Dynamics Analysis and Chaotic Control of a Fractional-Order Three-Species Food-Chain System

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Abstract: Based on Hastings and Powell's research, this paper extends a three-species food-chain system to fractional-order form, whose dynamics are analyzed and explored. The necessary conditions for generating chaos are confirmed by the stability theory of fractional-order systems, chaos is characterized by its phase diagrams, and bifurcation diagrams prove that the dynamic behaviors of the fractional-order food-chain system are affected by the order. Next, the chaotic control of the fractional-order system is realized by the feedback control method with a good effect in a relative short period. The stability margin of the controlled system is revealed by the theory and numerical analysis. Finally, the results of theory analysis are verified by numerical simulations.

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1. Introduction

Chaos is the common phenomenon in non-linear science, and it is a special motion of non-linear systems. The best-known example of chaos is the Butterfly Effect. A butterfly in the rainforest of the Amazon River in South America occasionally flapping its wings may trigger a tornado in Texas two weeks later, i.e., it will have a huge impact on the future after constant evolution, even there is a tiny change to the initial condition. This phenomenon is not only very interesting, but also has many applications in non-linear science. Some chaotic systems, such as the Lorenz system [1], Chen system [2], Rossler system [3] and so on, were studied synthetically, and excellent performances were shown in meteorology [4], circuits [5], communication [6], physiology [7], medicine [8,9], and finance [10], etc.

Some chaotic phenomena were found in the ecosystem, such as predation, competition, parasitism, mutual benefit, and symbiosis [11]. The food chain refers to the food-linked chain relationship in which various species in the ecosystem can maintain their own living activities and take other species as food. The predator–prey relationship between species constitutes the food chain, further forming the food-chain system. However, chaos in the system can bring adverse effects on the healthy development of the system. Therefore, the in-depth study of the dynamics of systems is a valuable and significance topic, which can provide theoretical support for the regulation of the food chain and of ecological balance.

In 1991, a teacup-type chaotic attractor was reported in a three-species food-chain system presented by Hastings and Powell [12]. Furthermore, two kinds of three-species ecological systems with hybrid functional responses were presented, showing complex dynamics behaviors [13]. Thereafter,

a modified Leslie–Gower-type three-species food-chain impulsive system with harvesting was reported, and the stronger impact of harvesting effort on the chaotic behaviors was analyzed [14]. Recently, the three-species system controlled by the Allee and Refugia parameters was presented, and the effects of the two parameters on the dynamics of the system were systematically discussed [15]. The food-chain predator–prey systems have become a topic worthy of further investigation [16–22]. These studies are based on the integer-order form and focus on theoretical analysis, but less on practical applications of fractional-order systems.

Due to the complexity and existence of non-linear effects from natural or unnatural factors, a fractional model of species systems provides a new feasibility to precisely describe dynamic behaviors of the multi-species food-chain ecosystems. The advantages of the fractional-order form are emerging in many ways, such as the meticulous depiction and accurate interpretation of operation rules. Based on this, traditional integer differential equations of ecosystems with predation, competition, and parasitism are replaced by fractional differential equations, which are used to further explore the dynamics of the systems. A fractional-order SIR model was studied to simulate the spatial spread of a hypothetical epidemic, which explored the dynamic evolution of the system [23]. The spatial spread of species following the Lévy motion was analyzed and simulated by a fractional-order diffusion–reaction model, which confirmed that fractional-order diffusion could lead to exponentially accelerating fronts of the system [24]. The dynamic behaviors of the fractional-order two-species cooperative systems with harvesting were studied, which provided several sufficient conditions to stabilize the system [25]. The fractional-order model of the system is introduced to more and more ecosystems to explore the corresponding dynamic behaviors [26–32]. Recently, the study of the fractional-order system has become a hotspot. However, there is less research on the effect of order on the dynamics of fractional-order ecosystems. To further explore the dynamics of the three-species food chain, the corresponding fractional-order system is presented in this paper.

Chaos is not conducive to the stability of biological species and easily leads to serious imbalances and even collapse of ecosystems. The stability control of the three-species food-chain system with the Holling I-type functional response was realized, which suppressed chaos of the system successfully [33]. Three different feedback control strategies were presented to stabilize a discrete-time prey–predator system at different P-periodic orbits [34]. Many results in recent years have been obtained in the control of the traditional integer-order ecosystems, which eliminates the influence of chaos on the stability of the systems [35–39]. Similar techniques were used in [40]. This indicates that manual intervention is necessary for realizing the balance and long-term development of ecosystems. Now, there is little work done in the stability control of the fractional-order food-chain ecosystem. To balance and develop the ecosystem better, the chaos control of the fractional-order three-species food-chain system is explored and realized in this paper.

To be specific, this paper extends an integer-order three-species food-chain system to fractional-order form, and determines the range of order with chaos in the fractional-order ecosystem. The dynamics of the derived fractional-order ecosystem is studied with the variation of the parameter value. Moreover, the chaos control is achieved by classical feedback control method. Finally, the stability margin of the fractional-order system is measured.

The paper is organized as follows. The food-chain model studied by Hastings and Powell is transformed in Section 2. The dynamic behaviors of the integer-order food-chain model are analyzed in Section 3. The dynamic behaviors of the corresponding fractional-order system are explored in Section 4. The chaos control of the fractional-order system is realized, and the stability margin of the controlled system is confirmed in Section 5. Conclusions are given in the last section.

2. The Food-Chain System

The dynamics of the food-chain system [12] are described by:

$$
\begin{cases}
\frac{dX}{dT} = R_0 X\left(1 - \frac{X}{K_0}\right) - C_1 F_1(X)Y, \\
\frac{dY}{dT} = F_1(X)Y - F_2(Y)Z - D_1 Y, \\
\frac{dZ}{dT} = C_2 F_2(Y)Z - D_2 Z,
\end{cases}
\tag{1}
$$

where the variables X, Y and Z represent the species of the prey, the intermediate predator and the top predator respectively. Parameters R_0 and K_0 are the intrinsic growth rate and the carrying capacity of species X respectively, parameters C_1^{-1} and C_2 represent the conversion rate of species X to species Y and Z respectively, parameters D_1 and D_2 are the death rates of the species Y and Z respectively. The A_i and B_i are the parameters of the functional response

$$
F_i(U) = \frac{A_i U}{B_i + U} (i = 1, 2),
\tag{2}
$$

where the functions $F_1(X)Y$ and $F_2(X)Y$ represent the interactions between species, respectively.

The modified variables are chosen as follows:

$$
x = \frac{X}{K_0}, y = \frac{C_1 Y}{K_0}, z = \frac{C_1 Z}{C_2 K_0}, t = R_0 T.
\tag{3}
$$

The system (1) is transformed to be:

$$
\begin{cases}
\frac{dx(t)}{dt} = x(t)(1 - x(t)) - \frac{a_1 x(t)y(t)}{1 + b_1 x(t)}, \\
\frac{dy(t)}{dt} = \frac{a_1 x(t)y(t)}{1 + b_1 x(t)} - \frac{a_2 y(t)z(t)}{1 + b_2 y(t)} - d_1 y(t), \\
\frac{dz(t)}{dt} = \frac{a_2 y(t)z(t)}{1 + b_2 y(t)} - d_2 z(t), \\
x(0) = x_0 > 0, \\
y(0) = y_0 > 0, \\
z(0) = z_0 > 0,
\end{cases}
\tag{4}
$$

where

$$
a_1 = \frac{K_0 A_1}{R_0 B_1}, a_2 = \frac{C_2 A_2 K_0}{C_1 R_0 B_2}, b_1 = \frac{K_0}{B_1}, b_2 = \frac{K_0}{C_1 B_2}, d_1 = \frac{D_1}{R_0}, d_2 = \frac{D_2}{R_0}.
\tag{5}
$$

Based on the three-species food-chain system, the initial conditions x_0, y_0 and z_0 are all positive. For the system (4), since species Z is more level predator than species Y, the natural death rate of species Z is lower than species Y in the competition for survival, i.e., $d_2 < d_1$, and parameters d_1 and d_2 are all positive.

3. Dynamics of the Integer-Order Ecosystem

3.1. Basic Characteristic Analysis

The dynamic behaviors of the system (4) are analyzed in this subsection, because the Lyapunov exponents are an important quantitative measure for describing the complexity of non-linear systems. It indicates the average exponential rate of convergence or divergence between adjacent orbits of non-linear systems in the phase space. The positive largest Lyapunov exponent means the existence of chaos.

Let the values of the parameters a_1, a_2, b_1, b_2, d_1 and d_2 be fixed as in Table 1, with the initial conditions $(x_0, y_0, z_0) = (0.75, 0.3, 9)$.

Table 1. The values of the parameters a_1, a_2, b_1, b_2, d_1, d_2.

Parameters	Values
a_1	5
a_2	0.1
b_1	2.9
b_2	2.15
d_1	0.4
d_2	0.01

The Lyapunov exponents of the system (4) are calculated as:

$$LE_1 = 0.0169, LE_2 = 0, \ LE_3 = -0.4646. \tag{6}$$

Based on the definition of the Lyapunov dimension:

$$D_L = j + \frac{1}{\left|LE_{(j+1)}\right|}\sum_{i=1}^{j} LE_j, \tag{7}$$

where j is a positive integer which satisfies

$$\begin{cases} LE_1 + LE_2 + \ldots + LE_j > 0, \\ LE_1 + LE_2 + \ldots + LE_j + LE_{j+1} < 0. \end{cases} \tag{8}$$

Substitute (6) into (7), the corresponding Lyapunov dimension is calculated below:

$$D_{L_0} = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 2 + \frac{0.0169 + 0}{|-0.4646|} = 2.0364. \tag{9}$$

Since the largest Lyapunov exponent is $LE_1 > 0$, and the Lyapunov dimension of the system is not an integer, the system (4) is chaotic. The corresponding chaotic attractor is obtained, as shown in Figure 1. It can be seen from Figure 1d that the system has a cup attractor.
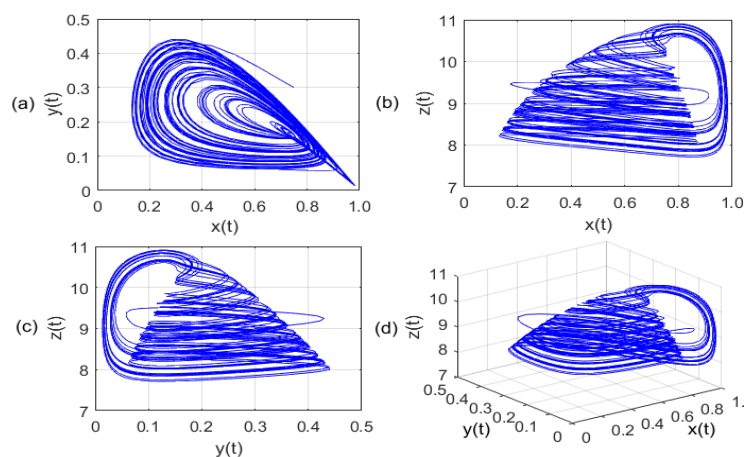
Figure 1. The chaotic attractor of system (4). (**a**) $x(t)$-$y(t)$; (**b**) $x(t)$-$z(t)$; (**c**) $y(t)$-$z(t)$; (**d**) $x(t)$-$y(t)$-$z(t)$.

The phase diagrams in different subspaces are given in Figure 1a–c, respectively. The trajectory of the system performs a complex motion similar to random in the specific parameter set, which corresponds to the variations of number of each species in the three-species ecosystems. The time-domain waves of the system further confirm that the trajectory of three-species ecosystem is chaotic, as shown in Figure 2. It can be seen from Figures 1 and 2 that the teacup attractor and the

time-domain waves of the system are similar to the counterparts in [12], though the values of the parameters b_1 and b_2 are different from the original ones.
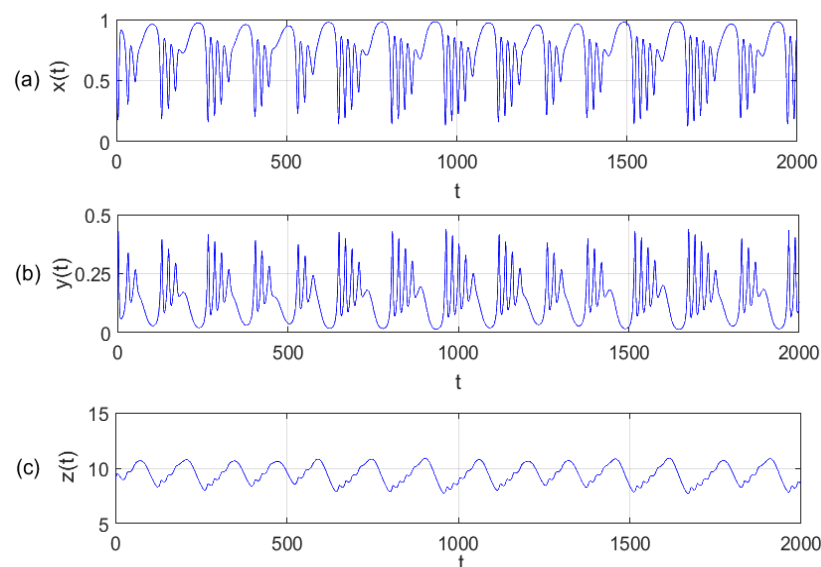

Figure 2. Time-domain waves of system (4). (**a**) $x(t)$-t; (**b**) $y(t)$-t; (**c**) $z(t)$-t.

To explore the effect of environmental factors on the ecosystems, the dynamics of system (4) with the varying parameters are analyzed by Lyapunov exponents and bifurcation diagrams.

3.2. Dynamics of the Ecosystem with the Varying Parameter b_1

The dynamics of the ecosystem with the varying parameter b_1 are explored in this subsection. Let the parameters a_1, a_2, b_2, d_1, d_2 be fixed as Table 1 and the parameter b_1 varies in the interval $[2,3]$, the bifurcation diagram of the system is calculated, as shown in Figure 3. It can be seen from this figure, the system goes through state transitions from period-1 to period-2, then to period-4, and enters the chaotic state as the value of parameter b_1 increases, i.e., the system can generate chaos by the period-doubling bifurcation, which can lead to the destruction of the whole ecosystem.
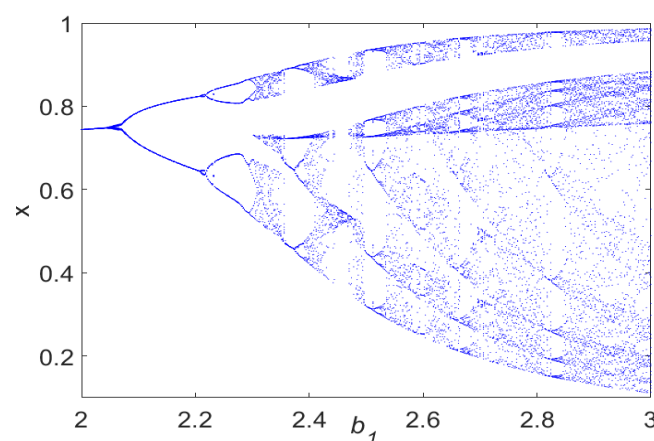

Figure 3. The bifurcation diagram with the varying parameter b_1.

Based on the relationship

$$b_1 = \frac{K_0}{B_1} \tag{10}$$

in Section 2, parameter b_1 is positively correlated with the carrying capacity K_0 of species X when parameter B_1 is fixed. Since the parameter b_1 reflects the effect of K_0 on the stability of the three-species food-chain system, the ecosystem generating chaos with the increase of b_1 means that the increase of K_0 can cause the ecosystem to lose balance. To keep the species of the ecosystem in balance, the carrying capacity K_0 of the species is limited. Once it is beyond the tolerance of nature, the ecosystem will be destroyed or even collapsed. Thus, the parameter b_1 can be used as a control parameter to adjust the ecosystem dynamics to bring it to a new equilibrium.

3.3. Dynamics of the Ecosystem with the Varying Parameter d_1

Since the impact of mortality of species Y has a key effect on the ecosystem, the dynamics of system (4) with the varying parameter d_1 (positive correlation with mortality) is analyzed in this subsection. Letting parameter d_1 be varying from 0 to 1, and parameters a_1, a_2, b_1, d_1, d_2 be fixed as in Table 1, the Lyapunov exponents of system (4) are calculated, as shown in Figure 4. LE_1 is the largest Lyapunov exponent, as shown in Figure 4a, the others are less than zero, as shown in Figure 4b,c, respectively. It can be seen from Figure 4a that LE_1 is positive in the interval (0.24, 0.51), and negative in the interval (0.52, 1). These indicate that the three-species food-chain ecosystem has chaotic and periodic behaviors with the different parameters d_1, which predicts that the balance of the ecosystem can be realized by adjusting the parameter d_1.
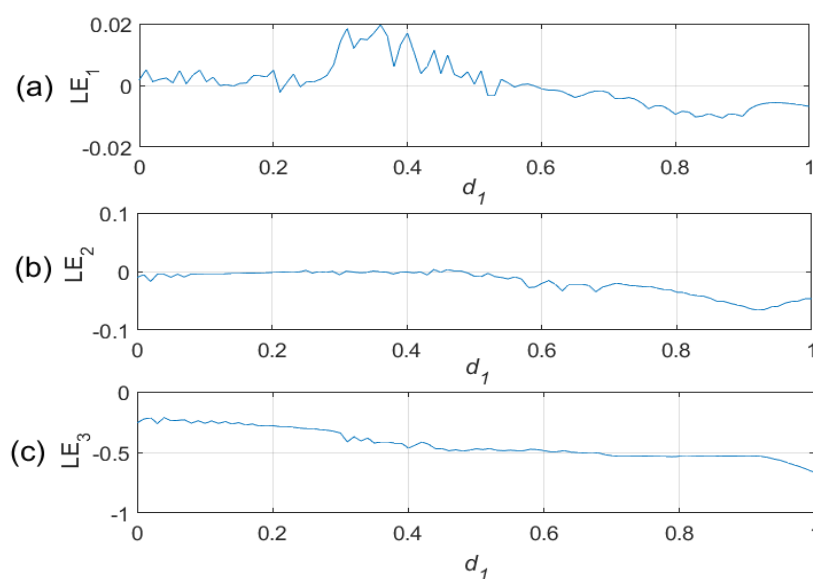

Figure 4. The Lyapunov exponents with the varying parameter d_1. (**a**) LE_1; (**b**) LE_2; (**c**) LE_3.

When the parameters d_1, D_1 and R_0 satisfy (in Section 2)

$$d_1 = \frac{D_1}{R_0} \tag{11}$$

and the intrinsic growth rate R_0 is a constant, the parameter d_1 is positively correlated with the natural death rate D_1 of the species Y. The parameter d_1 represents the natural death rate of species Y, and $d_1 = 0$ represents the species Y does not die due to its own factors and $d_1 = 1$ represents the species Y becomes extinct. It can be seen that the changes of the natural death rate of the species have a great effect on the dynamics of the three-species food-chain ecosystem.

4. The Dynamics of the Fractional-Order Ecosystem

4.1. A Necessary Condition for Generating Chaos

System (4) is next extended to fractional-order form, which is described by:

$$
\begin{cases}
D^{q_1}x(t) = x(t)(1 - x(t)) - \dfrac{5x(t)y(t)}{1+2.9x(t)}, \\
D^{q_2}y(t) = \dfrac{5x(t)y(t)}{1+2.9x(t)} - \dfrac{0.1y(t)z(t)}{1+2.15y(t)} - 0.4y(t), \\
D^{q_3}z(t) = \dfrac{0.1y(t)z(t)}{1+2.15y(t)} - 0.01z(t), \\
x(0) = x_0 > 0, \\
y(0) = y_0 > 0, \\
z(0) = z_0 > 0.
\end{cases}
\tag{12}
$$

Based on the Caputo definition of the fractional derivative:

$$
{}_{0}^{C}D_{t}^{q}f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds,
\tag{13}
$$

where $q > 0$, $n - 1 < q < n$, $n \in N^*$. Such $D^{q_i} = \frac{d^{q_i}}{dt^{q_i}}$, and $0 < q_i < 1 (i = 1, 2, 3)$. The dynamics of system (12) and the effect of the order of the fractional-order on the system will be are analyzed and explored in this section. Next, the equilibria and their stability of the system are discussed first.

Letting the right-hand side of the first three equations of system (12) be zero yields the six equilibria of the system as follows: $E_1 = (0.8098, 0.1274, 10.3085)$, $E_2 = (0.1042, 0.2333, 0)$, $E_3 = (1, 0, 0)$, $E_4 = (0, 0, 0)$, $E_5 = (0, 0.1274, -5.0955)$, $E_6 = (-0.1546, 0.1274, -22.9458)$.

The Jacobian matrix of the system is calculated at the equilibrium (x^*, y^*, z^*) as follows:

$$
J = \begin{bmatrix}
1 - 2x^* - \dfrac{5y^*}{(1+2.9x^*)^2} & -\dfrac{5x^*}{1+2.9x^*} & 0 \\
\dfrac{5y^*}{(1+2.9x^*)^2} & \dfrac{5x^*}{1+2.9x^*} - \dfrac{0.1z^*}{(1+2.15y^*)^2} - 0.4 & -\dfrac{0.1y^*}{1+2.15y^*} \\
0 & \dfrac{0.1z^*}{(1+2.15y^*)^2} & \dfrac{0.1y^*}{1+2.15y^*} - 0.01
\end{bmatrix}.
$$

The number of species is positive because zero means extinction. Only the equilibrium E_1 satisfies that all components are greater than zero, the stability of the system is to discuss here. The values of the equilibrium E_1 are substituted into J, the Jacobian matrix J_1 is

$$
J_1 = \begin{bmatrix}
-0.6764 & -1.2092 & 0 \\
0.0568 & 0.1740 & -0.01 \\
0 & 0.6352 & 0
\end{bmatrix}.
$$

Its eigenvalues are $\lambda_1 = -0.5875$, $\lambda_{2,3} = 0.0425 \pm 0.0742i$. The system has a negative real eigenvalue, and two conjugate complex eigenvalues with positive real parts, so E_1 is an unstable saddle-focus with index 2.

Based on the stability theory of fractional-order systems, a necessary condition for generating chaos is instability of the equilibria (See [41] for details). For the fractional-order ecosystem $D^{\alpha}F(X) = f(X)$, the parameter α satisfies:

$$
\tan(\alpha\frac{\pi}{2}) > \frac{|\text{Im}(\lambda)|}{\text{Re}(\lambda)} \Rightarrow \frac{\alpha\pi}{2} > \arctan(\frac{|\text{Im}(\lambda)|}{\text{Re}(\lambda)}),
\tag{14}
$$

where the order $\alpha = q_1 = q_2 = q_3$. According to the conjugate complex roots obtained above, one has:

$$\frac{\alpha\pi}{2} > \left|\arg(\lambda_2)\right| = \left|\arg(\lambda_3)\right| = \left|\arctan\left(\frac{0.0742}{0.0425}\right)\right| = \frac{0.6689\pi}{2}. \tag{15}$$

According the chaos theory [41], the necessary condition for chaos is the order $\alpha > 0.6689$, which is the minimum order for generating chaos. Letting $\alpha = 0.985$ yields a teacup chaotic attractor, as shown in Figure 5. The corresponding time-domain waves of the system are given in Figure 5a–c, respectively, and the orbit is shown in Figure 5d. This verifies the existence of chaos. When $\alpha < 0.6689$, the system is non-chaotic, as shown in Figure 6, with the order $\alpha = 0.6$. It can be seen from Figure 6a–c that the orbits of $x(t)$, $y(t)$ and $z(t)$ are asymptotically stable, and from Figure 6d the system tends to a fixed point. This indicates that the ecosystem of these three species can gradually reach a stable balance.
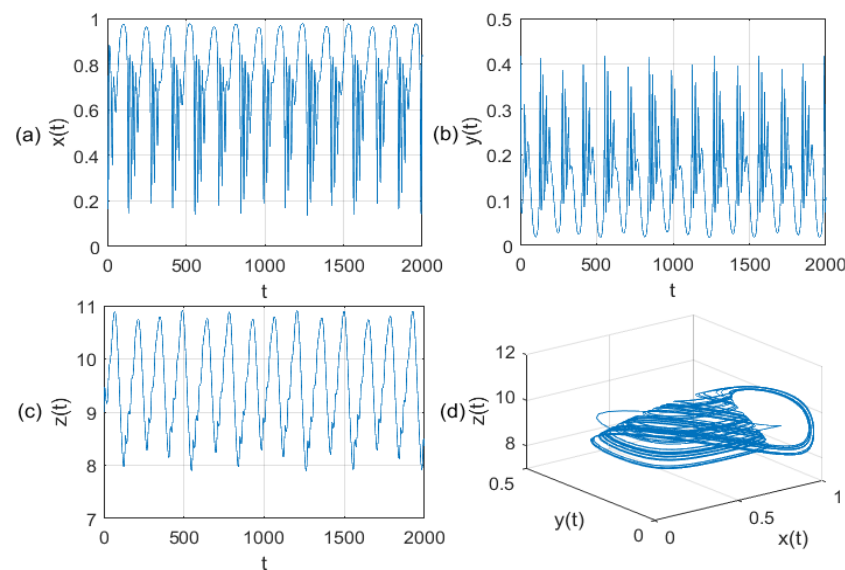

Figure 5. The teacup chaotic attractor and the time-domain waves of system (12) with $\alpha = 0.985$. (**a**) $x(t)$-t; (**b**) $y(t)$-t; (**c**) $z(t)$-t; (**d**) $x(t)$-$y(t)$-$z(t)$.
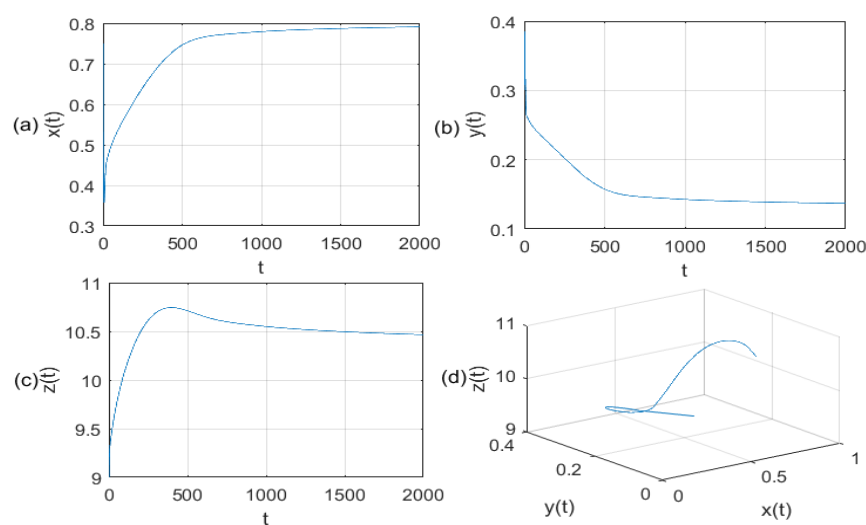

Figure 6. Evolution of the system states with $\alpha = 0.6$. (**a**) $x(t)$-t; (**b**) $y(t)$-t; (**c**) $z(t)$-t; (**d**) $x(t)$-$y(t)$-$z(t)$.

4.2. Dynamics of the Ecosystem with the Varying Parameter b_1

To analyze complex dynamics with different orders, the bifurcation diagrams are used in this subsection. Let the parameters a_1, a_2, b_2, d_1, d_2 be fixed as Table 1 and the parameter b_1 varies in the interval [2,3], the bifurcation diagrams with different orders are obtained, as shown in Figure 7.
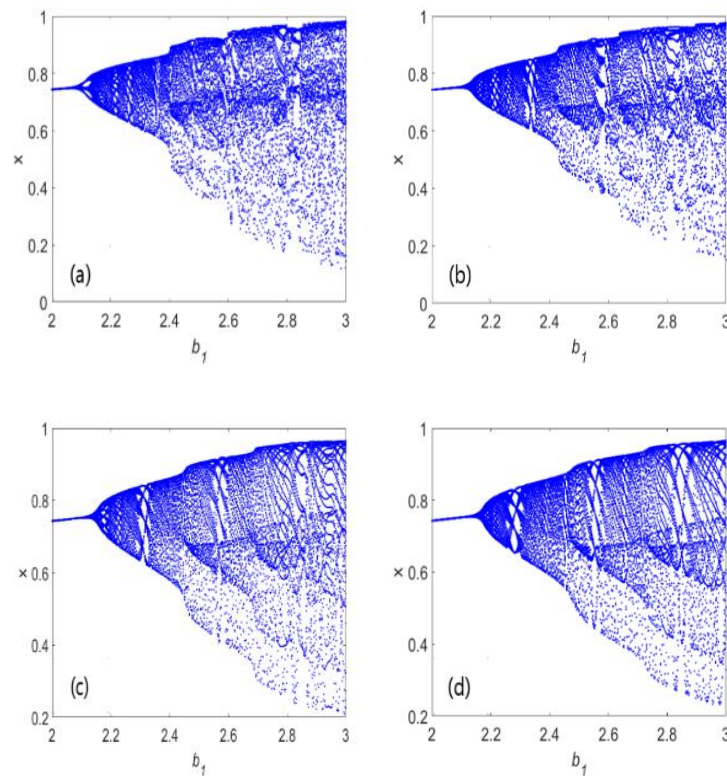

Figure 7. Bifurcation diagrams with the varying parameter b_1. (**a**) $\alpha = 0.985$; (**b**) $\alpha = 0.97$; (**c**) $\alpha = 0.96$; (**d**) $\alpha = 0.95$.

Comparing the corresponding bifurcation diagrams of different orders shows that the order has different effects on the dynamic characterization of the ecosystem. In the specific range of order (0.95, 0.985), the first period-doubling bifurcation point defers with the decrease of order, which causes the value of corresponding parameter b_1 to increase. The values of the bifurcation point are inversely proportional to the values of the order, and the carrying capacity K_0 of the species X increases with the decrease of the system order. Thus, the choice of order is a key to accurately describe the dynamics of the ecosystem.

4.3. Dynamics of the Ecosystem with the Varying Parameter d_1

Next, d_1 is used as the control parameter to explore the dynamic behaviors of the system. Let parameters a_1, a_2, b_1, b_2, d_2 be fixed as Table 1, the bifurcation diagrams are obtained, as shown in Figure 8, with the parameter d_1 being in the interval (0, 1). It can be seen from Figure 8 that the system enter chaos by the reverse period-doubling bifurcation and the first bifurcation point decreases with the decrease of the order in the specific range (0.95, 0.985), which causes the value of the corresponding parameter d_1 to decrease.

The value of the bifurcation point is proportional to the value of the order, and the natural death rate of the species Y decreases with the decrease of the system order. The choice of system order has effects on the natural death rate of the species Y.
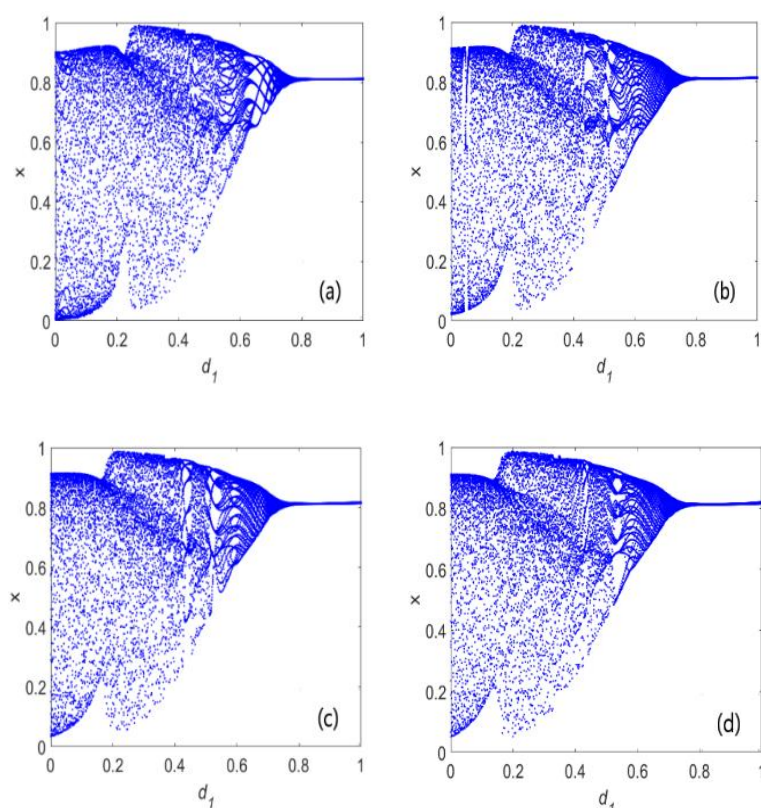
Figure 8. Bifurcation diagrams with the varying parameter d_1. (**a**) $\alpha = 0.985$, (**b**) $\alpha = 0.97$, (**c**) $\alpha = 0.96$, (**d**) $\alpha = 0.95$.

It is concluded that the choice of order is a key to accurately describe the dynamics of the ecosystem, and has a significant effect on the dynamic of the fractional-order three-species food-chain system.

5. Chaos Control of the Fractional-Order Ecosystem

To eliminate the negative impacts of chaos on the dynamic balance of species number, the control of the three-species food chain is implemented to realize the stability of the system by feedback control method in this section. The Routh–Hurwitz criterion is an algebraic criterion that determines the stability of the system, which is adopted to determine the position of the eigenvalues in the S-plane. Then the stability margin of the controlled system is measured by the Routh–Hurwitz criterion.

The controlled system is first given by

$$
\begin{cases}
D^{q_1}x(t) = x(t)(1 - x(t)) - \frac{5x(t)y(t)}{1+2.9x(t)} - u_1, \\
D^{q_2}y(t) = \frac{5x(t)y(t)}{1+2.9x(t)} - \frac{0.1y(t)z(t)}{1+2.15y(t)} - 0.4y(t) - u_2, \\
D^{q_3}z(t) = \frac{0.1y(t)z(t)}{1+2.15y(t)} - 0.01z(t) - u_3, \\
x(0) = x_0 > 0, \\
y(0) = y_0 > 0, \\
z(0) = z_0 > 0,
\end{cases}
\tag{16}
$$

where $u_i(i = 1, 2, 3)$ are the external control inputs, the control law has the following form:

$$
\begin{cases}
u_1 = k_1(x(t) - x^*(t)), \\
u_2 = k_2(y(t) - y^*(t)), \\
u_3 = k_3(z(t) - z^*(t)),
\end{cases}
\tag{17}
$$

where $(x^*(t), y^*(t), z^*(t))$ is the unstable equilibrium of system (4), and $k_i (i = 1, 2, 3)$ are the feedback gains.

The controlled differential equations have the following form

$$
\begin{cases}
D^{q_1} x(t) = x(t)(1 - x(t)) - \frac{5x(t)y(t)}{1+2.9x(t)} - k_1(x(t) - 0.8098), \\
D^{q_2} y(t) = \frac{5x(t)y(t)}{1+2.9x(t)} - \frac{0.1y(t)z(t)}{1+2.15y(t)} - 0.4y(t) - k_2(y(t) - 0.1274), \\
D^{q_3} z(t) = \frac{0.1y(t)z(t)}{1+2.15y(t)} - 0.01z(t) - k_3(z(t) - 10.3085), \\
x(0) = x_0 > 0, \\
y(0) = y_0 > 0, \\
z(0) = z_0 > 0.
\end{cases}
\tag{18}
$$

To apply the Routh–Hurwitz criterion, system (18) is linearized first. The Jacobian matrix of the system is

$$
J_2 =
\begin{bmatrix}
-2x - \frac{5y}{(-1+2.9x)^2} + (1 - k_1) & -\frac{5x}{1+2.9x} & 0 \\
\frac{5y}{(1+2.9x)^2} & \frac{5x}{1+2.9x} - \frac{0.1z}{(1+2.15y)^2} + (-0.4 - k_2) & -\frac{0.1y}{1+2.15y} \\
0 & \frac{0.1z}{(1+2.15y)^2} & \frac{0.1y}{1+2.15y} + (-0.01 - k_3)
\end{bmatrix}.
$$

The characteristic equation of the system is

$$
\begin{aligned}
f(\lambda) = {}& \lambda^3 + \left(k_1 + k_2 + k_3 + \tfrac{314}{625}\right)\lambda^2 \\
& + \left(k_1 k_2 + k_1 k_3 + k_2 k_3 + \tfrac{-87}{500}k_1 + \tfrac{1691}{2500}k_2 + \tfrac{314}{625}k_3 + \tfrac{-266619}{6250000}\right)\lambda \\
& + k_1 k_2 k_3 + \tfrac{-87}{500}k_1 k_3 + \tfrac{1691}{2500}k_2 k_3 + \tfrac{397}{62500}k_1 + \tfrac{-306319}{6250000}k_3 + \tfrac{4953518943508485}{1152921504606846976}.
\end{aligned}
\tag{19}
$$

The corresponding coefficients are

$$
\begin{cases}
a_3 = 1, \\
a_2 = k_1 + k_2 + k_3 + \tfrac{314}{625}, \\
a_1 = k_1 k_2 + k_1 k_3 + k_2 k_3 + \tfrac{-87}{500}k_1 + \tfrac{1691}{2500}k_2 + \tfrac{314}{625}k_3 + \tfrac{-266619}{6250000}, \\
a_0 = k_1 k_2 k_3 + \tfrac{-87}{500}k_1 k_3 + \tfrac{1691}{2500}k_2 k_3 + \tfrac{397}{62500}k_1 + \tfrac{-306319}{6250000}k_3 + \tfrac{4953518943508485}{1152921504606846976}.
\end{cases}
\tag{20}
$$

Letting the feedback gains $k_1 = 1$, $k_2 = 5$, $k_3 = 10$ of the system yields $a_3 = 1 > 0$, $a_2 = 16.5024 > 0$, $a_1 = 73.1893 > 0$, $a_0 = 81.6005 > 0$. Thus, ones have $f(\lambda) = \lambda^3 + 16.5024\lambda^2 + 73.1893\lambda + 81.6005$. All coefficients are positive real values, which meets the necessary conditions of stability of the system. The corresponding Routh array is obtained, as shown in Table 2.

Table 2. The Routh array with $k_1 = 1, k_2 = 5, k_3 = 10$.

Terms	Coefficients	
λ^3	1	73.1893
λ^2	16.5024	81.6005
λ^1	68.2446	0
λ^0	81.6005	

The values of the first column of the Routh array are all positive, which confirms that the system is asymptotically stable based on the Routh–Hurwitz criterion. In fact, the roots of the characteristic equation are $\lambda_1 = -9.9988$, $\lambda_2 = -0.4053$, and $\lambda_3 = -1.6984$, three roots are all negative real values. Further, the evolution of the controlled system (18) is obtained, as shown in Figure 9. It can be seen from Figure 9a–c that the orbits of states $x(t)$, $y(t)$ and $z(t)$ are asymptotically stable, and from Figure 9d the system tends to a stable point. These indicate that the system is asymptotically stable.
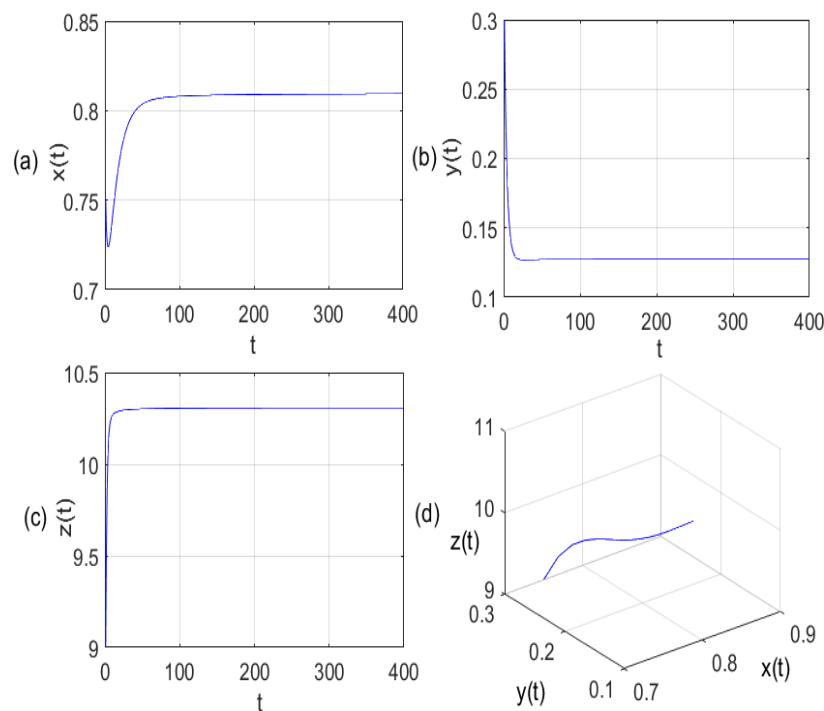
Figure 9. Evolution of the controlled system (18) with $k_1 = 1, k_2 = 5, k_3 = 10$. (**a**) $x(t)$-t; (**b**) $y(t)$-t; (**c**) $z(t)$-t; (**d**) $x(t)$-$y(t)$-$z(t)$.

The values of feedback gains k_1, k_2 and k_3 are all positive, it indicates the three species of the system needs artificial fishing to adjust it. The numbers of fishing species X, Y and Z are 1, 5, and 10 times of the number of existing species exceeding their equilibrium, respectively. The chaos control of the system is realized by selecting specific feedback gains, which confirms that the feedback control method is effective. That being said, manual intervention, such as artificial deliberate protection or control, system can realize the chaos of the food chain to the stable state and achieve the long-term development of the ecosystem.

The detection of the stability margin of the controlled system is explored in the subsection. Substituting $\lambda = s - m$ into $f(\lambda)$ yields the characteristic equation as follows:

$$
\begin{aligned}
f(s) \ & = (s - m)^3 + 16.5024(s - m)^2 + 73.1893(s - m) + 81.6005 \\
& = s^3 + (-3m + 16.5024)s^2 + (3m^2 - 33.0048m + 73.1893)s \\
& \ + (-m^3 + 16.5024m^2 - 73.1893m + 81.6005),
\end{aligned}
\tag{21}
$$

where the stability margin $m_i(i = 1, 2, 3, 4)$ are positive. To explore the stability of the system, the Routh array of Equation (21) is given, as shown in Table 3.

Table 3. The Routh array with $\lambda = s - m$.

Terms	Coefficients	
s^3	1	$3m^2 - 33.0048m + 73.1893$
s^2	$-3m + 16.5024$	$-m^3 + 16.5024m^2 - 73.1893m + 81.6005$
s^1	n	0
s^0	$-m^3 + 16.5024m^2 - 73.1893m + 81.6005$	

In Table 3, the symbol n is given below:

$$n = \frac{n_1 m^3 - n_2 m^2 + n_3 m - n_4}{n_5 m - n_6},$$ (22)

and

$$n_1 = 219902325552 \times 10^8, n_2 = 362891613724213248 \times 10^4,$$
$$n_3 = 18995080734745441759488, n_4 = 30956711515873565050625,$$ (23)
$$n_5 = 824633720832 \times 10^8, n_6 = 4536145175526656 \times 10^4.$$

When the coefficients of term s and all values of the first column of the Routh array are positive, the system is stable. Therefore, the stability range of the system is calculated as $0 < m < 1.6984$, which indicates that the stability margin of the controlled system is 1.6984.

6. Conclusions

In this paper, a fractional-order three-species food-chain ecosystem is presented, which shows some unique dynamic behaviors. The chaotic state and the range of order with chaos of the system are confirmed. The bifurcation analysis with different orders verifies that the choice of order is extremely important for accurately characterizing the dynamics of the system. The result shows that the carrying capacity and mortality rate of each species have a great influence on the stability and development of the ecosystem under the certain conditions, which has the potential significance in practical applications. The stability of the fractional-order chaotic ecosystem is adjusted by the feedback control method, which generate a good effect. Moreover, the stability margin of the controlled fractional-order ecosystem is obtained by theory analysis and numerical simulations.

These subjects about the discretization of the fractional-order system, control methods of fractional-order system, and the dynamic analysis of corresponding time delay fractional system and others deserve further study in the near future.

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