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Maggi's Equations Used in the Finite Element Analysis of the Multibody Systems with Elastic Elements

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Abstract: The main method used to determine the equations of motion of a multibody system (MBS) with elastic elements is the method of Lagrange's multipliers. The assembly of equations for the whole system represents an important step in the elastodynamic analysis of such a system. This paper presents a new method of approaching this stage, by applying Maggi's equations. In this way, the links that exist between the finite elements and the connections that exist between different bodies of the MBS system are conveniently taken into account, each body having a distinct velocity and acceleration field. Although Maggi's equations have been used, sporadically, in some applications so far, we are not aware that they have been used in the study of elastic systems using the finite element method. Finally, an algorithm is presented that uses the Maggi formalism to obtain the equations of motion for an MBS system.

Keywords: finite element; Lagrange; Maggi; FEA; multibody system; MBS; nonlinear system; elastic elements; analytical dynamics; robotics

1. Introduction

The consecrated method for analyzing an MBS system with elastic elements is the finite element method (FEM). Usually, for obtaining the equations of motion for a finite element, the Lagrange equations are used. Thus, the most important step is realized for the analysis of mechanical systems with elastic elements. The other procedures that follow are the classic ones, currently used in the usual FEM software and verified by practice. Using the known assembly methods and introducing the boundary conditions, we obtain a set of differential equations describing the behavior of the entire elastic MBS.

We can mention that the method of Lagrange's equations was, in these works, practically the only method to approach this type of problems. Although, theoretically, analytical mechanics offers many equivalent formulations, the familiarity of researchers and the ease of obtaining equations for systems with a large number of degrees of freedom have made Lagrange's method preferred in studies [1–14].

Another method to obtain the motion equations in such type of problems is the use of Gibbs–Appell (GA) equations. Compared with Lagrange's equations, this method has advantages, considering the number of operations required.

The necessity of Gibbs–Appell equations was due to the search for simpler methods to solve the problem of non-holonomic systems. This method was presented first by Gibbs in 1879 [15] and then, independently, by Appell in 1899 ([16]). Applying this method, the functions of Lagrange or Hamilton are replaced by the energy of accelerations. The basic principle used is well known Gauss' principle of least constraint. The Gibbs–Appell method was used for a wide class of applications [17–21]. The analytical methods of studying MBS and the possibilities of obtaining these equations are presented in [22–25]. The Gibbs–Appell method is a method rarely found in studies, but it may be of interest to researchers to use this method in analyzing MBS systems with elastic elements [26,27].

The following presents the method of Maggi equation, an alternative way to the two methods presented above to obtain the equations of motion. It is considered a mechanical system whose evolution is characterized by n parameters $q_1, q_2, \ldots, q_n$, which connect with each other through m linear relationships:

$$\sum_{j=1}^{n} a_{ij}(q_1, q_2, \ldots, q_n, t)\dot{q}_j + b_i(q_1, q_2, \ldots, q_n, t) = 0 \qquad , \quad i = \overline{1, m} \tag{1}$$

Problems arise when Lagrange's equations are applied and the constraints are nonholonomic (thus renouncing Lagrange's multipliers, used for holonomic constraints). One method of realizing this is represented by Maggi's equations [26]. These are used for mechanical systems with only nonholonomic constraints. When it is applied to systems with holonomic constraints, the method can fail. To avoid this, a tangent space ordinary differential equation (ODE) extension of the Maggi classical formulations is used [28].

In the Appendix B is a brief presentation of how to obtain Maggi's equations for a mechanical system defined by n parameters.

The form of these results:

$$\sum_{k=1}^{n} a_{kj}\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] = 0 \quad ; \quad j = \overline{1, n-m} \tag{2}$$

or:

$$[A]^T\{Ma\} = 0 \tag{3}$$

representing a number of $n{-}m$ independent equations called Maggi's equations.

Recent years have shown an interest of researchers in applying Maggi's equations to different engineering problems. In the case of the complex technical systems as robots and manipulators, there are situations in which the control system design of such systems must be ensured [29,30]. The method of feedback linearization technique for the nonlinear system naturally leads to the use of Maggi's equations.

Maggi's formulation proved to be a simple and stable way to solve the dynamic equations of constrained MBS. A problem using this method consists in an appropriate choice of independent coordinates. This leads to the high cost of computing and updating the basis of the tangent null space of constraint equations. Maggi's formulation is considered by [31] to be the most efficient way to solve the index-1 equations of Lagrange.

We can mention that the presence of nonholonomic constraints naturally leads to the use of Maggi's equations. In the case of these types of problems, especially for systems with a large number of degrees of freedom, the use of these equations can prove useful from the point of view of the required computation effort. However, the method has been little used and is not familiar to researchers. In this paper we would like to present the possibility of applying this method to the FEA case of MBS systems. The method can be proven effective in this case due to the large number of degrees of freedom that arise in solving such problems.

2. Motion Equations and Kinetic Energy for a Finite Element

The determination of kinetic energy for the studied system is essential in applying Maggi's method. So, it will be important to determine the kinetic energy for a finite element. The general case of a three-dimensional finite element will be considered. In the case of FEM, the displacements of any point of the finite element are defined, using an interpolation formula, by the nodal coordinates, which are considered to be independent coordinates.

A local coordinate system will be considered, to which the finite elementary studied is reported and which participates in the rigid motion of the MBS. The movements of all finite elements are related to the global reference frame (Figure 1). The following notations are used: $\overline{v}_{oe}(\dot{X}_{oe}, \dot{Y}_{oe}, \dot{Z}_{oe})$ is the speed of origin of the local reference system, $\overline{a}_{oe}(\ddot{X}_{oe}, \ddot{Y}_{oe}, \ddot{Z}_{oe})$ the acceleration of the origin of the local reference frame, $\overline{\omega}_e(\omega_{xe}, \omega_{ye}, \omega_{ze})$ the angular velocity and with $\overline{\varepsilon}_e(\varepsilon_{xe}, \varepsilon_{ye}, \varepsilon_{ze})$ the angular acceleration of the element numbered with e. In this paper, the index G will mark a size (vector or matrix), with the components expressed in the global reference system and the index L marking the same size, but with components expressed in a mobile reference frame. The non-indexed sizes are considered in the local reference frame. The orthonormal matrix $[R]$ transforms the components of an arbitrary vector $\{t\}_L$ from the local reference frame to the global reference frame.

$$\{t\}_G = [R]\{t\}_L \tag{4}$$

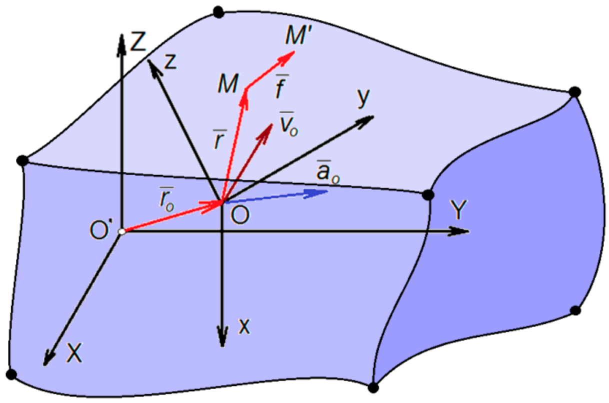

Figure 1. Three-dimensional finite element.

Considering this, the position vector $\{r_{M'}\}_G$ of the point M after deformation (becoming M'), has a displacement $\{f\}_L$, which is:

$$\{r_{M'}\}_G = \{r_O\}_G + [R]\Big(\{r\}_L + \{f\}_L\Big) \tag{5}$$

Here, $\{r_M\}_G$ is the position vector of point M before deformation. In FEA, a continuous displacement field $\{f\}_L$ will be approximated by the relation:

$$\{f\}_L = [N(x, y, z)]\{\delta\}_L \tag{6}$$

The shape function matrix $[N(x, y, z)]$ chosen determines, finally, the values of the matrix coefficient of the differential equations. We have denoted $\{\delta\}_L$ as the independent nodal coordinates vector. The velocity of an arbitrary point M' is:

$$\{v_{M'}\}_G = \left\{\dot{r}_{M'}\right\}_G = \left\{\dot{r}_O\right\}_G + \left[\dot{R}\right]\{r\}_L + \left[\dot{R}\right][N]\{\delta\}_L + [R][N]\left\{\dot{\delta}\right\}_L \tag{7}$$

and the kinetic energy:

$$E_c = \frac{1}{2}\int_V \rho\{v_{M'}\}_G^T\{v_{M'}\}_G dV \tag{8}$$

$$\begin{aligned}
E_c &= \tfrac{1}{2}\int_V \rho\Big(\big\{\dot{r}_O\big\}_G^T\big\{\dot{r}_O\big\}_G + 2\big\{\dot{r}_O\big\}_G^T\big[\dot{R}\big]\{r\}_L + 2\big\{\dot{r}_O\big\}_G^T\big[\dot{R}\big][N]\{\delta\}_L + 2\big\{\dot{r}_O\big\}_G^T[R][N]\big\{\dot{\delta}\big\}_L\Big)dV + \\
&+\tfrac{1}{2}\int_V \rho\Big(\{r\}_L^T\big[\dot{R}\big]^T\big[\dot{R}\big]\{r\}_L + 2\{r\}_L^T\big[\dot{R}\big]^T\big[\dot{R}\big][N]\{\delta\}_L + 2\{r\}_L^T\big[\dot{R}\big]^T[R][N]\big\{\dot{\delta}\big\}_L\Big)dV + \\
&+\tfrac{1}{2}\int_V \rho\Big(\{\delta\}_L^T[N]^T\big[\dot{R}\big]^T\big[\dot{R}\big][N]\{\delta\}_L + 2\{\delta\}_L^T[N]^T\big[\dot{R}\big]^T[R][N]\big\{\dot{\delta}\big\}_L + \big\{\dot{\delta}\big\}_L^T[N]^T[N]\big\{\dot{\delta}\big\}_L\Big)dV
\end{aligned} \tag{9}$$

These equations are related to the mobile reference frame. Similar formulae can be obtained as we consider the global reference frame. In this case, the relation between the components of a vector in the mobile reference frame and the components expressed in the global reference frame is due to the orthogonal matrix $[T]$ [6]:

$$\{\Delta_e\} = [T]\{\delta_e\} \quad ; \quad \{\delta_e\} = [T]^T\{\Delta_e\} \tag{10}$$

The kinetic energy becomes:

$$\begin{aligned}
E_c &= \tfrac{1}{2}\int_V \rho\Big(\big\{\dot{r}_O\big\}_G^T\big\{\dot{r}_O\big\}_G + 2\big\{\dot{r}_O\big\}_G^T\big[\dot{R}\big]\{r\}_L + 2\big\{\dot{r}_O\big\}_G^T\big[\dot{R}\big][N][T]^T\{\Delta_e\} + 2\big\{\dot{r}_O\big\}_G^T[R][N][T]^T\big\{\dot{\Delta}_e\big\}\Big)dV + \\
&+\tfrac{1}{2}\int_V \rho\Big(\{r\}_L^T\big[\dot{R}\big]^T\big[\dot{R}\big]\{r\}_L + 2\{r\}_L^T\big[\dot{R}\big]^T\big[\dot{R}\big][N][T]^T\{\Delta_e\} + 2\{r\}_L^T\big[\dot{R}\big]^T[R][N][T]^T\big\{\dot{\Delta}_e\big\}\Big)dV + \\
&+\tfrac{1}{2}\int_V \rho\Big(\{\Delta_e\}^T[T][N]^T\big[\dot{R}\big]^T\big[\dot{R}\big][N][T]^T\{\Delta_e\} + 2\{\Delta_e\}^T[T][N]^T\big[\dot{R}\big]^T[R][T]^T\big\{\dot{\Delta}_e\big\} + \big\{\dot{\Delta}_e\big\}^T[T][N]^T[N][T]^T\big\{\dot{\Delta}_e\big\}\Big)dV
\end{aligned} \tag{11}$$

To apply Lagrange's equations we must obtain:

$$\begin{aligned}
\left\{\frac{\partial E_c}{\partial \Delta_e}\right\} &= \big\{\dot{r}_O\big\}_G^T\big[\dot{R}\big]\Big(\int_V \rho([N])dV\Big)[T]^T\{\Delta_e\} + \{r\}_L^T\big[\dot{R}\big]^T\big[\dot{R}\big]\Big(\int_V \rho[N]dV\Big)[T]^T\{\Delta_e\} + \\
&+\int_V \rho\Big([T][N]^T\big[\dot{R}\big]^T\big[\dot{R}\big][N][T]^T\{\Delta_e\} + [T][N]^T\big[\dot{R}\big]^T[R][T]^T\big\{\dot{\Delta}_e\big\}\Big)dV
\end{aligned} \tag{12}$$

$$\begin{aligned}
\left\{\frac{\partial E_c}{\partial \dot{\Delta}_e}\right\} &= \big\{\dot{r}_O\big\}_G^T[R]\Big(\int_V \rho[N]dV\Big)[T]^T + \Big(\int_V \rho\{r\}_L^T\big[\dot{R}\big]^T[R][N]dV\Big)[T]^T + \\
&+[T]\Big(\int_V \rho[N]^T\big[\dot{R}\big]^T[R]dV\Big)[T]^T\{\Delta_e\} + [T]\Big(\int_V \rho[N]^T[N]dV\Big)[T]^T\big\{\dot{\Delta}_e\big\}
\end{aligned} \tag{13}$$

By $\left\{\frac{\partial E_c}{\partial \Delta_e}\right\}$ it is noted $\begin{bmatrix} \frac{\partial E_c}{\partial \Delta_{e,1}} & \frac{\partial E_c}{\partial \Delta_{e,2}} & \cdots & \cdots & \frac{\partial E_c}{\partial \Delta_{e,se}} \end{bmatrix}^T$ and by $\left\{\frac{\partial E_c}{\partial \dot{\Delta}_e}\right\}$ it is noted $\begin{bmatrix} \frac{\partial E_c}{\partial \dot{\Delta}_{e,1}} & \frac{\partial E_c}{\partial \dot{\Delta}_{e,2}} & \cdots & \cdots & \frac{\partial E_c}{\partial \dot{\Delta}_{e,se}} \end{bmatrix}^T$.

The orthonormal rotation matrix $[R]$ satisfies the relation:

$$[R][R]^T = [R]^T[R] = [E] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

where $[E]$ is the unit matrix. By differentiating (14) it obtains:

$$\big[\dot{R}\big][R]^T + [R]\big[\dot{R}\big]^T = \big[\dot{R}\big]^T[R] + [R]^T\big[\dot{R}\big] = [0] \tag{15}$$

We denote:

$$[\omega]_G = \big[\dot{R}\big][R]^T = -[R]\big[\dot{R}\big]^T = \begin{bmatrix} 0 & -\omega_{zG} & \omega_{yG} \\ \omega_{zG} & 0 & -\omega_{xG} \\ -\omega_{yG} & \omega_{xG} & 0 \end{bmatrix} \tag{16}$$

which is the skew symmetric operator angular velocity, with components expressed in the global coordinate system, corresponding to the vector angular velocity:

$$\{\omega\}_G = \begin{Bmatrix} \omega_{xG} \\ \omega_{yG} \\ \omega_{zG} \end{Bmatrix} \tag{17}$$

Obviously, we also have:

$$[\omega]_L = [R]^T\left[\dot{R}\right] = \begin{bmatrix} 0 & -\omega_{zL} & \omega_{yL} \\ \omega_{zL} & 0 & -\omega_{xL} \\ -\omega_{yL} & \omega_{xL} & 0 \end{bmatrix} \text{ and } \{\omega\}_L = \begin{Bmatrix} \omega_{xL} \\ \omega_{yL} \\ \omega_{zL} \end{Bmatrix}. \tag{18}$$

In a similar way, the operator angular acceleration in a global reference frame can be defined by:

$$[\varepsilon]_G = \left[\dot{\omega}\right]_G = \begin{bmatrix} 0 & -\dot{\omega}_{zG} & \dot{\omega}_{yG} \\ \dot{\omega}_{zG} & 0 & -\dot{\omega}_{xG} \\ -\dot{\omega}_{yG} & \dot{\omega}_{xG} & 0 \end{bmatrix} = \left[\ddot{R}\right][R]^T + \left[\dot{R}\right]\left[\dot{R}\right]^T = \begin{bmatrix} 0 & -\varepsilon_{zG} & \varepsilon_{yG} \\ \varepsilon_{zG} & 0 & -\varepsilon_{xG} \\ -\varepsilon_{yG} & \varepsilon_{xG} & 0 \end{bmatrix} \tag{19}$$

and in a local reference frame:

$$[\varepsilon]_L = \left[\dot{\omega}\right]_L = \begin{bmatrix} 0 & -\dot{\omega}_{zL} & \dot{\omega}_{yL} \\ \dot{\omega}_{zL} & 0 & -\dot{\omega}_{xL} \\ -\dot{\omega}_{yL} & \dot{\omega}_{xL} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\varepsilon_{zL} & \varepsilon_{yL} \\ \varepsilon_{zL} & 0 & -\varepsilon_{xL} \\ -\varepsilon_{yL} & \varepsilon_{xL} & 0 \end{bmatrix} \tag{20}$$

It will result, from (14), after some elementary calculus:

$$\left[\ddot{R}\right][R]^T = [\varepsilon]_G - \left[\dot{R}\right]\left[\dot{R}\right]^T = [\varepsilon]_G + [\omega]_G[\omega]_G \tag{21}$$

whereas:

$$[\omega]_G = -[\omega]_G^T \tag{22}$$

We have also, in a similar way:

$$[R]^T\left[\ddot{R}\right] = [\varepsilon]_L - \left[\dot{R}\right]^T\left[\dot{R}\right] = [\varepsilon]_L + [\omega]_L[\omega]_L \tag{23}$$

These relations will be useful in the following considerations.

Using notations presented in Appendix A, after some calculus it results:

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\partial E_c}{\partial \dot{\Delta}_e}\right\} - \left\{\frac{\partial E_c}{\partial \Delta_e}\right\} = [M_e]\left\{\ddot{\Delta}_e\right\} + [C_e]\left\{\dot{\Delta}_e\right\} + ([K_e(\varepsilon)] + [K_e(\omega)])\{\Delta_e\} \\ &+\left\{Q_e^i(\varepsilon)\right\} + \left\{Q_e^i(\omega)\right\} + \left[M_{Oe}^i\right]\left\{\ddot{r}_{Oe}\right\}_G \end{aligned} \tag{24}$$

3. Maggi's Method to FEM Assembly Procedures

Equation (24) is written for the finite element e. The total kinetic energy for the entire system is:

$$\begin{aligned} &E_{ce} = \frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\{v_{M'}\}_G^T\{v_{M'}\}_G dV = \frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\{v_{M'}\}_L^T[R]^T[R]\{v_{M'}\}_L dV = \frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\{v_{M'}\}_L^T\{v_{M'}\}_L dV = \\ &\frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\left(\left\{\dot{r}_O\right\}_G^T\left\{\dot{r}_O\right\}_G + 2\left\{\dot{r}_O\right\}_G^T\left[\dot{R}\right]\{r\}_L + 2\left\{\dot{r}_O\right\}_G^T\left[\dot{R}\right][N][T]^T\{\Delta_e\} + 2\left\{\dot{r}_O\right\}_G^T[R][N][T]^T\left\{\dot{\Delta}_e\right\}\right)dV+ \\ &\frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\left(\{r\}_L^T\left[\dot{R}\right]^T\left[\dot{R}\right]\{r\}_L + 2\{r\}_L^T\left[\dot{R}\right]^T\left[\dot{R}\right][N][T]^T\{\Delta_e\} + 2\{r\}_L^T\left[\dot{R}\right]^T[R][N][T]^T\left\{\dot{\Delta}_e\right\}\right)dV+ \\ &\frac{1}{2}\sum_{i=1}^{Ne}\int_V \rho\left(\{r\}_L^T\left[\dot{R}\right]^T\left[\dot{R}\right]\{r\}_L + 2\{r\}_L^T\left[\dot{R}\right]^T\left[\dot{R}\right][N][T]^T\{\Delta_e\} + 2\{r\}_L^T\left[\dot{R}\right]^T[R][N][T]^T\left\{\dot{\Delta}_e\right\}\right)dV \end{aligned} \tag{25}$$

If it is denoted:

$$[M] = \begin{bmatrix} M_1 & & & & 0 \\ & M_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & M_p \end{bmatrix} ; \quad [C] = \begin{bmatrix} C_1 & & & & 0 \\ & C_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & C_p \end{bmatrix};$$

$$[K] = \begin{bmatrix} K_1 & & & & 0 \\ & K_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & K_p \end{bmatrix} ; \quad [K(\varepsilon)] = \begin{bmatrix} K_1(\varepsilon) & & & & 0 \\ & K_2(\varepsilon) & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & K_p(\varepsilon) \end{bmatrix};$$

$$[K(\omega)] = \begin{bmatrix} K_1(\omega) & & & & 0 \\ & K_2(\omega) & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & K_p(\omega) \end{bmatrix}; \{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ \\ \vdots \\ Q_p \end{Bmatrix}; \{Q^*\} = \begin{Bmatrix} Q_1^* \\ Q_2^* \\ \vdots \\ \\ \vdots \\ Q_p^* \end{Bmatrix}; \tag{26}$$

$$\left\{Q^i(\varepsilon)\right\} = \begin{Bmatrix} Q_1^i(\varepsilon) \\ Q_2^i(\varepsilon) \\ \vdots \\ \\ \vdots \\ Q_p^i(\varepsilon) \end{Bmatrix}; \left\{Q^i(\omega)\right\} = \begin{Bmatrix} Q_1^i(\omega) \\ Q_2^i(\omega) \\ \vdots \\ \\ \vdots \\ Q_p^i(\omega) \end{Bmatrix}; \left\{Q_O^i\right\} = \begin{Bmatrix} \left[M_{O1}^i\right]\left\{\ddot{r}_{O1}\right\}_G \\ \left[M_{O2}^i\right]\left\{\ddot{r}_{O2}\right\}_G \\ \vdots \\ \\ \vdots \\ \left[M_{Op}^i\right]\left\{\ddot{r}_{Op}\right\}_G \end{Bmatrix}.$$

$$\{\Delta\} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \\ \vdots \\ \Delta_M \end{Bmatrix} ; \quad \left\{\dot{\Delta}\right\} = \begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \vdots \\ \\ \vdots \\ \dot{\Delta}_M \end{Bmatrix} ; \quad \left\{\ddot{\Delta}\right\} = \begin{Bmatrix} \ddot{\Delta}_1 \\ \ddot{\Delta}_2 \\ \vdots \\ \\ \vdots \\ \ddot{\Delta}_M \end{Bmatrix}$$

$$\begin{gathered} \frac{d}{dt}\left\{\frac{\partial E_c}{\partial \dot{\Delta}}\right\} - \left\{\frac{\partial E_c}{\partial \Delta}\right\} - \{Q\} - \{Q^*\} = \\ = [M]\left\{\ddot{\Delta}\right\} + [C]\left\{\dot{\Delta}\right\} + ([K] + [K(\varepsilon)] + [K(\omega)])\{\Delta\} - \{Q\} - \{Q^*\} + \left\{Q^i(\varepsilon)\right\} + \left\{Q^i(\omega)\right\} + \left\{Q_O^i\right\} \end{gathered} \tag{27}$$

Now, we must also consider the relation between finite elements. Some nodes may belong to two or more finite elements. It follows that the coordinates defining the kinematic elements of the nodes are not independent, but are in liaison by linear relations which, in this case, are written as:

$$\{\Delta\} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \\ \vdots \\ \Delta_M \end{Bmatrix} = [A]\{q\} \quad ; \quad \{\dot{\Delta}\} = \begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \vdots \\ \\ \vdots \\ \dot{\Delta}_M \end{Bmatrix} = [A]\{\dot{q}\} \quad ; \quad \{\ddot{\Delta}\} = \begin{Bmatrix} \ddot{\Delta}_1 \\ \ddot{\Delta}_2 \\ \vdots \\ \\ \vdots \\ \ddot{\Delta}_M \end{Bmatrix} = [A]\{\ddot{q}\}. \tag{28}$$

where $\{q\}$ is the vector of independent coordinates of the whole system. Introducing (28) in (27), it obtains:

$$\begin{gathered} \frac{d}{dt}\left\{\frac{\partial E_c}{\partial \dot{\Delta}}\right\} - \left\{\frac{\partial E_c}{\partial \Delta}\right\} - \{Q\} - \{Q^*\} = \\ [M][A]\{\ddot{q}\} + [C][A]\{\dot{q}\} + ([K] + [K(\varepsilon)] + [K(\omega)])[A]\{q\} \\ -\{Q\} - \{Q^*\} + \left\{Q^i(\varepsilon)\right\} + \left\{Q^i(\omega)\right\} + \left\{Q^i_O\right\} \end{gathered} \tag{29}$$

Applying Maggi's Equations from Appendix B, (A27) and (A28) to expression (29) it becomes, finally:

$$\begin{gathered} [A]^T[M][A]\{\ddot{q}\} + [A]^T[C][A]\{\dot{q}\} + [A]^T([K] + [K(\varepsilon)] + [K(\omega)])[A]\{q\} = \\ = [A]^T\{Q\} + [A]^T\{Q^*\} - [A]^T\left\{Q^i(\varepsilon)\right\} - [A]^T\left\{Q^i(\omega)\right\} - [A]^T\left\{Q^i_O\right\} \end{gathered} \tag{30}$$

that represents the motion equations for the studied MBS, in term of finite elements. In this way it is possible to obtain these equations in an alternative form. At the same time, this representation has a strong theoretical background.

From the equations presented above, we highlight an algorithm that can be used in the case of applying FEA to an MBS analysis with elastic elements, in order to obtain the evolution equations of such a system. The first step is to write the equations of motion for a single finite element. These equations will depend on the type of finite element used and the shape functions chosen. The second step is to refer all the equations of motion to a global (fix) reference system. The third step is the application of Maggi's equations to the set of differential equations obtained in the previous step. For this, it is necessary to describe the constraints between the different finite elements, using linear relations between the independent coordinates. By applying this step, one obtains the differential equations of the second order which, by solving, lead to the dynamic response of the system. After the assembly stage using Maggi's equations, the procedures for analyzing and solving the system of differential equations are the classical ones, applied for this type of problems.

4. Discussion and Conclusions

In accomplishing the procedures of assembling the equations of motion obtained for each finite element of a discretization for FEA of an elastic multibody system, different methods can be used. In the last decades, the analysis of an MBS is made using different methodologies: Newton–Euler equations, Lagrange equations, Kane's method [32] or Maggi's equations. The analytical mechanics also offer formulations equivalent to these, useful in different practical circumstances. An important advantage of the Newton–Euler method is that the equations will be, formally independent of the geometry, inertia or constraints. The disadvantage is that the constraint forces or torques must be determined. When the dimension of the system is high enough and there are many DOF to be considered, it will be difficult to determine the liaison forces. A largely used method is the Lagrange method, which makes it possible to obtain constraint-free differential equations. This is an important advantage if a comparison with Newton–Euler method is made. In the engineering practice, commercial software for the analysis of multibody dynamic systems (i.e., ADAMS, DADS, DYMAC) mainly use the Lagrange method. Other software, such as SD-EXACT, NBOD2, and SD/FAST, use the Kane's approach. Kane's method is a development of the Maggi's equations. In the first papers, Maggi's equations are developed as an extension of Lagrangian formalism. The main advantage of this method consists of an easy analysis of MBS with high DOF and nonholonomic constraints. The liaisons appearing in the system expressed by linear relation offer the possibility of applying a projection operator (orthogonal complement matrix), in order to eliminate the terms containing the multipliers. We have presented a new way of approaching this problem, using Maggi's equations. The use of these equations allows a simpler approach, from the point of view of the formal calculation of these systems. Thus, only the kinetic energy is calculated, after which, using the liaison conditions directly between the nodes of the finite elements, the equations of motion are obtained. In this way, the approach of such a system becomes simpler. At the same time, there is also a justification of the classical assembly methods, applied empirically in the FEA.

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Appendix A

After some calculus and denoting $\left[N_{(i)}\right]$ the row i of matrix $[N]$, we will denote:

$$\begin{aligned}[m_e] &= \int_V \rho [N]^T [N] dV = \int_V \rho \begin{bmatrix} N_{(1)}^T & N_{(2)}^T & N_{(3)}^T \end{bmatrix} \begin{bmatrix} N_{(1)} \\ N_{(2)} \\ N_{(3)} \end{bmatrix} dV = \\ &= \int_V \rho N_{(1)}^T N_{(1)} dV + \int_V \rho N_{(2)}^T N_{(2)} dV + \int_V \rho N_{(3)}^T N_{(3)} dV = [m_{11}] + [m_{22}] + [m_{33}],\end{aligned} \tag{A1}$$

where:

$$\left[m_{e,ij}\right] = \int_V \rho N_{(i)}^T N_{(j)} dV, \tag{A2}$$

which is:

$$[m_{e,11}] = \int_V \rho N_{(1)}^T N_{(1)} dV,\ [m_{e,22}] = \int_V \rho N_{(2)}^T N_{(2)} dV,\ [m_{e,33}] = \int_V \rho N_{(3)}^T N_{(3)} dV. \tag{A3}$$

We have denoted too:

$$\left[m_{eO}^i\right] = \int_V \rho [N]^T dV;\ \left\{q_e^i(\varepsilon)\right\}_L = \int_V \rho [N]^T [\varepsilon]_L \{r\}_L dV;\ \left\{q_e^i(\omega)\right\}_L = \int_V \rho [N]^T [\omega]_L [\omega]_L \{r\}_L dV; \tag{A4}$$

$$[k_e(\varepsilon)] = \int_V \rho [N]^T [\varepsilon][N] dV;\ [k_e(\varepsilon)] = \int_V \rho [N]^T [\omega]_L [\omega]_L [N] dV;\ [c_e] = \int_V \rho [N]^T [\omega]_L [N] dV; \tag{A5}$$

$$\left\{m_{e,ix}\right\} = \int_V \rho \left[N_{(i)}\right]^T x dV \quad ; \quad \left\{m_{e,iy}\right\} = \int_V \rho \left[N_{(i)}\right]^T y dV \quad ; \quad \left\{m_{e,iz}\right\} = \int_V \rho \left[N_{(i)}\right]^T z dV \tag{A6}$$

and:

$$\begin{gathered}[M_e] = [T][m_e][T]^T\ ;\ [C] = [T][c_e][T]^T\ ;\ [K] = [T][k_e][T]^T; \\ [K_e(\varepsilon)] = [T][k_e(\varepsilon)][T]^T\ ;\ [K_e(\omega)] = [T][k_e(\omega)][T]^T\ ;\ \{Q_e\} = [T]\{q_e\}\ ;\ \{Q_e^*\} = [T]\{q_e^*\}_L; \\ \left\{Q_e^i(\varepsilon)\right\} = [T]\left\{q_e^i(\varepsilon)\right\}\ ;\ \left\{Q_e^i(\omega)\right\} = [T]\left\{q_e^i(\omega)\right\}\ ;\ \left[M_{Oe}^i\right] = [T]\left[m_{Oe}^i\right][T]^T\end{gathered} \tag{A7}$$

Appendix B

It is considered to be a mechanical system whose evolution is characterized by n parameters q_1, q_2, . . . , q_n, which connect with each other through m linear relationships:

$$\sum_{j=1}^{n} a_{ij}(q_1, q_2, \ldots, q_n, t)\dot{q}_j + b_i(q_1, q_2, \ldots, q_n, t) = 0 \quad , \quad i = \overline{1, m} \tag{A8}$$

The following is a brief presentation of how to obtain Maggi's equations for a mechanical system defined by n parameters.

If we consider the d'Alembert–Lagrange equation [26]:

$$\sum_{i=1}^{N} \left(\overline{F}_i - m_i \overline{a}_i\right) \delta \overline{r}_i = 0 \quad , \tag{A9}$$

applied to a system of N material points, where:

$$\overline{r}_i = \overline{r}_i(q_1, q_2, \ldots, q_n, t) \quad , \quad i = \overline{1, N}. \tag{A10}$$

represents the position vector of the particle i, $q_1, q_2, \ldots, q_n$ are the generalized coordinates that define the position of the system. The virtual displacement $\delta \overline{r}_i$ can be expressed as:

$$\delta \overline{r}_i = \sum_{i=1}^{n} \frac{\partial \overline{r}_i}{\partial q_k} \delta q_k \qquad , \quad i = \overline{1, N}; \tag{A11}$$

Considering the differentials δq_k independents and introducing in (2), it results:

$$\sum_{i=1}^{N} \left(\overline{F}_i - m_i \overline{a}_i\right) \frac{\partial \overline{r}_i}{\partial q_k} = 0 \quad , \quad k = \overline{1, n}; \tag{A12}$$

If we denote with:

$$Q_k = \sum_{i=1}^{N} \overline{F}_i \frac{\partial \overline{r}_i}{\partial q_k} = 0 \quad , \quad k = \overline{1, n}, \tag{A13}$$

the components of the generalized forces, it obtains:

$$\sum_{i=1}^{N} m_i \overline{a}_i \frac{\partial \overline{r}_i}{\partial q_k} = Q_k \quad , \quad k = \overline{1, n}; \tag{A14}$$

or

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k} = Q_k \quad , \quad k = \overline{1, n}; \tag{A15}$$

which represents Lagrange's equations, where with E_c is denoted the kinetic energy. Using rel. (7) it is possible to write rel. (2) under the form:

$$\sum_{k=1}^{n} \left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] \delta q_k = 0 \quad , \tag{A16}$$

that represents the basis on the application of Maggi's method.

If between the coordinates $q_1, q_2, \ldots, q_n$ we have the rel. (1), written in the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{Bmatrix} + \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} = 0. \tag{A17}$$

If a virtual displacement of the coordinates $q_1, q_2, \ldots, q_n$ is considered, then one can write [26,27]:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{Bmatrix} = 0. \tag{A18}$$

In this case, $n-m$ virtual displacements can be written according to m-imposed conditions (11). Suppose we have renumbered the coordinates so that the dependent ones are renumbered as δq_{n-m+1}, $\delta q_{n-m+2}, \ldots$., δq_n. The dependent coordinates can be written in terms of independent coordinates as:

$$\begin{Bmatrix} \delta q_{n-m+1} \\ \delta q_{n-m+2} \\ \vdots \\ \delta q_n \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1,n-m} \\ c_{21} & c_{22} & \cdots & c_{2,n-m} \\ \vdots & & & \vdots \\ c_{n-m,1} & c_{n-m,2} & \cdots & c_{n-m,n-m} \end{bmatrix} \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_{n-m} \end{Bmatrix}, \tag{A19}$$

or

$$\begin{Bmatrix} \delta q_{n-m+1} \\ \delta q_{n-m+2} \\ \vdots \\ \delta q_n \end{Bmatrix} = [C] \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_{n-m} \end{Bmatrix} \tag{A20}$$

Using rel. (13) the coordinate vector $q_1, q_2, \ldots, q_n$ can be expressed as:

$$\begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{Bmatrix} = \begin{bmatrix} E_{n-m} \\ C_{mx(n-m)} \end{bmatrix} \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_{n-m} \end{Bmatrix} = \left[A_{nx(n-m)}\right] \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_{n-m} \end{Bmatrix} = \left[A_{nx(n-m)}\right]\{\delta q\} \tag{A21}$$

$$\delta q_k = \sum_{j=1}^{n-m} a_{kj} \delta q_j, \quad k = \overline{1, n}. \tag{A22}$$

We denote with $\{Ma\}$ the Maggi's vector with the components:

$$Ma(k) = \left(\frac{d}{dt} \left(\frac{\partial E_c}{\partial \dot{q}_k} \right) - \frac{\partial E_c}{\partial q_k} \right) - Q_k \quad k = 1,\ n. \tag{A23}$$

Rel. (9) becomes:

$$\{Ma\}^T \{\delta q\} = 0, \tag{A24}$$

equivalent to

$$\{\delta q\}^T\{Ma\} = 0. \tag{A25}$$

Using in rel. (9) the rel. (15), it obtains:

$$\sum_{k=1}^{n}\left\{\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right]\sum_{j=1}^{n-m} a_{kj}\delta q_j\right\} = 0. \tag{A26}$$

Coordinates $q_1, q_2, \ldots, q_{n-m}$ being independent it results:

$$\sum_{k=1}^{n} a_{kj}\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] = 0 \quad ; \quad j = \overline{1, n-m}, \tag{A27}$$

or:

$$[A]^T\{Ma\} = 0 \tag{A28}$$

Representing in a number of $n-m$ independent equations, called Maggi's equations.

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