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Oscillation Criteria of Higher-order Neutral Differential Equations with Several Deviating Arguments

Osama Moaaz [1,†], Ioannis Dassios [2,*,†] and Omar Bazighifan [3,4,†]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; o_moaaz@mans.edu.eg
2 AMPSAS, University College Dublin, D4 Dublin, Ireland
3 Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com
4 Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen
* Correspondence: ioannis.dassios@ucd.ie
† These authors contributed equally to this work.

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Abstract: This work is concerned with the oscillatory behavior of solutions of even-order neutral differential equations. By using the technique of Riccati transformation and comparison principles with the second-order differential equations, we obtain a new Philos-type criterion. Our results extend and improve some known results in the literature. An example is given to illustrate our main results.

Keywords: even-order differential equations; neutral delay; oscillation

1. Introduction

In this article, we investigate the asymptotic behavior of solutions of even-order neutral differential equation of the form

$$\left(b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma}\right)'+\sum_{i=1}^{k}q_{i}\left(t\right)u^{\gamma}\left(\delta_{i}\left(t\right)\right)=0, \tag{1}$$

where $t\geq t_0$, $n\geq 4$ is an even natural number, $k\geq 1$ is an integer and $z\left(t\right):=u\left(t\right)+p\left(t\right)u\left(\sigma\left(t\right)\right)$. Throughout this paper, we assume the following conditions to hold:

(P_1) γ is a quotient of odd positive integers;
(P_2) $b\in C[t_0,\infty)$, $b\left(t\right)>0$, $b'\left(t\right)\geq 0$;
(P_3) $\sigma\in C^1[t_0,\infty)$, $\delta_i\in C[t_0,\infty)$, $\sigma'\left(t\right)>0$, $\delta\left(t\right)\leq\delta_i\left(t\right)$, $\sigma\left(t\right)\leq t$ and $\lim_{t\to\infty}\sigma\left(t\right)=\lim_{t\to\infty}\delta_i\left(t\right)=\infty$, $i=1,2,...,k$;
(P_4) $p,q_i\in C[t_0,\infty)$, $q_i\left(t\right)>0$, $0\leq p\left(t\right)<p_0<\infty$ and

$$\int_{t_0}^{\infty}b^{-1/\gamma}\left(s\right)\mathrm{d}s=\infty \tag{2}$$

Definition 1. *The function $u\in C^3[t_u,\infty)$, $t_u\geq t_0$, is called a solution of (1), if $b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma}\in C^1[t_u,\infty)$, and $u\left(t\right)$ satisfies (1) on $[t_u,\infty)$. Moreover, a solution of (1) is called oscillatory if it has arbitrarily large zeros on $[t_u,\infty)$, and otherwise is called to be nonoscillatory.*

Definition 2. *Let*

$$D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \text{ and } D_0 = \{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

A kernel function $H_i \in p(D,\mathbb{R})$ *is said to belong to the function class* $\Im$*, written by* $H \in \Im$*, if, for* $i = 1,2$*,*

(i) $H_i(t,s) = 0$ *for* $t \geq t_0$, $H_i(t,s) > 0$, $(t,s) \in D_0$;
(ii) $H_i(t,s)$ *has a continuous and nonpositive partial derivative* $\partial H_i/\partial s$ *on* D_0 *and there exist functions* $\sigma, \vartheta \in C^1([t_0,\infty),(0,\infty))$ *and* $h_i \in C(D_0,\mathbb{R})$ *such that*

$$\frac{\partial}{\partial s}H_1(t,s) + \frac{\theta'(s)}{\theta(s)}H_1(t,s) = h_1(t,s)\,H_1^{\gamma/(\gamma+1)}(t,s) \tag{3}$$

and

$$\frac{\partial}{\partial s}H_2(t,s) + \frac{v'(s)}{v(s)}H_2(t,s) = h_2(t,s)\sqrt{H_2(t,s)}. \tag{4}$$

The oscillation theory of differential equations with deviating arguments was initiated in a pioneering paper [1] of Fite, which appeared in the first quarter of the twentieth century.
Delay equations play an important role in applications of real life. One area of active research in recent times is to study the sufficient criteria for oscillation of differential equations, see [1–11], and oscillation of neutral differential equations has become an important area of research, see [12–30]. Having in mind such applications, for instance, in electrical engineering, we cite models that describe electrical power systems, see [18]. Neutral differential equations also have wide applications in applied mathematics [31,32], physics [33], ecology [34] and engineering [35].

In the following, we show some previous results in the literature related to this paper: Moaaz et al. [23] proved that if there exist positive functions η, $\zeta \in C^1([t_0,\infty),\mathbb{R})$ such that the differential equations

$$\psi'(t) + \left(\frac{\mu\left(\delta^{-1}(\eta(t))\right)^{n-1}}{(n-1)!r^{1/\alpha}\left(\delta^{-1}(\eta(t))\right)}\right)^{\alpha} q(t)\,P_n^{\alpha}(\sigma(t))\,\psi\left(\delta^{-1}(\eta(t))\right) = 0$$

and

$$\phi'(t) + \delta^{-1}(\zeta(t))\,R_{n-3}(t)\,\phi\left(\delta^{-1}(\zeta(t))\right) = 0$$

are oscillatory, then (1) is oscillatory.

Zafer [29] proved that the even-order differential equation

$$z^{(n)}(t) + q(t)\,x(\sigma(t)) = 0 \tag{5}$$

is oscillatory if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)\,2^{(n-1)(n-2)}}{\mathrm{e}}, \tag{6}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)\,2^{(n-1)(n-2)}, \quad \sigma'(t) \geq 0.$$

where $Q(t) := \sigma^{n-1}(t)\,(1 - p(\sigma(t)))\,q(t)$.

Zhang and Yan [30] proved that (5) is oscillatory if either

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}}, \tag{7}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)!,\ \ \sigma(t) \geq 0.$$

It's easy to note that $(n-1)! < (n-1)\,2^{(n-1)(n-2)}$ for $n > 3$, and hence results in [30] improved results of Zafer in [29].

Xing et al. [28] proved that (1) is oscillatory if

$$\left(\delta^{-1}(t)\right)' \geq \delta_0 > 0,\ \sigma'(t) \geq \sigma_0 > 0,\ \sigma^{-1}(\delta(t)) < t$$

and

$$\liminf_{t\to\infty}\int_{\sigma^{-1}(\delta(t))}^{t} \frac{\widehat{q}(s)}{b(s)}\left(s^{n-1}\right)^{\gamma}\mathrm{d}s > \left(\frac{1}{\delta_0}+\frac{p_0^{\gamma}}{\delta_0\sigma_0}\right)\frac{((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{8}$$

where $\widehat{q}(t) := \min\left\{q\left(\delta^{-1}(t)\right), q\left(\delta^{-1}(\sigma(t))\right)\right\}$.

Hence, [28] improved the results in [29,30].

In our paper, by carefully observing and employing some inequalities of different type, we provide a new criterion for oscillation of differential Equation (1). Here, we provide different criteria for oscillation, which can cover a larger area of different models of fourth order differential equations. We introduce a Riccati substitution and comparison principles with the second-order differential equations to obtain a new Philos-type criteria. Finally, we apply the main results to one example.

2. Some Auxiliary Lemmas

We shall employ the following lemmas:

Lemma 1 ([5]). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Uu - Vu^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}}.$$

Lemma 2 ([6]). *If the function u satisfies $u^{(i)}(t) > 0$, $i = 0, 1, ..., n$, and $u^{(n+1)}(t) < 0$, then*

$$\frac{u(t)}{t^n/n!} \geq \frac{u'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 3 ([4]). *The equation*

$$\left(b(t)\left(u'(t)^{\gamma}\right)\right)' + q(t)\,u^{\gamma}(t) = 0, \tag{9}$$

where $b \in C[t_0,\infty)$, $b(t) > 0$ and $q(t) > 0$, is non-oscillatory if and only if there exist a $t \geq t_0$ and a function $v \in C^1[\ell,\infty)$ such that

$$v'(t) + \frac{\gamma}{b^{1/\gamma}(t)}v^{1+1/\gamma}(t) + q(t) \leq 0,$$

for $t \geq t_0$.

Lemma 4 ([2], Lemma 2.2.3). *Let $u \in C^n([t_0,\infty),(0,\infty))$. Assume that $u^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists a $t_1 \geq t_0$ such that $u^{(n-1)}(t)\,u^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} u(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_\mu \geq t_1$ such that*

$$u(t) \geq \frac{\mu}{(n-1)!}t^{n-1}\left|u^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

3. Main Results

In this section, we give the main results of the article. Here, we define the next notation:

$$\begin{aligned}
P_k(t) &= \frac{1}{p\left(\sigma^{-1}(t)\right)}\left(1-\frac{\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)^{k-1}}{\left(\sigma^{-1}(t)\right)^{k-1}p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}\right), \quad \text{for } k=2,n,\\
R_0(t) &= \left(\frac{1}{b(t)}\int_t^{\infty}\sum_{i=1}^{k}q_i(s)\,P_2^{\gamma}\left(\delta_i(s)\right)\mathrm{d}s\right)^{1/\gamma},\\
\Theta(t) &= \gamma\frac{\mu_1}{(n-2)!}\left(\frac{b(t)}{b\left(\sigma^{-1}\left(\delta_i(t)\right)\right)}\right)^{1/\gamma}\frac{\left(\sigma^{-1}\left(\delta_i(t)\right)\right)'\left(\delta_i(t)\right)'\left(\sigma^{-1}\left(\delta_i(t)\right)\right)^{n-2}}{(b\theta)^{1/\gamma}(t)},\\
\tilde{\Theta}(t) &= \frac{h_1^{\gamma+1}(t,s)\,H_1^{\gamma}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{\left((n-2)!\right)^{\gamma}b\left(\sigma^{-1}\left(\delta_i(t)\right)\right)\theta(t)}{\left(\mu_1\left(\sigma^{-1}\left(\delta_i(t)\right)\right)'\left(\delta_i(t)\right)'\left(\sigma^{-1}\left(\delta_i(t)\right)\right)^{n-2}\right)^{\gamma}}
\end{aligned}$$

and

$$R_m(t)=\int_t^{\infty}R_{m-1}(s)\,\mathrm{d}s,\quad m=1,2,...,n-3.$$

Lemma 5 ([8], Lemma 1.2). *Assume that u is an eventually positive solution of (1). Then, there exist two possible cases:*

$$\begin{array}{ll}
(\mathbf{S}_1) & z(t)>0,\ z'(t)>0, z''(t)>0,\ z^{(n-1)}(t)>0,\ z^{(n)}(t)<0,\\
(\mathbf{S}_2) & z(t)>0, z^{(j)}(t)>0, z^{(j+1)}(t)<0 \text{ for all odd integer}\\
 & j\in\{1,3,...,n-3\}, z^{(n-1)}(t)>0, z^{(n)}(t)<0,
\end{array}$$

for $t\geq t_1$, where $t_1\geq t_0$ is sufficiently large.

Lemma 6. *Let u be an eventually positive solution of (1) and*

$$\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)^{n-1}<\left(\sigma^{-1}(t)\right)^{n-1}p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right). \tag{10}$$

Then

$$u(t)\geq\frac{z\left(\sigma^{-1}(t)\right)}{p\left(\sigma^{-1}(t)\right)}-\frac{1}{p\left(\sigma^{-1}(t)\right)}\frac{z\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}{p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}. \tag{11}$$

Proof. Let u be an eventually positive solution of (1) on $[t_0,\infty)$. From the definition of $z(t)$, we see that

$$p(t)\,u(\sigma(t))=z(t)-u(t)$$

and so

$$p\left(\sigma^{-1}(t)\right)u(t)=z\left(\sigma^{-1}(t)\right)-z\left(\sigma^{-1}(t)\right).$$

Repeating the same process, we obtain

$$u(t)=\frac{1}{p\left(\sigma^{-1}(t)\right)}\left(z\left(\sigma^{-1}(t)\right)-\left(\frac{z\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}{p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}-\frac{u\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}{p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}\right)\right),$$

which yields

$$u(t)\geq\frac{z\left(\sigma^{-1}(t)\right)}{p\left(\sigma^{-1}(t)\right)}-\frac{1}{p\left(\sigma^{-1}(t)\right)}\frac{z\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}{p\left(\sigma^{-1}\left(\sigma^{-1}(t)\right)\right)}.$$

Thus, (11) holds. This completes the proof. □

Lemma 7. *Assume that u is an eventually positive solution of (1) and*

$$\left(b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma}\right)' \leq -z^{\gamma}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)\sum_{i=1}^{k} q_i\left(t\right)P_n^{\gamma}\left(\delta_i\left(t\right)\right), \text{ if } z \text{ satisfies } \left(\mathbf{S}_1\right) \tag{12}$$

and

$$z''\left(t\right)+R_{n-3}\left(t\right)z\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)\leq 0, \text{ if } z \text{ satisfies } \left(\mathbf{S}_2\right). \tag{13}$$

Proof. Let u be an eventually positive solution of (1) on $[t_0,\infty)$. It follows from Lemma 5 that there exist two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Suppose that Case $(\mathbf{S}_1)$ holds. From Lemma 2, we obtain $z\left(t\right)\geq\frac{1}{(n-1)}tz'\left(t\right)$ and hence the function $t^{1-n}z\left(t\right)$ is nonincreasing, which with the fact that $\sigma\left(t\right)\leq t$ gives

$$\left(\sigma^{-1}\left(t\right)\right)^{n-1}z\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)\leq\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)^{n-1}z\left(\sigma^{-1}\left(t\right)\right). \tag{14}$$

Combining (11) and (14), we conclude that

$$\begin{aligned} u\left(t\right) &\geq \frac{1}{p\left(\sigma^{-1}\left(t\right)\right)}\left(1-\frac{\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)^{n-1}}{\left(\sigma^{-1}\left(t\right)\right)^{n-1}p\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)}\right)z\left(\sigma^{-1}\left(t\right)\right)\\ &= P_n\left(t\right)z\left(\sigma^{-1}\left(t\right)\right). \end{aligned} \tag{15}$$

From (1) and (15), we obtain

$$\begin{aligned} \left(b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma}\right)' &\leq -\sum_{i=1}^{k} q_i\left(t\right)P_n^{\gamma}\left(\delta_i\left(t\right)\right)z^{\gamma}\left(\sigma^{-1}\left(\delta_i\left(t\right)\right)\right)\\ &\leq -z^{\gamma}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)\sum_{i=1}^{k} q_i\left(t\right)P_n^{\gamma}\left(\delta_i\left(t\right)\right). \end{aligned}$$

Thus, (12) holds.

Suppose that Case $(\mathbf{S}_2)$ holds. From Lemma 2, we find

$$z\left(t\right)\geq tz'\left(t\right) \tag{16}$$

and thus the function $t^{-1}z\left(t\right)$ is nonincreasing, eventually. Since $\sigma^{-1}\left(t\right)\leq\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)$, we obtain

$$\sigma^{-1}\left(t\right)z\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)\leq\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)z\left(\sigma^{-1}\left(t\right)\right). \tag{17}$$

Combining (11) and (17), we find

$$\begin{aligned} u\left(t\right) &\geq \frac{1}{p\left(\sigma^{-1}\left(t\right)\right)}\left(1-\frac{\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)}{\left(\sigma^{-1}\left(t\right)\right)p\left(\sigma^{-1}\left(\sigma^{-1}\left(t\right)\right)\right)}\right)z\left(\sigma^{-1}\left(t\right)\right)\\ &= P_2\left(t\right)z\left(\sigma^{-1}\left(t\right)\right), \end{aligned}$$

which with (1) yields

$$\left(b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma}\right)'+\sum_{i=1}^{k} q_i\left(t\right)P_2^{\gamma}\left(\delta_i\left(t\right)\right)z^{\gamma}\left(\sigma^{-1}\left(\delta_i\left(t\right)\right)\right)\leq 0. \tag{18}$$

Integrating the (18) from t to ∞, we obtain

$$z^{(n-1)}(t) \geq b_0(t) z\left(\sigma^{-1}(\delta(t))\right).$$

Integrating this inequality from t to ∞ a total of $n-3$ times, we obtain

$$z''(t) + R_{n-3}(t) z\left(\sigma^{-1}(\delta(t))\right) \leq 0.$$

Thus, (13) holds. This completes the proof. □

Theorem 1. *Let (2) and (10) hold. If there exist positive functions $\theta, v \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty} \frac{1}{H_1(t,t_1)} \int_{t_1}^{t} \left(H_1(t,s)\psi(s) - \tilde{\Theta}(s)\right) ds = \infty \tag{19}$$

and

$$\limsup_{t\to\infty} \frac{1}{H_2(t,t_1)} \int_{t_1}^{t} \left(H_2(t,s)\psi^*(s) - \frac{v(s) h_2^2(t,s)}{4}\right) ds = \infty, \tag{20}$$

where

$$\psi(s) = \theta(t) \sum_{i=1}^{k} q_i(t) P_n^{\gamma}(\delta_i(t)), \ \psi^*(s) = v(t) b_{n-3}(t) \left(\frac{\sigma^{-1}(\delta(t))}{t}\right)$$

and

$$\tilde{\Theta}(s) = \frac{h_1^{\gamma+1}(t,s) H_1^{\gamma}(t,s)}{(\gamma+1)^{\gamma+1}} \frac{((n-2)!)^{\gamma} b\left(\sigma^{-1}(\delta(t))\right)\theta(t)}{\left(\mu_1 \left(\sigma^{-1}(\delta(t))\right)' (\delta(t))' \left(\sigma^{-1}(\delta(t))\right)^{n-2}\right)^{\gamma}},$$

then (1) is oscillatory.

Proof. Let u be a non-oscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we can assume that u is eventually positive. It follows from Lemma 5 that there exist two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let $(\mathbf{S}_1)$ hold. From Lemma 7, we arrive at (12). Next, we define a function ξ by

$$\xi(t) := \theta(t) \frac{b(t)\left(z^{(n-1)}(t)\right)^{\gamma}}{z^{\gamma}\left(\sigma^{-1}(\delta(t))\right)} > 0.$$

Differentiating and using (12), we obtain

$$\begin{aligned} \xi'(t) \leq & \frac{\theta'(t)}{\theta(t)}\xi(t) - \theta(t) \sum_{i=1}^{k} q_i(t) P_n^{\gamma}(\delta_i(t)) \\ & -\gamma\theta(t) \frac{b(t)\left(z^{(n-1)}(t)\right)^{\gamma} \left(\sigma^{-1}(\delta(t))\right)' (\delta(t))' z_u'\left(\sigma^{-1}(\delta(t))\right)}{z_u^{\gamma+1}\left(\sigma^{-1}(\delta(t))\right)}. \end{aligned} \tag{21}$$

Recalling that $b(t)\left(z^{(n-1)}(t)\right)^{\gamma}$ is decreasing, we get

$$b\left(\sigma^{-1}(\delta(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\delta(t))\right)\right)^{\gamma} \geq b(t)\left(z^{(n-1)}(t)\right)^{\gamma}.$$

This yields

$$\left(z^{(n-1)}\left(\sigma^{-1}(\delta(t))\right)\right)^{\gamma} \geq \frac{b(t)}{b\left(\sigma^{-1}(\delta(t))\right)}\left(z^{(n-1)}(t)\right)^{\gamma}. \tag{22}$$

It follows from Lemma 4 that

$$z'\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right) \geq \frac{\mu_1}{(n-2)!}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)^{n-2} z^{(n-1)}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right), \tag{23}$$

for all $\mu_1 \in (0,1)$ and every sufficiently large t. Thus, by (21), (22) and (23), we get

$$\begin{aligned}\xi'\left(t\right) \leq\ & \frac{\theta'\left(t\right)}{\theta\left(t\right)}\xi\left(t\right) - \theta\left(t\right)\sum_{i=1}^{k} q_i\left(t\right) P_n^{\gamma}\left(\delta_i\left(t\right)\right) \\ & -\gamma\theta\left(t\right)\frac{\mu_1}{(n-2)!}\left(\frac{b\left(t\right)}{b\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)}\right)^{1/\gamma} \frac{b\left(t\right)\left(z^{(n-1)}\left(t\right)\right)^{\gamma+1}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)'\left(\delta\left(t\right)\right)'\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)^{n-2}}{z^{\gamma+1}\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)}.\end{aligned}$$

Hence,

$$\begin{aligned}\xi'\left(t\right) \leq\ & \frac{\theta'\left(t\right)}{\theta\left(t\right)}\xi\left(t\right) - \theta\left(t\right)\sum_{i=1}^{k} q_i\left(t\right) P_n^{\gamma}\left(\delta_i\left(t\right)\right) \\ & -\Theta\left(t\right)\xi^{\frac{\gamma+1}{\gamma}}\left(t\right).\end{aligned} \tag{24}$$

Multiplying (24) by $H_1\left(t,s\right)$ and integrating the resulting inequality from t_1 to t; we find that

$$\begin{aligned}\int_{t_1}^{t} H_1\left(t,s\right)\psi\left(s\right)\mathrm{d}s \leq\ & \xi\left(t_1\right)H_1\left(t,t_1\right) + \int_{t_1}^{t}\left(\frac{\partial}{\partial s}H_1\left(t,s\right) + \frac{\theta'\left(s\right)}{\theta\left(s\right)}H_1\left(t,s\right)\right)\xi\left(s\right)\mathrm{d}s \\ & -\int_{t_1}^{t}\Theta\left(s\right)H_1\left(t,s\right)\xi^{\frac{\gamma+1}{\gamma}}\left(s\right)\mathrm{d}s.\end{aligned}$$

From (3), we get

$$\begin{aligned}\int_{t_1}^{t} H_1\left(t,s\right)\psi\left(s\right)\mathrm{d}s \leq\ & \xi\left(t_1\right)H_1\left(t,t_1\right) + \int_{t_1}^{t} h_1\left(t,s\right)H_1^{\gamma/(\gamma+1)}\left(t,s\right)\xi\left(s\right)\mathrm{d}s \\ & -\int_{t_1}^{t}\Theta\left(s\right)H_1\left(t,s\right)\xi^{\frac{\gamma+1}{\gamma}}\left(s\right)\mathrm{d}s.\end{aligned} \tag{25}$$

Using Lemma 1 with $V = \Theta\left(s\right)H_1\left(t,s\right)$, $U = h_1\left(t,s\right)H_1^{\gamma/(\gamma+1)}\left(t,s\right)$ and $u = \xi\left(s\right)$, we get

$$\begin{aligned}& h_1\left(t,s\right)H_1^{\gamma/(\gamma+1)}\left(t,s\right)\xi\left(s\right) - \Theta\left(s\right)H_1\left(t,s\right)\xi^{\frac{\gamma+1}{\gamma}}\left(s\right) \\ \leq\ & \frac{h_1^{\gamma+1}\left(t,s\right)H_1^{\gamma}\left(t,s\right)}{\left(\gamma+1\right)^{\gamma+1}}\frac{\left((n-2)!\right)^{\gamma} b\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)\theta\left(t\right)}{\left(\mu_1\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)'\left(\delta\left(t\right)\right)'\left(\sigma^{-1}\left(\delta\left(t\right)\right)\right)^{n-2}\right)^{\gamma}},\end{aligned}$$

which with (25) gives

$$\frac{1}{H_1\left(t,t_1\right)}\int_{t_1}^{t}\left(H_1\left(t,s\right)\psi\left(s\right) - \tilde{\Theta}\left(s\right)\right)\mathrm{d}s \leq \xi\left(t_1\right),$$

which contradicts (19).

On the other hand, let $(\mathbf{S}_2)$ hold. Using Lemma 7, we get that (13) holds. Now, we define

$$\varphi\left(t\right) = v\left(t\right)\frac{z'\left(t\right)}{z\left(t\right)}. \tag{26}$$

Then $\varphi(t) > 0$ for $t \geq t_1$. By differentiating φ and using (13), we find

$$\begin{aligned} \varphi'(t) &= \frac{v'(t)}{v(t)}\varphi(t) + v(t)\frac{z''(t)}{z(t)} - v(t)\left(\frac{z'(t)}{z(t)}\right)^2 \\ &\leq \frac{v'(t)}{v(t)}\varphi(t) - v(t)\, b_{n-3}(t)\frac{z\left(\sigma^{-1}(\delta(t))\right)}{z(t)} - \frac{1}{v(t)}\varphi^2(t). \end{aligned} \tag{27}$$

By using Lemma 2, we find that

$$z(t) \geq tz'(t). \tag{28}$$

From (28), we get that

$$z\left(\sigma^{-1}(\delta(t))\right) \geq \frac{\sigma^{-1}(\delta(t))}{t} z(t). \tag{29}$$

Thus, from (27) and (29), we obtain

$$\varphi'(t) \leq \frac{v'(t)}{v(t)}\varphi(t) - v(t)\, R_{n-3}(t)\left(\frac{\sigma^{-1}(\delta(t))}{t}\right) - \frac{1}{v(t)}\varphi^2(t). \tag{30}$$

Multiplying (30) by $H_2(t,s)$ and integrating the resulting from t_1 to t, we obtain

$$\begin{aligned} \int_{t_1}^{t} H_2(t,s)\,\psi^*(s)\,\mathrm{d}s &\leq \varphi(t_1)\,H_2(t,t_1) \\ &\quad + \int_{t_1}^{t}\left(\frac{\partial}{\partial s}H_2(t,s) + \frac{v'(s)}{v(s)}H_2(t,s)\right)\varphi(s)\,\mathrm{d}s \\ &\quad - \int_{t_1}^{t}\frac{1}{v(s)}H_2(t,s)\,\varphi^2(s)\,\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{t_1}^{t} H_2(t,s)\,\psi^*(s)\,\mathrm{d}s &\leq \varphi(t_1)\,H_2(t,t_1) + \int_{t_1}^{t} h_2(t,s)\sqrt{H_2(t,s)}\,\varphi(s)\,\mathrm{d}s \\ &\quad - \int_{t_1}^{t}\frac{1}{v(s)}H_2(t,s)\,\varphi^2(s)\,\mathrm{d}s \\ &\leq \varphi(t_1)\,H_2(t,t_1) + \int_{t_1}^{t}\frac{v(s)\,h_2^2(t,s)}{4}\mathrm{d}s \end{aligned}$$

and so

$$\frac{1}{H_2(t,t_1)}\int_{t_1}^{t}\left(H_2(t,s)\,\psi^*(s) - \frac{v(s)\,h_2^2(t,s)}{4}\right)\mathrm{d}s \leq \varphi(t_1),$$

which contradicts (20). This completes the proof. □

In the next theorem, we establish new oscillation results for (1) by using the theory of comparison with a second order differential equation.

Theorem 2. *Assume that the equation*

$$y''(t) + y(t)\sum_{i=1}^{k} q_i(t)\,P_n^{\gamma}(\delta_i(t)) = 0 \tag{31}$$

and

$$\left[b(t)\left(y'(t)\right)^{\gamma}\right]' + R_{n-3}(t)\left(\frac{\sigma^{-1}(\delta(t))}{t}\right)y^{\gamma}(t) = 0, \tag{32}$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a eventually positive solution u and by virtue of Lemma 3. From Theorem 1, we set $\theta(t) = 1$ in (24), then we get

$$\xi'(\ell) + \Theta(t)\,\xi^{\frac{\gamma+1}{\gamma}} + \sum_{i=1}^{k} q_i(t)\,P_n^{\gamma}(\delta_i(t)) \leq 0.$$

Thus, we can see that Equation (31) is nonoscillatory, which is a contradiction. If we now set $v(t) = 1$ in (30), then we obtain

$$\varphi'(t) + R_{n-3}(t)\left(\frac{\sigma^{-1}(\delta(t))}{t}\right) + \varphi^2(t) \leq 0.$$

Hence, Equation (32) is nonoscillatory, which is a contradiction.
Theorem 2 is proved. □

Corollary 1. *If conditions (19) and (20) in Theorem 1 are replaced by the following conditions:*

$$\limsup_{t\to\infty} \frac{1}{H_1(t,t_1)} \int_{t_1}^{t} H_1(t,s)\,\psi(s)\,\mathrm{d}s = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H_1(t,t_1)} \int_{t_1}^{t} \tilde{\Theta}(s)\,\mathrm{d}s < \infty.$$

Moreover,

$$\limsup_{t\to\infty} \frac{1}{H_2(t,t_1)} \int_{t_1}^{t} H_2(t,s)\,\psi^*(s)\,\mathrm{d}s = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H_2(t,t_1)} \int_{t_1}^{t} v(s)\,h_2^2(t,s)\,\mathrm{d}s < \infty,$$

then (1) is oscillatory.

Corollary 2. *Let (10) holds. If there exist positive functions $v, \theta \in^1 ([t_0,\infty),\mathbb{R})$ such that*

$$\int_{t_0}^{\infty} \left(\theta(s)\sum_{i=1}^{k} q_i(s)\,P_n^{\gamma}(\delta_i(s)) - \varpi(s)\right)\mathrm{d}s = \infty \tag{33}$$

and

$$\int_{t_0}^{\infty} \left(P_1 v(s) \int_{t}^{\infty} \left(\frac{1}{r(\varrho)} \int_{\varrho}^{\infty} \sum_{i=1}^{k} q_i(s) \left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\alpha} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}\varrho - \pi(s)\right)\mathrm{d}s = \infty, \tag{34}$$

where

$$\varpi(t) := \frac{(n-2)!^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r\left(\tau^{-1}(\sigma(t))\right)(\theta'(t))^{\alpha+1}}{\left(\mu_1\theta(t)\left(\tau^{-1}(\sigma(t))\right)'\left(\tau^{-1}(\sigma(t))\right)^{n-2}\right)^{\alpha}}$$

and

$$\pi(t) := \frac{(v'(s))^2}{4v(s)},$$

then (1) is oscillatory.

Example 1. *Consider the equation*

$$\left(x(t) + 16x\left(\frac{1}{2}t\right)\right)^{(4)} + \frac{q_0}{t^4}x\left(\frac{1}{3}t\right) = 0, \quad t \geq 1, \tag{35}$$

where $q_0 > 0$. *We note that* $r(t) = 1$, $p(t) = 16$, $\tau(t) = t/2$, $\sigma(t) = t/3$ *and* $q(t) = q_0/t^4$.

Thus, we have

$$P_1(t) = \frac{1}{32}, \; P_2(t) = \frac{7}{128}.$$

Now, we obtain

$$\int_{t_0}^{\infty} \left(\theta(s) \sum_{i=1}^{k} q_i(s) P_n^{\gamma}(\delta_i(s)) - \varpi(s) \right) \mathrm{d}s = \infty$$

and

$$\begin{aligned} & \int_{t_0}^{\infty} \left(P_1 v(s) \int_{t}^{\infty} \left(\frac{1}{r(\varrho)} \int_{\varrho}^{\infty} \sum_{i=1}^{k} q_i(s) \left(\frac{\tau^{-1}(\sigma(s))}{s} \right)^{\alpha} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}\varrho - \pi(t) \right) \mathrm{d}s \\ = & \int_{t_0}^{\infty} \left(\frac{7q_0}{1152} - \frac{1}{4} \right) \mathrm{d}s, \\ = & \infty, \quad \textit{if } q_0 > 41.14. \end{aligned}$$

Thus, by using Corollary 2, Equation (35) is oscillatory if $q_0 > 41.14$.

4. Conclusions

The aim of this article was to provide a study of asymptotic nature for a class of even-order neutral delay differential equations. We used a generalized Riccati substitution and the integral averaging technique to ensure that every solution of the studied equation is oscillatory. The results presented here complement some of the known results reported in the literature.

A further extension of this article is to use our results to study a class of systems of higher order neutral differential equations as well as of fractional order. For all these there is already some research in progress.

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