [5], Theorem 4.18) map, and $H^J(\,\cdot\,,\eta(\,\cdot\,))\in\mathbf{A}(\overline{U},E)$ for any continuous function $\eta:\overline{U}\to[0,1]$ and $\Phi(x)\cap H^J_t(x)=\emptyset$ for $x\in\partial U$ and $t\in(0,1)$ (if $x\in\partial U$ and $t\in(0,1)$ then since $J|_{\partial U}=F|_{\partial U}$ we note that $\Phi(x)\cap H^J_t(x)=\Phi(x)\cap[t\,F(x)+(1-t)\,G(x)]$) so as a result $G\cong J$ in $A_{\partial U}(\overline{U},E)$ (i.e., (1) holds). (Note E being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space E if the space E has the property that the closed convex hull of a compact set in E is compact. In fact it is easy to see, if we argue differently, that all we need to assume is that E is a topological vector space).

With this simple result we now present the topological transversality theorem. Assume

$$\cong \quad \text{in } A_{\partial U}(\overline{U},E) \text{ is an equivalence relation} \tag{2}$$

and

$$\text{if } F,\,G\in A_{\partial U}(\overline{U},E) \text{ with } F|_{\partial U}=G|_{\partial U} \text{ then } F\cong G \text{ in } A_{\partial U}(\overline{U},E). \tag{3}$$

In our next theorem E will be a completely regular topological space and U will be an open subset of E.

Theorem 2. *Assume (2) and (3) hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U},E)$ with $F\cong G$ in $A_{\partial U}(\overline{U},E)$. Now F is Φ–essential in $A_{\partial U}(\overline{U},E)$ if and only if G is Φ–essential in $A_{\partial U}(\overline{U},E)$.*

Proof. Assume G is Φ–essential in $A_{\partial U}(\overline{U},E)$. Let $J\in A_{\partial U}(\overline{U},E)$ with $J|_{\partial U}=F|_{\partial U}$. We will show $G\cong J$ in $A_{\partial U}(\overline{U},E)$ (i.e., we will show (1)) and then Theorem 1 guarantees that F is Φ–essential in $A_{\partial U}(\overline{U},E)$. Note $G\cong J$ in $A_{\partial U}(\overline{U},E)$ is immediate since from (3) we have $J\cong F$ in $A_{\partial U}(\overline{U},E)$ and since $F\cong G$ in $A_{\partial U}(\overline{U},E)$ then (2) guarantees that $G\cong J$ in $A_{\partial U}(\overline{U},E)$. Similarly if F is Φ–essential in $A_{\partial U}(\overline{U},E)$ then G is Φ–essential in $A_{\partial U}(\overline{U},E)$. □

Remark 4. *Suppose E is a (Hausdorff) topological vector space, U is a open convex subset of E and $F\in\mathbf{A}(\overline{U},E)$ means $F:\overline{U}\to K(E)$ has acyclic values then immediately (2) holds (we use the definition of $\cong$ in $A_{\partial U}(\overline{U},E)$ from Definition 3). Suppose*

$$\text{there exists a retraction } r:\overline{U}\to\partial U. \tag{4}$$

(Note (4) is satisfied if E is an infinite dimensional Banach space).

Then (3) holds (we use the definition of $\cong$ in $A_{\partial U}(\overline{U},E)$ from Definition 3). To see this let r be in (4), $F,\,G\in A_{\partial U}(\overline{U},E)$ with $F|_{\partial U}=G|_{\partial U}$. Consider $F^\star$ given by $F^\star(x)=F(r(x))$, $x\in\overline{U}$. Note $F^\star(x)=G(r(x))$, $x\in\overline{U}$ since $F|_{\partial U}=G|_{\partial U}$. Now take

$$\Lambda(x,\lambda)=G(2\,\lambda\,r(x)+(1-2\,\lambda)\,x)=G\circ j\,(x,\lambda) \;\text{ for }\; (x,\lambda)\in\overline{U}\times\left[0,\frac{1}{2}\right]$$

(here $j:\overline{U}\times\left[0,\frac{1}{2}\right]\to\overline{U}$ (note $\overline{U}$ is convex) is given by $j(x,\lambda)=2\,\lambda\,r(x)+(1-2\,\lambda)\,x$) it is easy to see that

$$G\cong F^\star \;\text{ in }\; A_{\partial U}(\overline{U},E);$$

note $\Lambda:\overline{U}\times\left[0,\frac{1}{2}\right]\to K(E)$ is a u.s.c. compact map and also for a fixed $x\in\overline{U}$ note $\Lambda(x,\mu(x))=G(j(x,\mu(x)))$ has acyclic values and so $\Lambda(\,\cdot\,,\eta(\,\cdot\,))\in\mathbf{A}(\overline{U},E)$ for any continuous function $\eta:\overline{U}\to[0,1]$

with $\eta(\partial U) = 0$*, and finally note* $\Phi(x) \cap \Lambda_t(x) = \emptyset$ *for* $x \in \partial U$ *and* $t \in \left[0, \frac{1}{2}\right]$ *(note if* $x \in \partial U$ *and* $t \in \left[0, \frac{1}{2}\right]$ *then since* $r(x) = x$ *we have* $\Phi(x) \cap \Lambda_t(x) = \Phi(x) \cap G(x)$*). Similarly with*

$$\Theta(x, \lambda) = F((2 - 2\lambda)\, r(x) + (2\lambda - 1)\, x) \ \ \textit{for} \ \ (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$F^\star \cong F \ \ \textit{in} \ \ A_{\partial U}(\overline{U}, E).$$

Consequently $F \cong G$ *in* $A_{\partial U}(\overline{U}, E)$ *so (3) holds.*

It is easy to present examples of Φ–essential maps if one uses coincidence result from the literature. In our next theorem E will be a (Hausdorff) topological space and U will be an open subset of E.

Theorem 3. *Let* $\Phi \in B(\overline{U}, E)$ *and* $F \in A_{\partial U}(\overline{U}, E)$*. Assume the following conditions hold:*

$$\begin{cases} \textit{there exists a retraction} \ \ r : E \to \overline{U} \\ \textit{with} \ \ r(w) \in \partial U \ \ \textit{if} \ \ w \in E \setminus U \end{cases} \tag{5}$$

and

$$\begin{cases} \textit{for any map} \ \ J \in A_{\partial U}(\overline{U}, E) \ \ \textit{with} \ \ J|_{\partial U} = F|_{\partial U} \\ \textit{(i). there exists a} \ \ w \in \overline{U} \ \ \textit{with} \ \ r\,J(w) \cap \Phi(w) \neq \emptyset, \ \ \textit{and} \\ \textit{(ii). there is no} \ \ z \in E \setminus U \ \ \textit{and} \ \ y \in \overline{U} \ \ \textit{with} \ \ z \in J(y) \ \ \textit{and} \ \ r(z) \in \Phi(y). \end{cases} \tag{6}$$

Then F *is* Φ*–essential in* $A_{\partial U}(\overline{U}, E)$*.*

Proof. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. Now (6) (i) implies there exists a $w \in \overline{U}$ with $r\,J(w) \cap \Phi(w) \neq \emptyset$. Then there exists a $z \in J(w)$ with $r(z) \in \Phi(w)$. Note $z \in E \setminus U$ or $z \in U$. If $z \in E \setminus U$ then $z \in J(w)$, $w \in \overline{U}$ and $r(z) \in \Phi(w)$ which contradicts (6) (ii). Thus $z \in U$ so $r(z) = z$ and as a result $z \in J(w)$ and $z\,(= r(z)) \in \Phi(w)$ that is, $\Phi(w) \cap J(w) \neq \emptyset$. □

Remark 5. *(i). Suppose* $\Phi = i$ *(identity) and* $F \in \mathbf{A}(\overline{U}, E)$ *means* $F : \overline{U} \to K(E)$ *has acyclic values. Then (6) (i) holds (i.e., there exists a* $w \in \overline{U}$ *with* $w \in r\,J(w)$*) from a theorem of Eilenberg and Montgomery [6,7] (note* r *is continuous and* J *is an acyclic u.s.c. compact map).*

(ii). Now let us consider (5) and (6) (ii). Now in addition assume E *is a locally convex topological vector space,* $0 \in U$ *and* U *an open convex subset of* E*. Let*

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \ \ \textit{for} \ \ x \in E,$$

where μ *is the Minkowski functional on* $\overline{U}$ *(i.e.,* $\mu(x) = \inf\{\alpha > 0 : \ x \in \alpha\,\overline{U}\}$*). Now (5) holds*

First let $\Phi = i$*. If we assume a Leray–Schauder type condition*

$$x \notin \lambda\, F(x) \ \ \textit{for} \ \ x \in \partial U \ \ \textit{and} \ \ \lambda \in (0, 1) \tag{7}$$

then (6) (ii) holds. To see this let $J \in A_{\partial U}(\overline{U}, E)$ *with* $J|_{\partial U} = F|_{\partial U}$*. Suppose there is a* $z \in E \setminus U$ *and* $y \in \overline{U}$ *with* $z \in J(y)$ *and* $r(z) \in \Phi(y)$ *(i.e* $r(z) = y$ *since* $\Phi = i$*). Now*

$$y = r(z) = \frac{z}{\mu(z)} \ \ \textit{with} \ \ \mu(z) \geq 1 \ \ \textit{since} \ \ z \in E \setminus U,$$

so $y \in \lambda\, J(y)$ *with* $0 < \lambda = \frac{1}{\mu(y)} \leq 1$*. Note* $y = r(z) \in \partial U$ *since* $z \in E \setminus U$ *so* $y \in \lambda\, F(y)$ *since* $J|_{\partial U} = F|_{\partial U}$*. This contradicts (7).*

Next we do not assume $\Phi = i$. Assume

$$\begin{cases} \text{for any map } \ J \in A_{\partial U}(\overline{U}, E) \ \text{ with } \ J|_{\partial U} = F|_{\partial U} \\ \text{if } \ y \in \overline{U}, \ z \in E \setminus U \ \text{ with } \ z \in J(y) \\ \text{and } \ r(z) \in \Phi(y) \ \text{ then } \ y \in \partial U \end{cases} \tag{8}$$

and

$$\Phi(x) \cap \lambda F(x) = \emptyset \ \text{ for } \ x \in \partial U \ \text{ and } \ \lambda \in (0,1). \tag{9}$$

Then (6) (ii) holds. To see this let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. Suppose there is a $z \in E \setminus U$ and $y \in \overline{U}$ with $z \in J(y)$ and $r(z) \in \Phi(y)$. Now (8) guarantees that $y \in \partial U$. Also $r(z) = \frac{z}{\mu(z)}$ with $\mu(z) \geq 1$, so $r(z) \in \Phi(y)$ and $r(z) \in \frac{1}{\mu(z)} J(y)$. Thus $\emptyset \neq \Phi(y) \cap \lambda J(y) = \Phi(y) \cap \lambda F(y)$ (since $J|_{\partial U} = F|_{\partial U}$) with $0 < \lambda = \frac{1}{\mu(y)} \leq 1$, and this contradicts (9).

(iii). One also has a "dual" version of Theorem 3 if we consider $J\,r$ instead of $r\,J$. Let $\Phi \in B(E,E)$ (i.e., $\Phi \in \mathbf{B}(E,E)$ and $\Phi : E \to K(E)$ is a u.s.c. map), $F \in A_{\partial U}(\overline{U}, E)$ and assume (5) holds. In addition suppose

$$\begin{cases} \text{for any map } \ J \in A_{\partial U}(\overline{U}, E) \ \text{ with } \ J|_{\partial U} = F|_{\partial U} \\ \text{there exists a } \ w \in E \ \text{ with } \ J\,r(w) \cap \Phi(w) \neq \emptyset \end{cases} \tag{10}$$

and

$$\begin{cases} \text{there is no } \ y \in E \setminus U \ \text{ and } \ z \in \partial U \ \text{ with} \\ z = r(y) \ \text{ and } \ F(z) \cap \Phi(y) \neq \emptyset. \end{cases} \tag{11}$$

Then F is Φ–essential in $A_{\partial U}(\overline{U}, E)$.

The proof is immediate since for any $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ from (10) there exists a $y \in E$ with $J\,r(y) \cap \Phi(y) \neq \emptyset$, so if $z = r(y)$ then $J(z) \cap \Phi(y) \neq \emptyset$. If $y \in E \setminus U$ then $z \in \partial U$ and $\emptyset \neq J(z) \cap \Phi(y) = F(z) \cap \Phi(y)$ (since $J|_{\partial U} = F|_{\partial U}$), a contradiction. Thus $y \in U$ so $z = r(y) = y$ and $J(y) \cap \Phi(y) \neq \emptyset$.

In our next theorem E will be a (Hausdorff) topological space and U will be an open subset of E.

Theorem 4. *Let $\Phi \in B(E,E)$ and assume:*

$$0 \in \mathbf{A}(\overline{U}, E) \ \text{ where } \ 0 \ \text{ denotes the zero map} \tag{12}$$

$$\begin{cases} \text{for any map } \ J \in A_{\partial U}(\overline{U}, E) \ \text{ with } \ J|_{\partial U} = \{0\} \ \text{ and} \\ R(x) = \begin{cases} J(x), & x \in \overline{U} \\ \{0\}, & x \in E \setminus \overline{U}, \end{cases} \\ \text{there exists a } \ y \in E \ \text{ with } \ \Phi(y) \cap R(y) \neq \emptyset \end{cases} \tag{13}$$

and

$$\text{there is no } \ z \in E \setminus U \ \text{ with } \Phi(z) \cap \{0\} \neq \emptyset. \tag{14}$$

Then the zero map is Φ–essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Note $0 \in A_{\partial U}(\overline{U}, E)$ (see (12) and (14)). Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \{0\}$. Let R be as in (13) so there exists a $y \in E$ with $\Phi(y) \cap R(y) \neq \emptyset$. We have two cases, namely $y \in U$ and $y \in E \setminus U$. If $y \in E \setminus U$ then $R(y) = \{0\}$ so $\Phi(y) \cap \{0\} \neq \emptyset$, and this contradicts (14). Thus $y \in U$ so $\Phi(y) \cap J(y) \neq \emptyset$. □

Remark 6. *(i). Suppose $F \in \mathbf{A}(\overline{U}, E)$ means $F : \overline{U} \to K(E)$ has acyclic values. If $\Phi \in B(E,E)$ and (13) and (14) are satisfied then Theorem 4 guarantees that zero map is Φ–essential in $A_{\partial U}(\overline{U}, E)$.*

Suppose E is a completely metrizable locally convex space, U is an open convex subset of E, $0 \in U$, $F \in A_{\partial U}(\overline{U}, E)$, $\Phi \in B(E,E)$ and assume (4), (9) (namely $\Phi(x) \cap \lambda F(x) = \emptyset$ for $x \in \partial U$ and $\lambda \in (0,1)$),

(13) and (14) hold. Then Theorem 2 and Remark 4 guarantees that F is Φ–essential in $A_{\partial U}(\overline{U}, E)$. This is immediate since a homotopy (Definition 3) from F to $\{0\}$ is $\Psi(x,t) = t\,F(x)$ (here $t \in [0,1]$ and $x \in \overline{U}$). To see this note $\Psi : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous compact (see [5], Theorem 4.18) map and also note for a fixed $t \in [0,1]$ and a fixed $x \in \overline{U}$ that $\Psi_t(x)$ is acyclic valued (recall homeomorphic spaces have isomorphic homology groups) so $\Psi_t \in A_{\partial U}(\overline{U}, E)$ and this immediately implies $\Psi(\,\cdot\,, \eta(\,\cdot\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$, $\eta(\partial U) = 0$ since for $x \in \overline{U}$ fixed note $\Psi(x, \mu(x)) = \Psi_{\mu(x)}(x) = \Psi_t(x)$ with $t = \mu(x) \in [0,1]$. Note E being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space E if the space E has the property that the closed convex hull of a compact set in E is compact. In fact it is easy to see, if we argue differently, that all we need to assume is that E is a topological vector space.

(ii). It is very easy to extend the above ideas to the (L,T) Φ–essential maps in [2].

Now we consider d–Φ–essential maps. Let E be a completely regular topological space and U an open subset of E. For any map $F \in A(\overline{U}, E)$ write $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\,(B) \right\} \cup \{\emptyset\} \;\to\; \Omega \tag{15}$$

be any map with values in the nonempty set Ω where $B = \left\{ (x, \Phi(x)) : \; x \in \overline{U} \right\}$.

Definition 5. *Let $F \in A_{\partial U}(\overline{U}, E)$ and write $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–Φ–essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ (write $J^\star = I \times J$) with $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}\,(B) \right) = d\left((J^\star)^{-1}\,(B) \right) \neq d(\emptyset)$.*

Remark 7. *If $F^\star$ is d–Φ–essential then*

$$\emptyset \neq (F^\star)^{-1}\,(B) = \{x \in \overline{U} : \; (x, F(x)) \cap (x, \Phi(x)) \neq \emptyset\},$$

so there exists a $x \in U$ with $(x, \Phi(x)) \cap (x, F(x)) \neq \emptyset$ (i.e., $\Phi(x) \cap F(x) \neq \emptyset$).

In our next theorem E will be a completely regular topological space and U will be an open subset of E.

Theorem 5. *Let $B = \left\{ (x, \Phi(x)) : \; x \in \overline{U} \right\}$, d is defined in (15), $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ (write $F^\star = I \times F$ and $G^\star = I \times G$). Suppose $G^\star$ is d–Φ–essential and*

$$\begin{cases} \text{for any map } \; J \in A_{\partial U}(\overline{U}, E) \;\; \text{with} \;\; J|_{\partial U} = F|_{\partial U} \\ \text{we have } \; G \cong J \;\; \text{in} \;\; A_{\partial U}(\overline{U}, E) \;\; \text{and} \\ d\left((F^\star)^{-1}\,(B) \right) = d\left((G^\star)^{-1}\,(B) \right). \end{cases} \tag{16}$$

Then $F^\star$ is d–Φ–essential.

Proof. In the proof below we assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 3. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ (write $J^\star = I \times J$) and $J|_{\partial U} = F|_{\partial U}$. From (16) there exists a u.s.c. compact map $H^J : \overline{U} \times [0,1] \to K(E)$ with $H^J(\,\cdot\,, \eta(\,\cdot\,)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H^J_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$, $H^J_1 = J$ and $d\left((F^\star)^{-1}\,(B) \right) = d\left((G^\star)^{-1}\,(B) \right)$. Let $(H^J)^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $(H^J)^\star(x,t) = (x, H^J(x,t))$ and let

$$K = \left\{ x \in \overline{U} : \; (x, \Phi(x)) \cap (H^J)^\star(x,t) \neq \emptyset \;\; \text{for some} \;\; t \in [0,1] \right\}.$$

Now $K \neq \emptyset$ is closed, compact and $K \cap \partial U = \emptyset$ so since E is Tychonoff there exists a Urysohn map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x, \mu(x))$ and write $R^\star = I \times R$. Now $R \in A_{\partial U}(\overline{U}, E)$ (if $x \in \partial U$ then $\mu(x) = 0$ so $R(x) = G(x)$) with $R|_{\partial U} = G|_{\partial U}$. Since $G^\star$ is d–Φ–essential then

$$d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{17}$$

Now since $\mu(K) = 1$ we have

$$\begin{aligned}(R^\star)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H^J(x, \mu(x))) \neq \emptyset\right\} = \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H^J(x, 1)) \neq \emptyset\right\} \\ &= (J^\star)^{-1}(B),\end{aligned}$$

so from (17) we have $d\left((G^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now combine with the above and we have $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. $\square$

Also note one could adjust the proof in Theorem 5 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ in Remark 2.

In our next theorem E will be a completely regular topological space and U will be an open subset of E.

Theorem 6. *Let $B = \left\{(x, \Phi(x)) : x \in \overline{U}\right\}$, d is defined in (15) and assume (2) and (3) hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ (write $F^\star = I \times F$ and $G^\star = I \times G$) and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then $F^\star$ is d–Φ–essential if and only if $G^\star$ is d–Φ–essential.*

Proof. In the proof below we assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 3. Assume $G^\star$ is d–Φ–essential. Let $J \in A_{\partial U}(\overline{U}, E)$ (write $J^\star = I \times J$) and $J|_{\partial U} = F|_{\partial U}$. If we show (16) then $F^\star$ is d–Φ–essential from Theorem 5. Now (3) implies $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ and (2) guarantees that $G \cong J$ in $A_{\partial U}(\overline{U}, E)$. It remains to show $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. Note since $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ let $H : \overline{U} \times [0,1] \to K(E)$ be a u.s.c. compact map with $H(\cdot, \eta(\cdot)) \in \mathbf{A}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = G$ and $H_1 = F$. Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $H^\star(x,t) = (x, H(x,t))$ and let

$$K = \left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Now $K \neq \emptyset$ and there exists a Urysohn map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H(x, \mu(x))$ and write $R^\star = I \times R$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ so since $G^\star$ is d–Φ–essential then $d\left((G^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now since $\mu(K) = 1$ we have

$$\begin{aligned}(R^\star)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\right\} = \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\right\} \\ &= (F^\star)^{-1}(B),\end{aligned}$$

so $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. $\square$

Also note one could adjust the proof in Theorem 6 if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ in Remark 2.

Remark 8. *It is very easy to extend the above ideas to the (L, T) d–Φ–essential maps in [3].*

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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