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Coupled Fixed Point Theorems Employing CLR-Property on $\mathcal{V}$ -Fuzzy Metric Spaces

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Received: 19 February 2020; Accepted: 8 March 2020; Published: 12 March 2020

Abstract: The introduction of the common limit range property on $\mathcal{V}$ -fuzzy metric spaces is the foremost aim of this paper. Furthermore, significant results for coupled maps are proven by employing this property on $\mathcal{V}$ -fuzzy metric spaces. More precisely, we introduce the notion of CLR_{Ω} -property for the mappings $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$. We utilize our new notion to present and prove our new fixed point results.

Keywords: $\mathcal{V}$ -fuzzy metric space ($\mathcal{VFM}$ -space); weakly compatible mappings; coupled fixed point; common limit range property (CLR property)

1. Introduction and Preliminaries

Mustafa and Sims [1] brought the thought of the notion of G-metric spaces as a generalization of metric spaces. Moreover, Sedghi et al. [2] introduced the concept of S-metric spaces as one of the generalizations of the metric spaces. Abbas et al. [3] extended the notion of S-metric spaces to A-metric spaces by extending the definition to n-tuple.

In 1965, Zadeh [4] initially introduced the concept of fuzzy sets. After that, several influential mathematicians considered the notion of fuzzy sets to introduce many exciting notions in the field of mathematics, such as fuzzy differential equations, fuzzy logic and fuzzy metric spaces. A fuzzy metric space is well known to be an important generalization of the metric space. In 1975, Kramosil and Michalek [5] employed the notion of fuzzy sets to introduce the notion of fuzzy metric spaces. George and Veeramani [6] modified the concept of fuzzy metric spaces in the sense of Kramosil and Michalek [5].

Sun and Yang [7] coined the idea of $\mathcal{G}$ -fuzzy metric spaces. Aamri and D. El Moutawakil [8] generalized the concept of non compatibility by defining E.A. property for self mappings. Sintunavarat and Kumam [9] gave the definition of the common limit in the range property on fuzzy metric spaces.

After the exhaustive review of the previous literature, Gupta and Kanwar [10] introduced the notion of $\mathcal{V}$ -fuzzy metric spaces.

Definition 1 ([10]). Let $\mathcal{M}$ be a non-empty set and $\mathcal{V}$ be a fuzzy set on $\mathcal{M}^n \times (0, \infty)$. Let $*$ be a continuous t -norm. A 3-tuple $(\mathcal{M}, \mathcal{V}, *)$ is said to be a $\mathcal{VFM}$ -space if for all $t, s > 0$, the following conditions hold:

- i. $\mathcal{V}(\beta, \beta, \beta, \dots, \beta, \eta, t) > 0$ for all $\beta, \eta \in \mathcal{M}$ with $\beta \neq \eta$,

- ii $\mathcal{V}(\beta_1, \beta_1, \beta_1, \dots, \beta_1, \beta_2, t) \geq \mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t)$ for all $\beta_1, \beta_2, \beta_3, \dots, \beta_n \in \mathcal{M}$ with $\beta_2 \neq \beta_3 \neq \dots \neq \beta_n$,
- iii $\mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t) = 1$ if and only if $\beta_1 = \beta_2 = \beta_3 = \dots = \beta_n$,
- iv $\mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t) = \mathcal{V}(p(\beta_1, \beta_2, \beta_3, \dots, \beta_n), t)$, where p is a permutation function,
- v $\mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t+s) \geq \mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_{n-1}, a, t) * \mathcal{V}(a, a, a, \dots, a, \beta_n, s)$,
- vi $\lim_{t \rightarrow \infty} \mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t) = 1$,
- vii $\mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Lemma 1 ([10]). Let $(\mathcal{M}, \mathcal{V}, *)$ be a $\mathcal{VFM}$ -space such that

$\mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, kt) \geq \mathcal{V}(\beta_1, \beta_2, \beta_3, \dots, \beta_n, t)$, with $k \in (0, 1)$. Then $\beta_1 = \beta_2 = \beta_3 = \dots = \beta_n$.

Definition 2 ([10]). Let $(\mathcal{M}, \mathcal{V}, *)$ be a $\mathcal{VFM}$ -space. A sequence $\{\beta_r\}$ is said to be a Cauchy sequence if $\mathcal{V}(\beta_r, \beta_r, \beta_r, \dots, \beta_r, \beta_q, t) \rightarrow 1$ as $r, q \rightarrow \infty$ for all $t > 0$; that is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $r, q \geq n_0$, we have $\mathcal{V}(\beta_r, \beta_r, \beta_r, \dots, \beta_r, \beta_q, t) > 1 - \epsilon$.

Definition 3 ([10]). The $\mathcal{VFM}$ -space $(\mathcal{M}, \mathcal{V}, *)$ is called complete if every Cauchy sequence in $\mathcal{M}$ is convergent.

Definition 4. The mappings $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ are said to compatible on $\mathcal{V}$ -fuzzy metric spaces if

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Omega\Theta(\beta_r, \eta_r), \Omega\Theta(\beta_r, \eta_r), \dots, \Omega\Theta(\beta_r, \eta_r), \Theta(\Omega\beta_r, \Omega\eta_r), t) = 1$$

and

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Omega\Theta(\eta_r, \beta_r), \Omega\Theta(\eta_r, \beta_r), \dots, \Omega\Theta(\eta_r, \beta_r), \Theta(\Omega\eta_r, \Omega\beta_r), t) = 1,$$

whenever $\{\beta_r\}$ and $\{\eta_r\}$ are sequences in $\mathcal{M}$ such that $\lim_{r \rightarrow +\infty} \Omega(\beta_r) = \lim_{r \rightarrow +\infty} \Theta(\beta_r, \eta_r) = \beta$

and $\lim_{r \rightarrow +\infty} \Omega(\eta_r) = \lim_{r \rightarrow +\infty} \Theta(\eta_r, \beta_r) = \eta$ for all $\beta, \eta \in \mathcal{M}$, $t > 0$.

In order to study some more significant fixed point results on fuzzy metric spaces, one can see the research papers [11–18].

Bhaskar and Lakshmikantham [19] initiated the study of the coupled fixed point and mixed monotone property on the notion of metric spaces. For more theorems on coupled fixed point, see [20,21].

Motivated by different concepts introduced by many eminent mathematicians in [22–27], we investigate and give the concept of CLR_Ω -property on $\mathcal{V}$ -fuzzy metric spaces. Further, we study some fixed point theorems for the pair of mappings by using CLR_Ω -property. More precisely, under some conditions based on CLR_Ω -property on $\mathcal{V}$ -fuzzy metric spaces, we prove a fixed point for the mapping $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ and a coupled fixed point for the mapping $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ of the form $(u, u) \in \mathcal{M} \times \mathcal{M}$.

2. Results

We start with the definition of CLR_Ω -property:

Definition 5. Let $(\mathcal{M}, \mathcal{V}, *)$ be a $\mathcal{VFM}$ -space. The mappings $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ satisfy CLR_Ω -property if there exist $\{\beta_r\}$ and $\{\eta_r\}$ in $\mathcal{M}$ such that

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\beta_r, \eta_r), \Theta(\beta_r, \eta_r), \dots, \Theta(\beta_r, \eta_r), \Omega(\ell), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\beta_r), \Omega(\beta_r), \dots, \Omega(\beta_r), \Omega(\ell), t)$$

and

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\eta_r, \beta_r), \Theta(\eta_r, \beta_r), \dots, \Theta(\eta_r, \beta_r), \Omega(\hbar), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\eta_r), \Omega(\eta_r), \dots, \Omega(\eta_r), \Omega(\hbar), t)$$

for some $\ell, \hbar \in \mathcal{M}$.

In the rest of this paper, we call $v \in \mathcal{M}$ a common fixed point for the mappings $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ and $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ if $\Theta(v, v) = \Omega(v) = v$.

Theorem 1. Let $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ be weakly compatible mappings on a $\mathcal{VFM}$ -space $(\mathcal{M}, \mathcal{V}, *)$. Suppose the pair (Θ, Ω) holds CLR_{Ω} -property. Moreover, assume that for all $\beta, \eta, c, d \in \mathcal{M}, k \in (0, 1)$ and $t > 0$, we have

$$\mathcal{V}(\Theta(\beta, \eta), \Theta(\beta, \eta), \dots, \Theta(\beta, \eta), \Theta(c, d), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Omega(c), t), \\ \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Theta(\beta, \eta), t), \\ \mathcal{V}(\Omega(c), \Omega(c), \dots, \Omega(c), \Theta(c, d), t) \end{array} \right\}. \quad (1)$$

Then Θ and Ω have a unique common fixed point in $\mathcal{M}$.

Proof. The CLR_{Ω} -property for the (Θ, Ω) implies that

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\beta_r, \eta_r), \Theta(\beta_r, \eta_r), \dots, \Theta(\beta_r, \eta_r), \Omega(\ell), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\beta_r), \Omega(\beta_r), \dots, \Omega(\beta_r), \Omega(\ell), t)$$

and

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\eta_r, \beta_r), \Theta(\eta_r, \beta_r), \dots, \Theta(\eta_r, \beta_r), \Omega(\hbar), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\eta_r), \Omega(\eta_r), \dots, \Omega(\eta_r), \Omega(\hbar), t) \quad (2)$$

for sequences $\{\beta_r\}, \{\eta_r\}$ in $\mathcal{M}$ and some $\ell, \hbar \in \mathcal{M}$.

By using (1) and (2), we get

$$\mathcal{V}(\Theta(\beta_r, \eta_r), \Theta(\beta_r, \eta_r), \dots, \Theta(\beta_r, \eta_r), \Theta(\ell, \hbar), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\beta_r), \Omega(\beta_r), \dots, \Omega(\beta_r), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\beta_r), \Omega(\beta_r), \dots, \Omega(\beta_r), \Theta(\beta_r, \eta_r), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t) \end{array} \right\}.$$

Letting $r \rightarrow +\infty$, we get

$$\mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t) \end{array} \right\}.$$

So

$$\Omega(\ell) = \Theta(\ell, \hbar). \quad (3)$$

From (1) and (3), one can have

$$\mathcal{V}(\Theta(\eta_r, \beta_r), \Theta(\eta_r, \beta_r), \dots, \Theta(\eta_r, \beta_r), \Theta(\hbar, \ell), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\eta_r), \Omega(\eta_r), \dots, \Omega(\eta_r), \Omega(\hbar), t), \\ \mathcal{V}(\Omega(\eta_r), \Omega(\eta_r), \dots, \Omega(\eta_r), \Theta(\eta_r, \beta_r), t), \\ \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Theta(\hbar, \ell), t) \end{array} \right\}.$$

Letting $r \rightarrow +\infty$, we obtain

$$\mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Theta(\hbar, \ell), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Omega(\hbar), t), \\ \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Omega(\hbar), t), \\ \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Theta(\hbar, \ell), t) \end{array} \right\}.$$

Hence,

$$\Omega(\hbar) = \Theta(\hbar, \ell). \quad (4)$$

From (3) and (4), we have $\Omega(\ell) = \Theta(\ell, \hbar)$ and $\Omega(\hbar) = \Theta(\hbar, \ell)$.

Suppose that

$$\Omega(\ell) = \Theta(\ell, \hbar) = \sigma_1, \Omega(\hbar) = \Theta(\hbar, \ell) = \sigma_2. \quad (5)$$

The weakly compatible notion for the pair (Θ, Ω) implies that

$$\mathcal{V}(\Omega\Theta(\ell, \hbar), \Omega\Theta(\ell, \hbar), \dots, \Omega\Theta(\ell, \hbar), \Theta(\Omega(\ell), \Omega(\hbar)), t) = 1$$

and

$$\mathcal{V}(\Omega\Theta(\hbar, \ell), \Omega\Theta(\hbar, \ell), \dots, \Omega\Theta(\hbar, \ell), \Theta(\Omega(\hbar), \Omega(\ell)), t) = 1.$$

This gives

$$\Omega(\sigma_1) = \Theta(\sigma_1, \sigma_2), \quad \Omega(\sigma_2) = \Theta(\sigma_2, \sigma_1). \quad (6)$$

From (1), one can get

$$\mathcal{V}(\Theta(\sigma_1, \sigma_2), \Theta(\sigma_1, \sigma_2), \dots, \Theta(\sigma_1, \sigma_2), \Theta(\ell, \hbar), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Theta(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t), \end{array} \right\}.$$

From (5) and (6), one can have

$$\Omega(\sigma_1) = \sigma_1 = \Theta(\sigma_1, \sigma_2), \quad \Omega(\sigma_2) = \sigma_2 = \Theta(\sigma_2, \sigma_1). \quad (7)$$

Let us suppose that $\sigma_1 \neq \sigma_2$.

By using (1) and (7), we get

$$\begin{aligned} \mathcal{V}(\sigma_1, \sigma_1, \dots, \sigma_2, kt) &= \mathcal{V}(\Theta(\sigma_1, \sigma_2), \Theta(\sigma_1, \sigma_2), \dots, \Theta(\sigma_1, \sigma_2), \Theta(\sigma_2, \sigma_1), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Omega(\sigma_2), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Theta(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Omega(\sigma_2), \Omega(\sigma_2), \dots, \Omega(\sigma_2), \Theta(\sigma_2, \sigma_1), t), \end{array} \right\}. \end{aligned}$$

This shows that $\sigma_1 = \sigma_2$ and hence, $\Omega(\sigma_1) = \sigma_1 = \Theta(\sigma_1, \sigma_2)$.

To prove the uniqueness, suppose that v and λ are such that $v \neq \lambda$, $\Omega(v) = v = \Theta(v, v)$ and $\Omega(\lambda) = \lambda = \Theta(\lambda, \lambda)$. Then condition (1) implies that

$$\begin{aligned} \mathcal{V}(v, v, \dots, v, \lambda, kt) &= \mathcal{V}(\Theta(v, v), \Theta(v, v), \dots, \Theta(v, v), \Theta(\lambda, \lambda), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(v), \Omega(v), \dots, \Omega(v), \Omega(\lambda), t), \\ \mathcal{V}(\Omega(v), \Omega(v), \dots, \Omega(v), \Theta(v, v), t), \\ \mathcal{V}(\Omega(\lambda), \Omega(\lambda), \dots, \Omega(\lambda), \Theta(\lambda, \lambda), t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \mathcal{V}(v, v, \dots, v, \lambda, t), \\ \mathcal{V}(v, v, \dots, v, v), \\ \mathcal{V}(\lambda, \lambda, \dots, \lambda, \lambda) \end{array} \right\}. \end{aligned}$$

Hence, we get $v = \lambda$ and we conclude the uniqueness of the fixed point. $\square$

Example 1. Let $(\mathcal{M}, \mathcal{V}, *)$ be a $\mathcal{VFM}$ -space with $\mathcal{M} = [-1, 1]$, where

$$\mathcal{V}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n, t) = \frac{t}{t + (|\alpha_1 - \alpha_2| + |\alpha_2 - \alpha_3| + \dots + |\alpha_{n-1} - \alpha_n|)}.$$

Suppose $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ be mappings defined as $\Theta(\beta, \eta) = \beta + \eta$ and $\Omega(\beta) = \beta$ for all $\beta, \eta \in \mathcal{M}$. Now consider the sequences $\{\beta_r\} = \frac{1}{r}$ and $\{\eta_r\} = -\frac{1}{r}$. One can have

$$\lim_{r \rightarrow +\infty} \Theta(\beta_r, \eta_r) = \lim_{r \rightarrow +\infty} \Omega(\beta_r) = 0 = \Omega(0)$$

and

$$\lim_{r \rightarrow +\infty} \Theta(\eta_r, \beta_r) = \lim_{r \rightarrow +\infty} \Omega(\eta_r) = 0 = \Omega(0).$$

This implies that the pair (Θ, Ω) holds CLR_{Ω} -property. In addition, the pair (Θ, Ω) is weakly compatible. All the conditions of Theorem 1 are satisfied. Thus, the two mappings Θ and Ω have $\beta = 0$ as a unique common fixed point in $\mathcal{M}$.

Theorem 2. Let $\Theta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega : \mathcal{M} \rightarrow \mathcal{M}$ be weakly compatible mappings on a $\mathcal{VFM}$ -space $(\mathcal{M}, \mathcal{V}, *)$. Suppose the pair (Θ, Ω) holds the property (E.A). Moreover, assume that the range of Ω is a closed subspace of $\mathcal{M}$. Assume that for all $\beta, \eta, c, d \in \mathcal{M}, k \in (0, 1)$ and $t > 0$, we have

$$\mathcal{V}(\Theta(\beta, \eta), \Theta(\beta, \eta), \dots, \Theta(\beta, \eta), \Theta(c, d), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Omega(c), t), \\ \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Theta(\beta, \eta), t), \\ \mathcal{V}(\Omega(c), \Omega(c), \dots, \Omega(c), \Theta(c, d), t) \end{array} \right\}.$$

Then, Θ and Ω have a unique common fixed point in $\mathcal{M}$.

Proof. The property (E.A) for (Θ, Ω) implies that there exist $\{\beta_r\}$ and $\{\eta_r\}$ in $\mathcal{M}$ such that

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\beta_r, \eta_r), \Theta(\beta_r, \eta_r), \dots, \Theta(\beta_r, \eta_r), \Omega(\ell), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\beta_r), \Omega(\beta_r), \dots, \Omega(\beta_r), \Omega(\ell), t)$$

and

$$\lim_{r \rightarrow +\infty} \mathcal{V}(\Theta(\eta_r, \beta_r), \Theta(\eta_r, \beta_r), \dots, \Theta(\eta_r, \beta_r), \Omega(\hbar), t) = 1 = \lim_{r \rightarrow +\infty} \mathcal{V}(\Omega(\eta_r), \Omega(\eta_r), \dots, \Omega(\eta_r), \Omega(\hbar), t)$$

hold for all $\ell, \hbar \in \mathcal{M}$.

Moreover, the property of closed subspace of $\mathcal{M}$ implies that there exists $\alpha, \beta \in \mathcal{M}$ such that $\Omega(\alpha) = \ell, \Omega(\beta) = \hbar$. One can get that (Θ, Ω) holds CLR_{Ω} -property. The result follows from the previous theorem. $\square$

Theorem 3. Let $\Theta, \psi : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega, \Phi : \mathcal{M} \rightarrow \mathcal{M}$ be mappings on a $\mathcal{VFM}$ -space $(\mathcal{M}, \mathcal{V}, *)$. Suppose that (Θ, Ω) and (ψ, Φ) are weakly compatible as well as share CLR_{Ω} -property. Moreover, assume the following conditions:

i. For all $\beta, \eta, c, d \in \mathcal{M}, k \in (0, 1)$ and $t > 0$, we have

$$\mathcal{V}(\Theta(\beta, \eta), \Theta(\beta, \eta), \dots, \Theta(\beta, \eta), \psi(c, d), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Phi(c), t), \\ \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \Theta(\beta, \eta), t), \\ \mathcal{V}(\Phi(c), \Phi(c), \dots, \Phi(c), \psi(c, d), t), \\ \mathcal{V}(\Omega(\beta), \Omega(\beta), \dots, \Omega(\beta), \psi(c, d), t), \\ \mathcal{V}(\Phi(c), \Phi(c), \dots, \Phi(c), \Theta(\beta, \eta), t) \end{array} \right\},$$

ii. $\Theta(\mathcal{M} \times \mathcal{M}) \subset \Phi(\mathcal{M})$ (or $\psi(\mathcal{M} \times \mathcal{M}) \subset \Omega(\mathcal{M})$).

Then, Θ, Ω, ψ and Φ have a unique common fixed point in $\mathcal{M}$.

Proof. The CLR_{Ω} -property for the pairs (Θ, Ω) and (ψ, Φ) implies that there exist $\{\beta_r\}$, $\{\eta_r\}$, $\{c_r\}$ and $\{d_r\}$ in $\mathcal{M}$ such that

$$\lim_{r \rightarrow +\infty} \Theta(\beta_r, \eta_r) = \lim_{r \rightarrow +\infty} \Omega(\beta_r) = \lim_{r \rightarrow +\infty} \psi(c_r, d_r) = \lim_{r \rightarrow +\infty} \Phi(c_r) = \Omega(\ell)$$

and

$$\lim_{r \rightarrow +\infty} \Theta(\eta_r, \beta_r) = \lim_{r \rightarrow +\infty} \Omega(\eta_r) = \lim_{r \rightarrow +\infty} \psi(d_r, c_r) = \lim_{r \rightarrow +\infty} \Phi(d_r) = \Omega(\hbar), \quad (8)$$

hold for all $\ell, \hbar \in \mathcal{M}$.

By using (i), one can have

$$\mathcal{V}(\Theta(\ell, \hbar), \Theta(\ell, \hbar), \dots, \Theta(\ell, \hbar), \psi(c_r, d_r), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Phi(c_r), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t), \\ \mathcal{V}(\Phi(c_r), \Phi(c_r), \dots, \Phi(c_r), \psi(c_r, d_r), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \psi(c_r, d_r), t), \\ \mathcal{V}(\Phi(c_r), \Phi(c_r), \dots, \Phi(c_r), \Theta(\ell, \hbar), t) \end{array} \right\}.$$

By using (8) and letting $r \rightarrow \infty$, we have

$$\mathcal{V}(\Theta(\ell, \hbar), \Theta(\ell, \hbar), \dots, \Theta(\ell, \hbar), \Omega(\ell), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Omega(\ell), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t) \end{array} \right\}.$$

Thus, we have

$$\Omega(\ell) = \Theta(\ell, \hbar). \quad (9)$$

From (i), we have

$$\mathcal{V}(\Theta(\hbar, \ell), \Theta(\hbar, \ell), \dots, \Theta(\hbar, \ell), \psi(d_r, c_r), kt) \geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Phi(d_r), t), \\ \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \Theta(\hbar, \ell), t), \\ \mathcal{V}(\Phi(d_r), \Phi(d_r), \dots, \Phi(d_r), \psi(d_r, c_r), t), \\ \mathcal{V}(\Omega(\hbar), \Omega(\hbar), \dots, \Omega(\hbar), \psi(d_r, c_r), t), \\ \mathcal{V}(\Phi(d_r), \Phi(d_r), \dots, \Phi(d_r), \Theta(\hbar, \ell), t) \end{array} \right\}.$$

As $r \rightarrow \infty$ and using (8), one can obtain

$$\Omega(\hbar) = \Theta(\hbar, \ell). \quad (10)$$

Since $\Theta(\mathcal{M} \times \mathcal{M}) \subset \Phi(\mathcal{M})$, there exists $\alpha, \beta \in \mathcal{M}$ such that

$$\Phi(\alpha) = \Theta(\ell, \hbar) \text{ and } \Phi(\beta) = \Theta(\hbar, \ell). \quad (11)$$

By using (i), (9), (10) and (11), we get

$$\begin{aligned} \mathcal{V}(\Phi(\alpha), \Phi(\alpha), \dots, \Phi(\alpha), \psi(\alpha, \beta)) &= \mathcal{V}(\Theta(\ell, \hbar), \Theta(\ell, \hbar), \dots, \Theta(\ell, \hbar), \psi(\alpha, \beta), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Phi(\alpha), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t), \\ \mathcal{V}(\Phi(\alpha), \Phi(\alpha), \dots, \Phi(\alpha), \psi(\alpha, \beta), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \psi(\alpha, \beta), t), \\ \mathcal{V}(\Phi(\alpha), \Phi(\alpha), \dots, \Phi(\alpha), \Theta(\ell, \hbar), t) \end{array} \right\}. \end{aligned}$$

This implies that $\Phi(\alpha) = \psi(\alpha, \beta)$. Similarly, one can show that

$$\Phi(\beta) = \psi(\beta, \alpha). \quad (12)$$

Now consider that

$$\Phi(\alpha) = \psi(\alpha, \beta) = \Theta(\ell, \hbar) = \Omega(\ell) = \sigma_1 \text{ and } \Phi(\beta) = \psi(\beta, \alpha) = \Theta(\hbar, \ell) = \Omega(\hbar) = \sigma_2. \quad (13)$$

The notion of weakly compatible for mappings (Θ, Ω) and (ψ, Φ) gives

$$\Theta(\sigma_1, \sigma_2) = \Theta(\Omega(\ell), \Omega(\hbar)) = \Omega(\Theta(\ell, \hbar)) = \Omega(\sigma_1), \quad \Theta(\sigma_2, \sigma_1) = \Theta(\Omega(\hbar), \Omega(\ell)) = \Omega(\Theta(\hbar, \ell)) = \Omega(\sigma_2) \quad (14)$$

and

$$\psi(\sigma_1, \sigma_2) = \psi(\Phi(\alpha), \Phi(\beta)) = \Phi(\psi(\alpha, \beta)) = \Phi(\sigma_1), \quad \psi(\sigma_2, \sigma_1) = \psi(\Phi(\beta), \Phi(\alpha)) = \Phi(\psi(\beta, \alpha)) = \Phi(\sigma_2). \quad (15)$$

Finally, we assert that $\Theta(\sigma_1, \sigma_2) = \sigma_1, \Theta(\sigma_2, \sigma_1) = \sigma_2$.

Using (i), we have

$$\begin{aligned} \mathcal{V}(\Theta(\sigma_1, \sigma_2), \Theta(\sigma_1, \sigma_2), \dots, \Theta(\sigma_1, \sigma_2), \sigma_1, kt) &= \mathcal{V}(\Theta(\sigma_1, \sigma_2), \Theta(\sigma_1, \sigma_2), \dots, \Theta(\sigma_1, \sigma_2), \psi(\alpha, \beta), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Phi(\alpha), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Theta(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Phi(\alpha), \Phi(\alpha), \dots, \Phi(\alpha), \psi(\alpha, \beta), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \psi(\alpha, \beta), t), \\ \mathcal{V}(\Phi(\alpha), \Phi(\alpha), \dots, \Phi(\alpha), \Theta(\sigma_1, \sigma_2), t) \end{array} \right\}. \end{aligned}$$

By considering (13)–(15), we obtain $\Theta(\sigma_1, \sigma_2) = \Omega(\sigma_1)$.

In the same way, we get

$$\Theta(\sigma_1, \sigma_2) = \Omega(\sigma_1) = \sigma_1, \quad \Theta(\sigma_2, \sigma_1) = \Omega(\sigma_2) = \sigma_2. \quad (16)$$

Again using (i) and (12)–(16), one can get

$$\begin{aligned} \mathcal{V}(\sigma_1, \sigma_1, \dots, \sigma_1, \psi(\sigma_1, \sigma_2), kt) &= \mathcal{V}(\Theta(\ell, \hbar), \Theta(\ell, \hbar), \dots, \Theta(\ell, \hbar), \psi(\sigma_1, \sigma_2), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Phi(\sigma_1), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \Theta(\ell, \hbar), t), \\ \mathcal{V}(\Phi(\sigma_1), \Phi(\sigma_1), \dots, \Phi(\sigma_1), \psi(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Omega(\ell), \Omega(\ell), \dots, \Omega(\ell), \psi(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Phi(\sigma_1), \Phi(\sigma_1), \dots, \Phi(\sigma_1), \Theta(\ell, \hbar), t) \end{array} \right\}. \end{aligned}$$

This implies that

$$\Phi(\sigma_1) = \sigma_1 = \psi(\sigma_1, \sigma_2), \quad \Phi(\sigma_2) = \sigma_2 = \psi(\sigma_2, \sigma_1). \quad (17)$$

Assume $\sigma_1 \neq \sigma_2$.

By using (i) and (15)–(17), we have

$$\begin{aligned} \mathcal{V}(\sigma_1, \sigma_1, \dots, \sigma_1, \sigma_2, kt) &= \mathcal{V}(\Theta(\sigma_1, \sigma_2), \Theta(\sigma_1, \sigma_2), \dots, \Theta(\sigma_1, \sigma_2), \psi(\sigma_1, \sigma_2), kt) \\ &\geq \min \left\{ \begin{array}{l} \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Phi(\sigma_2), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \Theta(\sigma_1, \sigma_2), t), \\ \mathcal{V}(\Phi(\sigma_2), \Phi(\sigma_2), \dots, \Phi(\sigma_2), \psi(\sigma_2, \sigma_1), t), \\ \mathcal{V}(\Omega(\sigma_1), \Omega(\sigma_1), \dots, \Omega(\sigma_1), \psi(\sigma_2, \sigma_1), t), \\ \mathcal{V}(\Phi(\sigma_2), \Phi(\sigma_2), \dots, \Phi(\sigma_2), \Theta(\sigma_1, \sigma_2), t) \end{array} \right\}, \end{aligned}$$

this implies $\sigma_1 = \sigma_2$; a contradiction. Therefore Θ, Ω, ψ and Φ have a unique common fixed point in $\mathcal{M}$. $\square$

Example 2. Let $(\mathcal{M}, \mathcal{V}, *)$ be a $\mathcal{VFM}$ -space with $\mathcal{M} = [-2, 2]$, where

$$\mathcal{V}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n, t) = \frac{t}{t + (|\alpha_1 - \alpha_2| + |\alpha_2 - \alpha_3| + \dots + |\alpha_{n-1} - \alpha_n|)}.$$

Suppose $\Theta, \psi : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\Omega, \Phi : \mathcal{M} \rightarrow \mathcal{M}$ be mappings defined as

$$\Theta(\beta, \eta) = \frac{\beta + \eta}{2}, \Omega(\beta) = \beta, \psi(\beta, \eta) = \beta - \eta + 1, \Phi(\beta) = 1.$$

Consider the sequences $\{\beta_r\} = 1 + \frac{1}{r}$, $\{\eta_r\} = 1 - \frac{1}{r}$, $\{c_r\} = \frac{1}{r}$ and $\{d_r\} = -\frac{1}{r}$. Note that

$$\lim_{r \rightarrow +\infty} \Theta(\beta_r, \eta_r) = \lim_{r \rightarrow +\infty} \Omega(\beta_r) = \lim_{r \rightarrow +\infty} \psi(c_r, d_r) = \lim_{r \rightarrow +\infty} \Phi(c_r) = 1 = \Omega(1)$$

and

$$\lim_{r \rightarrow +\infty} \Theta(\eta_r, \beta_r) = \lim_{r \rightarrow +\infty} \Omega(\eta_r) = \lim_{r \rightarrow +\infty} \psi(d_r, c_r) = \lim_{r \rightarrow +\infty} \Phi(d_r) = 1 = \Omega(1).$$

This implies that pairs (Θ, Ω) and (ψ, Φ) hold CLR_Ω -property. In addition, pairs (Θ, Ω) and (ψ, Φ) are weakly compatible. All the conditions of Theorem 3 are satisfied. Thus, the mappings Θ, ψ, Ω and Φ have $\beta = 1$ as a unique common fixed point in $\mathcal{M}$.

Author Contributions: Formal analysis, W.S.; Investigation, V.G., W.S. and A.K.; Methodology, V.G., W.S. and A.K.; Supervision, V.G. and W.S.; Writing—original draft, V.G. and A.K.; Writing—review & editing, W.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors thanks all reviewers for their useful remarks which made our paper complete and significant.

Conflicts of Interest: The authors declare no conflict of interest.

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