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Article

On Pata–Suzuki-Type Contractions

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Abstract: In this manuscript, we introduce two notions, Pata–Suzuki $\mathcal{Z}$-contraction and Pata $\mathcal{Z}$-contraction for the pair of self-mapping g, f in the context of metric spaces. For such types of contractions, both the existence and uniqueness of a common fixed point are examined. We provide examples to illustrate the validity of the given results. Further, we consider ordinary differential equations to apply our obtained results.

Keywords: simulation function; Pata–Suzuki $\mathcal{Z}$-contraction; $\mathcal{Z}$-contraction; C-condition

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1. Introduction and Preliminaries

One of the interesting approach to extending existing fixed point results is to involve an auxiliary function into the hypotheses of theorems. In this paper, we consider the notion of the simulation function that is defined by Khojasteh et al. [1].

Definition 1 (See [1]). *A* simulation function *is a mapping* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(t,s) < s - t$ *for all* $t, s > 0$;
(ζ_2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

Notice that the axiom (ζ_1) yields that

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{2}$$

Note that in the original definition of the *simulation function*, there was a superfluous condition $\zeta(0,0) = 0$. From now on, the letter $\mathcal{Z}$-presents the class of all functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies (ζ_1) and (ζ_2). An immediate example of a simulation function is $\zeta(t,s) := ks - t$ where $k \in [0,1)$ for all $s, t \in [0,\infty)$. For more significant examples and applications of simulation functions, we refer e.g., [1–6].

From now on, the pairs (X, d) and (X^*, d) denote metric space and complete metric spaces, respectively. Furthermore, both f and g are self-mapping defined on (X^*, d). We say that f is *$\mathcal{Z}$-contraction* with respect to $\zeta \in \mathcal{Z}$ [1], if

$$\zeta(d(f\nu, f\omega), d(\nu, \omega)) \geq 0 \qquad \text{for all } \nu, \omega \in X. \tag{3}$$

By using this definition, the following result was proved in [1]:

Theorem 1. *Each $\mathcal{Z}$-contraction on a (X^*, d) possesses a unique fixed point.*

It is clear that Theorem 1 reduces Banach's contraction mapping principle if take $\zeta(t, s) := ks - t$, for all $s, t \in [0, \infty)$, where $k \in [0, 1)$.

The aim of Suzuki [7] is to extend the well-known Edelstein's Theorem by using the notion of C-condition.

Definition 2 (See [8])**.** *We say that f, defined on a (X, d), satisfies C-condition if*

$$\frac{1}{2}d(\nu, f\nu) \leq d(\nu, \omega) \Longrightarrow d(f\nu, f\omega) \leq d(\nu, \omega), \text{ for all } \nu, \omega \in X.$$

Next, we shall mention the impressive result of V.Pata [9] on the existence of a fixed point in the setting of in a complete metric space. Suppose ν_0 is an arbitrary but a fixed in X. We say that ν_0 is a zero of X, if

$$\|\nu\| = d(\nu, \nu_0), \text{ for all } \nu \in X.$$

We presumed that $\psi : [0, 1] \to [0, \infty)$ is continuous at zero with $\psi(0) = 0$ and is also increasing. Under these settings, recently, Pata [9] proposed the following result:

Theorem 2 (See [9])**.** *f, defined on (X^*, d), possesses a unique fixed point if*

$$d(f\nu, f\omega) \quad \leq (1 - \varepsilon)d(\nu, \omega) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|\nu\| + \|\omega\|\right]^{\beta},$$

fulfils for every $\nu, \omega \in X$, for each $\varepsilon \in [0, 1]$, where $\alpha \geq 1$, $\Lambda \geq 0$, and $\beta \in [0, \alpha]$, are fixed constants.

Theorem 2 has been investigated densely and it has been extended by [10–20]. We also refer to [21–25] for the basics of fixed point theory.

The main goal of this paper is to combine the notion of simulation functions, the concept of C-distance and Pata type contraction so that the obtained notions (namely, Pata–Suzuki $\mathcal{Z}$-contraction and Pata $\mathcal{Z}$-contraction) unify, extend and generalize several existing results in the literature of fixed point theory.

2. Main Results

Definition 3. *A pair (g, f) , on a (X, d), is called Pata–Suzuki $\mathcal{Z}$-contraction whenever the following is fulfilled (P)*

$$\text{either } \frac{1}{2}d(\nu, g\nu) \leq d(\nu, \omega) \text{ or } \frac{1}{2}d(\omega, f\omega) \leq d(\nu, \omega)$$

implies

$$\zeta(d(g\nu, f\omega), C_{g,f}(\nu, \omega)) \geq 0, \tag{4}$$

for every $\varepsilon \in [0, 1]$ and all $\nu, \omega \in X$, where $\zeta \in \mathcal{Z}$, $\alpha \geq 1$, $\Lambda \geq 0$, and $\beta \in [0, \alpha]$ are constants, and

$$
\begin{aligned}
C_{g,f}(\nu,\omega) &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(f\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega)+d(f\omega,\nu)\right]\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu\|+\|\omega\|+\|g\nu\|+\|f\omega\|\right]^{\beta}.
\end{aligned}
$$

Theorem 3. *If a pair (g,f), on a (X^*,d), forms Pata–Suzuki $\mathcal{Z}$-contraction, and g,f are continuous, then g,f have a common fixed point $\nu_* \in X$.*

Proof. Take an arbitrary $\nu \in X$ and rename as ν_0. Let $\nu_1 = g\nu_0$ and construct a sequence $\{\nu_n\}$ by

$$g\nu_{2n} = \nu_{2n+1} \text{ and } f\nu_{2n+1} = \nu_{2n+2} \text{ for all } n \in \mathbb{N}\cup\{0\}.$$

To winnow out the trivial cases, throughout the proof, we suppose that $\nu_{m+1} \neq \nu_m$ for all $m \in \mathbb{N}$. Indeed, if we suppose, on the contrary, that $\nu_{m_0+1} = \nu_{m_0}$ for some $m_0 \in \mathbb{N}$, then we conclude a common fixed point of f and g without any effort. Without loss of generality we may assume $\nu_{2n_0+1} = \nu_{2n_0}$.

Since $\frac{1}{2}d(\nu_{2n_0}, g\nu_{2n_0}) \leq d(\nu_{2n_0}, \nu_{2n_0+1})$ we have implies

$$\zeta(d(g\nu_{2n_0}, f\nu_{2n_0+1}), C_{g,f}(\nu_{2n_0}, \nu_{2n_0+1})) \geq 0,$$

which implies that

$$
\begin{aligned}
d(\nu_{2n_0+1},\nu_{2n_0+2}) &= d(g\nu_{2n_0}, f\nu_{2n_0+1}) \\
&\leq (1-\varepsilon)\max\left\{d(\nu_{2n_0},\nu_{2n_0+1}), d(g\nu_{2n_0},\nu_{2n_0}), d(f\nu_{2n_0+1},\nu_{2n_0+1}),\right. \\
&\qquad \left.\tfrac{1}{2}\left[d(g\nu_{2n_0},\nu_{2n_0+1}) + d(f\nu_{2n_0+1},\nu_{2n_0})\right]\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu_{2n_0}\|+\|\nu_{2n_0+1}\|+\|g\nu_{2n_0}\|+\|f\nu_{2n_0+1}\|\right]^{\beta} \\
&\leq (1-\varepsilon)d(\nu_{2n_0+1},\nu_{2n_0+2}) + K\varepsilon^{\alpha}\psi(\varepsilon)
\end{aligned}
$$

for some $K > 0$. Thus, we have

$$d(\nu_{2n_0+1},\nu_{2n_0+2}) \leq K\varepsilon^{\alpha-1}\psi(\varepsilon),$$

is true for all $\varepsilon > 0$. This yields $d(\nu_{2n_0+1},\nu_{2n_0+2}) = 0$. Consequently, we get $\nu_{2n_0} = \nu_{2n_0+1} = \nu_{2n_0+2}$ which implies $g\nu_{2n_0} = f\nu_{2n_0} = \nu_{2n_0}$. Hence ν_{2n_0} is a common fixed point of g and f which is observed without any difficulty. Analogously, one can derive that the case $\nu_{2n_0+1} = \nu_{2n_0+2}$ implies the same conclusion. For this reason, throughout the proof, we winnow out the trivial case and assume that

$$\nu_{m+1} \neq \nu_m \text{ for all } m \in \mathbb{N}. \tag{5}$$

Now, we claim that the sequence $\{d(\nu_m,\nu_{m-1})\}$ is non-increasing. First we observe that the sequence $\{d(\nu_{2n},\nu_{2n+1})\}$ is non-increasing. Suppose, on the contrary, that

$$d(\nu_{2n_0},\nu_{2n_0+1}) > d(\nu_{2n_0},\nu_{2n_0-1}) \text{ for some } n_0 \in \mathbb{N}. \tag{6}$$

Since $\frac{1}{2}d(f\nu_{2n_0-1},\nu_{2n_0-1}) \leq d(\nu_{2n_0-1},\nu_{2n_0})$ the expression (4) yields that

$$\zeta(d(f\nu_{2n_0-1}, g\nu_{2n_0}), C_{g,f}(\nu_{2n_0-1},\nu_{2n_0})) \geq 0,$$

which is equivalent to

$$\begin{aligned}
d(\nu_{2n_0},\nu_{2n_0+1}) &= d(f\nu_{2n_0-1},g\nu_{2n_0}) \le C_{g,f}(\nu_{2n_0-1},\nu_{2n_0}) \\
&= (1-\varepsilon)\max\Big\{d(\nu_{2n_0-1},\nu_{2n_0}),d(f\nu_{2n_0-1},\nu_{2n_0-1}),d(g\nu_{2n_0},\nu_{2n_0}), \\
&\qquad \tfrac{1}{2}\left[d(f\nu_{2n_0-1},\nu_{2n_0})+d(g\nu_{2n_0},\nu_{2n_0-1})\right]\Big\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu_{2n_0-1}\|+\|\nu_{2n_0}\|+\|f\nu_{2n_0-1}\|+\|g\nu_{2n_0}\|\right]^{\beta} \\
&\le (1-\varepsilon)d(\nu_{2n_0},\nu_{2n_0+1}) \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu_{2n_0-1}\|+\|\nu_{2n_0}\|+\|f\nu_{2n_0-1}\|+\|g\nu_{2n_0}\|\right]^{\beta}.
\end{aligned}$$

Since the inequality above holds for each $\varepsilon \ge 0$, it follows that $d(\nu_{2n_0},\nu_{2n_0+1}) = 0$. It contradicts (5) and hence the assumption (6) fails. Accordingly, $\{d(\nu_{2n_0},\nu_{2n_0+1})\}$ is a non-increasing sequence. Analogously, we find that $\{d(\nu_{2n_0+1},\nu_{2n_0+2})\}$ is a non-increasing sequence. So, we conclude that the sequence $\{d(\nu_m,\nu_{m-1})\}$ non-increasing.

We shall indicate that the set $\{C_n\}$ is bounded. Fix $n \in \mathbb{N}$. Since the sequence $\{d(\nu_m,\nu_{m-1})\}$ non-increasing, we have

$$d(\nu_{2n+1},\nu_{2n+2}) \le d(\nu_{2n},\nu_{2n+1}) \le \cdots \le d(\nu_0,\nu_1).$$

By the above and the triangle inequality we have

$$\begin{aligned}
C_{2n+1} = d(\nu_{2n+1},\nu_0) &\le d(\nu_{2n+1},\nu_{2n+2}) + d(\nu_{2n+2},\nu_1) + d(\nu_1,\nu_0) \\
&= d(\nu_{2n+2},\nu_1) + 2C_1 \\
&= d(g\nu_0,f\nu_{2n+1}) + 2C_1.
\end{aligned} \tag{7}$$

If $d(\nu_{2n+1},\nu_0) < \frac{1}{2}d(\nu_{2n+1},f\nu_{2n+1})$ then due to above observation we conclude that $C_{2n+1} = d(\nu_{2n+1},\nu_0) < \frac{C_1}{2}$ and it shows C_{2n+1} is bounded by $\frac{C_1}{2}$. Otherwise, we have $\frac{1}{2}d(\nu_{2n+1},f\nu_{2n+1}) \le d(\nu_{2n+1},\nu_0)$ and by (4) we have

$$\zeta(d(g\nu_0,f\nu_{2n+1}),C_{g,f}(\nu_{2n+1},\nu_0)) \ge 0. \tag{8}$$

Thus, by combining (7) and (8) together with $\beta \le \alpha$ we get

$$\begin{aligned}
C_{2n+1} &\le 2C_1 + C_{g,f}(\nu_0,\nu_{2n+1}) \\
&\le 2C_1 + (1-\varepsilon)(C_{2n+1}+C_1) \\
&\quad +\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+C_{2n+1}+\|\nu_1\|+\|\nu_{2n+2}\|\right]^{\beta}.
\end{aligned} \tag{9}$$

Notice that $C_{g,f}(\nu_{2n+1},\nu_0)$ is estimated by $C_{2n+1}+C_1$ as follows:

$$\begin{aligned}
C_{g,f}(\nu_0,\nu_{2n+1}) &= (1-\varepsilon)\max\left\{\begin{array}{c} d(\nu_0,\nu_{2n+1}),d(\nu_0,g\nu_0),d(\nu_{2n+1},f\nu_{2n+1}), \\ \frac{1}{2}\left[d(g\nu_0,\nu_{2n+1})+d(\nu_0,f\nu_{2n+1})\right] \end{array}\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu_0\|+\|\nu_{2n+1}\|+\|g\nu_0\|+\|f\nu_{2n+1}\|\right]^{\beta}. \\
&= (1-\varepsilon)\max\left\{\begin{array}{c} d(\nu_0,\nu_{2n+1}),d(\nu_0,\nu_1),d(\nu_{2n+1},\nu_{2n+2}), \\ \frac{1}{2}\left[d(\nu_1,\nu_{2n+1})+d(\nu_0,\nu_{2n+2})\right] \end{array}\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+C_{2n+1}+C_1+\|f\nu_{2n+1}\|\right]^{\beta} \\
&\le (1-\varepsilon)\max\left\{C_{2n+1},C_1,C_1,\tfrac{1}{2}[2(C_1+C_{2n+1})]\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+2C_{2n+1}+2C_1\right]^{\beta} \\
&= (1-\varepsilon)(C_1+C_{2n+1}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+2C_{2n+1}+2C_1\right]^{\beta}.
\end{aligned} \tag{10}$$

where

$$d(\nu_1,\nu_{2n+1}) \le d(\nu_1,\nu_0) + d(\nu_0,\nu_{2n+1}) = C_1 + C_{2n+1}$$

and

$$d(\nu_0, \nu_{2n+2}) \le d(\nu_0, \nu_{2n+1}) + d(\nu_{2n+1}, \nu_{2n+2}) \le C_{2n+1} + C_1$$

Attendantly, from (9) and (10), we conclude that

$$\varepsilon C_{2n+1} \le K(\varepsilon)^{\alpha}\psi(\varepsilon)C_{2n+1}^{\alpha} + L,$$

for some $K, L > 0$. If there is a subsequence $C_{2n_k+1} \to \infty$, the choice $\varepsilon = \varepsilon_1 = (1+L)/C_{2n_k+1}$ leads to the contradiction

$$1 \le K(1+L)^{\alpha}\psi(\varepsilon_1) \to 0.$$

As in the previous estimation (7) on C_{2n+1}, we derive the following estimation:

$$C_{2n+2} \le d(\nu_{2n+3}, \nu_1) + d(\nu_2, \nu_1) + 2C_1 \le d(\nu_{2n+3}, \nu_2) + 3C_1 \tag{11}$$

If $d(\nu_{2n+1}, \nu_0) < \frac{1}{2}d(\nu_{2n+1}, f\nu_{2n+1})$ then due to above observation we conclude that $C_{2n+1} = d(\nu_{2n+1}, \nu_0) < \frac{C_1}{2}$ and it shows C_{2n+1} is bounded by $\frac{C_1}{2}$. Otherwise, we have $\frac{1}{2}d(\nu_{2n+1}, f\nu_{2n+1}) \le d(\nu_{2n+1}, \nu_0)$ and by (4) we have

$$\zeta(d(g\nu_{2n+2}, f\nu_0), C_{g,f}(\nu_{2n+1}, \nu_0)) \ge 0. \tag{12}$$

Thus, by combining (7) and (8) together with $\beta \le \alpha$ we get
Therefore,

$$\begin{aligned}
C_{2n+2} \quad &\le d(\nu_{2n+3}, \nu_2) + 3C_1 \\
&= d(g\nu_{2n+2}, f\nu_1) + 3C_1 \\
&\le C_{g,f}(\nu_{2n+2}, \nu_1) + 3C_1 \\
&\le (1-\varepsilon)(C_{2n+2} + 2C_1) + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + 2C_{2n+2} + 4C_1\right]^{\beta} + 3C_1 \\
&\le (1-\varepsilon)C_{2n+2} + K'(\varepsilon)^{\alpha}\psi(\varepsilon)C_{2n+2}^{\alpha} + L'
\end{aligned}$$

for some $K', L' > 0$. Accordingly,

$$\varepsilon C_{2n+2} \le K'(\varepsilon)^{\alpha}\psi(\varepsilon)C_{2n+2}^{\alpha} + L'.$$

If there is a subsequence $C_{2n_k+2} \to \infty$, the choice $\varepsilon = \varepsilon_2 = (1+L)/C_{2n_k+2}$ leads to the contradiction

$$1 \le K'(1+L')^{\alpha}\psi(\varepsilon_2) \to 0.$$

Set

$$C = \sup_{n\in\mathbb{N}} \Lambda(1+2C_n)^{\beta} < \infty.$$

In the next step, we shall indicate that the sequence $\{\nu_n\}$ is Cauchy. Since $\{d(\nu_{2n}, \nu_{2n+1})\}$ is bounded by zero and non-increasing, we note that $d(\nu_{2n}, \nu_{2n+1}) \to r \ge 0$. If $r > 0$, then

$$\begin{aligned}
d(\nu_{2n}, \nu_{2n+1}) \quad &= d(f\nu_{2n-1}, g\nu_{2n}) \\
&\le (1-\varepsilon)C_{g,f}(\nu_{2n}, \nu_{2n+1}) + C\varepsilon^{\alpha}\psi(\varepsilon) \\
&\le (1-\varepsilon)d(\nu_{2n}, \nu_{2n+1}) + C\varepsilon^{\alpha}\psi(\varepsilon)
\end{aligned}$$

for all $n \in \mathbb{N}$, and $\varepsilon(0,1]$. As $n \to \infty$, we have

$$r \le (1-\varepsilon)r + C(\varepsilon)^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0,1]$. So

$$r < C\varepsilon^{(\alpha-1)}\psi(\varepsilon)$$

for all $\varepsilon \in (0,1]$. As $\varepsilon \to 0$ we get $r = 0$ and this is a contradiction, therefore $r = 0$.

Hence

$$\lim_{n\to\infty} d(\nu_{2n}, \nu_{2n+1}) = 0. \tag{13}$$

To show that $\{\nu_n\}$ is Cauchy sequence, it is sufficient to show that the subsequence $\{\nu_{2n}\}$ of $\{\nu_n\}$ is a Cauchy sequence in view of (13). If $\{\nu_n\}$ is not Cauchy, there exist an $\delta > 0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$,

$$d(\nu_{2m_k}, \nu_{2n_k}) \geq \delta \text{ and } d(\nu_{m_k}, \nu_{2n_k-2}) < \delta. \tag{14}$$

From (14), we get

$$\begin{aligned}
\delta &\leq d(\nu_{2m_k}, \nu_{2n_k}) \\
&\leq d(\nu_{2m_k}, \nu_{2n_k-2}) + d(\nu_{2n_k-2}, \nu_{2n_k-1}) + d(\nu_{2n_k-1}, \nu_{2n_k}) \\
&\leq \delta + d(\nu_{2n_k-2}, \nu_{2n_k-1}) + d(\nu_{2n_k-1}, \nu_{2n_k}).
\end{aligned}$$

As $k \to \infty$ together with (13), we have

$$\lim_{k\to\infty} d(\nu_{2m_k}, \nu_{2n_k}) = \delta. \tag{15}$$

Letting $k \to \infty$ and using (13)–(15), we get

$$|d(\nu_{2n_k+1}, \nu_{2m_k}) - d(\nu_{2n_k}, \nu_{2m_k})| \leq d(\nu_{2n_k+1}, \nu_{2n_k}).$$

Accordingly, we have

$$\lim_{k\to\infty} d(\nu_{2n_k+1}, \nu_{2m_k}) = \delta. \tag{16}$$

Taking $k \to \infty$ in the combinations of the expressions (13) and (16), we find

$$|d(\nu_{2n_k}, \nu_{2m_k-1}) - d(\nu_{2n_k}, \nu_{2m_k})| \leq d(\nu_{2m_k-1}, \nu_{2m_k}),$$

which implies that

$$\lim_{k\to\infty} d(\nu_{2n_k}, \nu_{2m_k-1}) = \delta. \tag{17}$$

Notice that $\frac{1}{2}d(\nu_{2n_k}, g\nu_{2n_k}) \leq d(\nu_{2n_k}, \nu_{2m_k-1})$. (Indeed, if not, we have $d(\nu_{2n_k}, \nu_{2m_k-1}) < \frac{1}{2}d(\nu_{2n_k}, g\nu_{2n_k})$ and by letting $k \to \infty$, we find $\delta \leq 0$, a contradiction.) Thus, by setting $x = \nu_{2n_k}$ and $y = \nu_{2m_k-1}$, in (4) we have

$$\zeta(d(g\nu_{2n_k}, f\nu_{2m_k-1}), C_{g,f}(\nu_{2n_k}, \nu_{2m_k-1})) \geq 0,$$

which is equivalent to

$$\begin{aligned}
d(g\nu_{2n_k}, f\nu_{2m_k-1}) &\leq C_{g,f}(\nu_{2n_k}, \nu_{2m_k-1}) \\
&= (1-\varepsilon)\max\left\{\begin{array}{c} d(\nu_{2n_k}, \nu_{2m_k-1}), d(\nu_{2n_k}, \nu_{2n_k+1}), d(\nu_{2m_k-1}, \nu_{2m_k}), \\ \frac{1}{2}[d(\nu_{2m_k}, \nu_{2n_k}) + d(\nu_{2m_k-1}, \nu_{2n_k+1})] \end{array}\right\} + C\varepsilon^{\alpha}\psi(\varepsilon) \\
&\leq (1-\varepsilon)\max\left\{\begin{array}{c} d(\nu_{2n_k}, \nu_{2m_k-1}), d(\nu_{2n_k}, \nu_{2n_k+1}), d(\nu_{2m_k-1}, \nu_{2m_k}), \\ \frac{1}{2}[d(\nu_{2m_k}, \nu_{2n_k}) + d(\nu_{2m_k-1}, \nu_{2m_k}) + d(\nu_{2m_k}, \nu_{2n_k+1})] \end{array}\right\} + C\varepsilon^{\alpha}\psi(\varepsilon)
\end{aligned}$$

for all $\varepsilon \in (0,1]$. Letting $k \to \infty$ and using (13)–(17) we get

$$\delta \leq (1-\varepsilon)\delta + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0,1]$. Thus

$$\delta \leq C\varepsilon^{\alpha-1}\psi(\varepsilon).$$

If $\varepsilon \to 0$ then we have $\delta = 0$ and it is a contradiction, therefore $\{\nu_{2n}\}$ is a Cauchy sequence.

Since X is complete, there exists $\nu_* \in X$ such that $\nu_n \to \nu_*$ as $n \to \infty$. So, we have $\nu_{2n} \to \nu_*$ and $\nu_{2n+1} \to \nu_*$. Due to continuity of g and f we have $f\nu_* = \nu_* = g\nu_*$.

As a last step, we shall show that ν_* is the unique common fixed point of g and f . Suppose that there exists $\omega_* \in X$ that $\omega_* = g\omega_* = f\omega_*$ and $\nu_* \neq \omega_*$. It is clear that $0 = \frac{1}{2}d(\nu_*, g\nu_*) \leq d(\nu_*, \omega_*)$ and by (4) we have

$$\zeta(d(g\nu_*, f\omega_*), C_{g,f}(\nu_*, \omega_*)) \geq 0, \tag{18}$$

which is equivalent to

$$d(\nu_*, \omega_*) = d(g\nu_*, f\omega_*) \leq (1-\varepsilon)d(\nu_*, f\omega_*) + k\varepsilon\psi(\varepsilon) = (1-\varepsilon)d(\nu_*, \omega_*) + k\varepsilon^{\alpha}\psi(\varepsilon).$$

Setting $\varepsilon = 0$, $d(\nu_*, \omega_*) = 0$, a contradiction. Hence, $\nu_* = \omega_*$. □

In Theorem 3, to provide C-condition, we need to suppose that both g and f are continuous. We realize that in case of removing C-condition, we relax the continuity conditions on g and f. In the following, we introduce Pata $\mathcal{Z}$-contraction which is more relaxed than Pata–Suzuki $\mathcal{Z}$-contraction

Definition 4. *A pair (g, f), defined on a (X, d) , is said to be a Pata $\mathcal{Z}$-contraction if for every $\varepsilon \in [0,1]$ and all $\nu, \omega \in X$, fulfills*

$$\zeta(d(g\nu, f\omega), C_{g,f}(\nu, \omega)) \geq 0, \tag{19}$$

where $\zeta \in \mathcal{Z}$, $\alpha \geq 1$, $\Lambda \geq 0$, and $\beta \in [0, \alpha]$ are constants, and,

$$\begin{aligned} C_{g,f}(\nu, \omega) &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(f\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega) + d(f\omega,\nu)\right]\right\} \\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|\nu\| + \|\omega\| + \|g\nu\| + \|f\omega\|\right]^{\beta}. \end{aligned}$$

This is the second main results of this paper.

Theorem 4. *If a pair (g, f), on a (X^*, d), forms a Pata $\mathcal{Z}$-contraction, then g, f have a common fixed point $\nu_* \in X$.*

Notice that in Pata–Suzuki $\mathcal{Z}$-contraction we need to satisfy the C-condition ($\frac{1}{2}d(\nu, g\nu) \leq d(\nu, \omega)$), but in Pata $\mathcal{Z}$-contraction, we do not need to check it. Therefore, we can repeat the proof of Theorem 3 by ignoring the C-condition.

Proof. We follow the lines in the proof of Theorem 3 step by step and we deduce that the constructive sequence $\{\nu_n\}$ is Cauchy sequence. Since X is complete, there exists $\nu_* \in X$ such that $\nu_n \to \nu_*$ as $n \to \infty$. So, we have $\nu_{2n} \to \nu_*$ and $\nu_{2n+1} \to \nu_*$. Due to assumption (19), for all $\varepsilon \in (0,1]$, we have

$$d(g\nu_*, f\nu_{2n+1}) \leq (1-\varepsilon)C_{g,f}(\nu_*, \nu_{2n+1}) + C\varepsilon^{\alpha}\psi(\varepsilon),$$

where

$$C_{g,f}(\nu_*, \nu_{2n+1}) = \max\{d(\nu_*, \nu_{2n+1}), d(\nu_*, g\nu_*), d(\nu_{2n+1}, \nu_{2n+2}), \frac{d(\nu_*, \nu_{2n+2}) + d(\nu_{2n+2}, g\nu_*)}{2}\}.$$

As $n \to \infty$ we have

$$d(\nu_*, g\nu_*) \leq (1-\varepsilon)d(\nu_*, g\nu_*) + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0,1]$. So

$$d(\nu_*, g\nu_*) \leq C\varepsilon^{\alpha-1}\psi(\varepsilon)$$

for all $\varepsilon \in (0,1]$. If $\varepsilon \to 0$ then we get $d(\nu_*, g\nu_*) \to 0$. Hence $g\nu_* = \nu_*$.

Claim that ν_* forms a fixed point of f too. Again by (19), we find that

$$\begin{aligned} 0 &< d(\nu_*, f\nu_*) \\ &= d(g\nu_*, f\nu_*) \\ &\leq (1-\varepsilon)\max\{d(\nu_*, \nu_*), d(\nu_*, g\nu_*), d(\nu_*, f\nu_*), \tfrac{d(\nu_*, f\nu_*)+d(\nu_*, g\nu_*)}{2}\} + k\varepsilon\psi(\varepsilon) \end{aligned}$$

where $k > 0$. So,

$$d(\nu_*, f\nu_*) \leq (1-\varepsilon)d(\nu_*, f\nu_*) + k\varepsilon^{\alpha}\psi(\varepsilon).$$

This implies that $d(\nu_*, f\nu_*) \leq k\psi(\varepsilon)$, where $\varepsilon \in (0,1]$. Since ψ is increasing and continuous at zero, then $\psi(0) = 0$ and $d(\nu_*, f\nu_*) = 0$.

Therefore $\nu_* = f\nu_*$.

The uniqueness of the common fixed point of g and f is derived from the proof Theorem 3. □

Theorem 5. *Let g, f be continuous mappings on (X^*, d). Assume that $\phi : [0,\infty) \to [0,\infty)$ is a continuous function satisfying the inequality $\phi(r) < r$ for every $r > 0$. If*

$$d(g\nu, f\omega) \leq \phi(C_{g,f}(\nu,\omega)), \tag{20}$$

for every ν, ω where

$$C_{g,f}(\nu,\omega) = \max\left\{d(\nu,\omega), d(\nu, g\nu), d(\omega, f\omega), \frac{1}{2}\left[d(\nu, f\omega) + d(y, g\nu)\right]\right\},$$

then, g and f have a unique common fixed point ν_* and $d(\nu_*, \nu_n) \to 0$, where $\{\nu_n\}$ is the sequence is defined in Theorem 3.

Proof. Note that $\zeta(t,s) := \phi(s) - t$ is a simulation function, see e.g., [2,6]. Hence, the result follows from Theorem 3 by letting $\zeta(t,s) := \phi(s) - t$. □

Corollary 1. *Suppose that a mapping g, defined on (X^*, d), satisfies*

$$\frac{1}{2}d(\nu, g\nu) \leq d(\nu,\omega) \text{ implies } \zeta(d(g\nu, g\omega), C_g(\nu,\omega)) \geq 0, \tag{21}$$

for every $\varepsilon \in [0,1]$ and all $\nu, \omega \in X$, where $\zeta \in \mathcal{Z}$, $\alpha \geq 1$, $\Lambda \geq 0$, and $\beta \in [0,\alpha]$ are constants, and

$$\begin{aligned} C_g(\nu,\omega) &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(g\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega) + d(g\omega,\nu)\right]\right\} \\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|\nu\| + \|\omega\| + \|g\nu\| + \|g\omega\|\right]^{\beta}. \end{aligned}$$

If g is continuous, then g possesses a unique fixed point $z \in X$.

Proof. It is sufficient to take $g = f$ in Theorem 3. □

In the following Corollary, we relax the continuity restriction

Corollary 2. *Suppose that a mapping g, defined on (X^*, d), satisfies*

$$\zeta(d(g\nu, g\omega), C_g(\nu,\omega)) \geq 0, \tag{22}$$

for every $\varepsilon \in [0,1]$ and all $\nu, \omega \in X$, where $\zeta \in \mathcal{Z}$, $\alpha \geq 1$, $\Lambda \geq 0$, and $\beta \in [0,\alpha]$ are constants, and

$$\begin{aligned} C_g(\nu,\omega) &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(g\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega)+d(g\omega,\nu)\right]\right\} \\ &+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu\|+\|\omega\|+\|g\nu\|+\|g\omega\|\right]^{\beta}. \end{aligned}$$

Then g possesses a unique fixed point $z \in X$.

Example 1. *Let $X = [0,\infty)$ is a metric space defined as*

$$d(\nu,\omega) = \begin{cases} \max\{\nu,\omega\}, & if \nu \neq \omega; \\ 0 & if \nu = \omega. \end{cases}$$

Let $g, f : X \to X$ be mappings defined by

$$g\nu = \frac{\nu}{4} \text{ and } f\nu = \frac{\nu}{9}.$$

Let $\zeta(t,s) = s - t$, for all $s, t \in [0,\infty)$. Let $\Lambda = \frac{1}{2}$, $\alpha = 1$ and $\beta = 1$ and $\psi(\varepsilon) = \varepsilon^{\frac{1}{2}}$ for every $\varepsilon \in [0,1]$. Now

$$\begin{aligned} \tfrac{1}{2}d(g\nu,\nu) &= \tfrac{1}{2}\max\{\tfrac{\nu}{4},\nu\} \\ &\leq \tfrac{1}{2}\max\{\nu,\omega\} \\ &\leq \max\{\nu,\omega\} \\ &= d(\nu,\omega) \end{aligned}$$

implies

$$\begin{aligned} &\zeta(d(g\nu,f\omega), C_{g,f}(\nu,\omega)) \\ &\quad= C_{g,f}(\nu,\omega) - d(g\nu,f\omega) \\ &\quad= C_{g,f}(\nu,\omega) - \max\{g\nu,f\omega\} \\ &\quad= C_{g,f}(\nu,\omega) - \max\{\tfrac{\nu}{4},\tfrac{\omega}{9}\} \\ &\quad\leq C_{g,f}(\nu,\omega) - \max\{\tfrac{\nu}{2},\tfrac{\nu}{2}\} \\ &\quad= C_{g,f}(\nu,\omega) - \tfrac{1}{2}\max\{\nu,\omega\} \\ &\quad\leq C_{g,f}(\nu,\omega) - \tfrac{1}{2}C_{g,f}(\nu,\omega) \\ &\quad= \tfrac{1}{2}C_{g,f}(\nu,\omega) > 0 \end{aligned}$$

where

$$\begin{aligned} C_{g,f}(\nu,\omega) &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(f\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega)+d(f\omega,\nu)\right]\right\} \\ &+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu\|+\|\omega\|+\|g\nu\|+\|f\omega\|\right]^{\beta}. \end{aligned}$$

Hence, g and f is a Pata - Suzuki $\mathcal{Z}$-contraction. Thus, g and f have a unique common fixed point in X.

Example 2. *Let $X = [0,\infty)$ is a metric space defined as*

$$d(\nu,\omega) = \begin{cases} \max\{\nu,\omega\}, & if \nu \neq \omega; \\ 0 & if \nu = \omega. \end{cases}$$

Let $g, f : X \to X$ be mappings defined by $g\nu = \frac{\nu}{6}$ and $f\nu = \frac{\nu}{12}$. Let $\zeta(t,s) = s - t$, for all $s, t \in [0,\infty)$. Let $\Lambda = \frac{1}{2}$, $\alpha = 1$ and $\beta = 1$ and $\psi(\varepsilon) = \varepsilon^{\frac{1}{2}}$ for every $\varepsilon \in [0,1]$.

Now

$$
\begin{aligned}
&\zeta(d(g\nu, f\omega), C_{g,f}(\nu,\omega)) \\
&\quad = C_{g,f}(\nu,\omega) - d(g\nu, f\omega) \\
&\quad = C_{g,f}(\nu,\omega) - \max\{Fx, Ty\} \\
&\quad = C_{g,f}(\nu,\omega) - \max\{\tfrac{\nu}{6}, \tfrac{\omega}{12}\} \\
&\quad \leq C_{g,f}(\nu,\omega) - \tfrac{1}{2}\max\{\nu,\omega\} \\
&\quad \leq C_{g,f}(\nu,\omega) - \tfrac{1}{2}C_{g,f}(\nu,\omega) \\
&\quad = \tfrac{1}{2}C_{g,f}(\nu,\omega) > 0
\end{aligned}
$$

where

$$
\begin{aligned}
C_{g,f}(\nu,\omega) \quad &= (1-\varepsilon)\max\left\{d(\nu,\omega), d(g\nu,\nu), d(f\omega,\omega), \tfrac{1}{2}\left[d(g\nu,\omega)+d(f\omega,\nu)\right]\right\} \\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|\nu\|+\|\omega\|+\|g\nu\|+\|f\omega\|\right]^{\beta}.
\end{aligned}
$$

Hence, g and f is a Pata-$\mathcal{Z}$-contraction. Thus, g and f have a unique common fixed point in X.

3. Application to Ordinary Differential Equations

We consider the following initial boundary value problem of second order differential equation:

$$
-\frac{d^2x}{dt^2} = f(t,\nu(t)),\ t\in[0,1],\ \ \nu(0)=\nu(1)=0, \tag{23}
$$

where $f:[0,1]\times R\to R$ is a continuous function.

Recall that the Green function associated to (23) is given by

$$
H(t,s)=\begin{cases} t(1-s), & 0\leq t\leq s\leq 1,\\ s(1-t), & 0\leq s\leq t\leq 1.\end{cases}
$$

Let $X=(C[0,1])$ be the space of all continuous functions defined on interval $[0,1]$ with the metric

$$
d(\nu,\omega) = \underset{t\in[0,1]}{sup}\ |\nu(t)-\omega(t)|.
$$

is a complete metric space. We consider the following conditions: there exists $\varepsilon\in[0,1]$ such that

$$
\frac{1}{2}|\nu(s)-\int_0^1 H(t,s)f(s,\nu(s))ds| \leq |\nu(s)-\omega(s)| \tag{24}
$$

implies

$$
|f(s,\nu(s))-f(s,\omega(s))| \leq (1-\varepsilon)|\nu(s)-\omega(s)|,\ \text{for all } \nu,\omega\in X, \tag{25}
$$

where $\sup\limits_{t\in[0,1]}\displaystyle\int_0^1 H(t,s)ds=\frac{1}{8}$.

Theorem 6. *Suppose that the conditions (24) and (25) are satisfied. Then (24) has solution $x^*\in C^2[0,1]$.*

Proof. It is known that $\nu\in C^2([0,1])$ is a solution of (23) if and only if $\nu\in C([0,1])$ is a solution of integral equation

$$
\nu(t) = \int\limits_0^1 H(t,s)f(s,\nu(s))ds, \quad t\in[0,1].
$$

We define $F: C[0,1]\to C[0,1]$ by

$$g\nu(t) = \int_0^1 H(t,s)f(s,\nu(s))ds \text{ for all } t \in [0,1].$$

Then, problem (23) is equivalent to finding $x^* \in C^2[0,1]$ that is fixed point of g. It follows that

$$\frac{1}{2}|\nu(s) - \int_0^1 H(t,s)f(s,\nu(s))ds| \leq (1-\varepsilon)|\nu(s) - \omega(s)|$$

implies

$$\begin{aligned}
d(g\nu, F\omega) &= \sup_{t\in[0,1]} |g\nu(t) - F\omega(t)| \\
&= \left|\int_0^1 H(t,s)[f(s,\nu(s)) - f(s,\omega(s))]ds\right| \\
&\leq \int_0^1 H(t,s)\, |f(s,\nu(s)) - f(s,\omega(s))]|\, ds \\
&\leq \int_0^1 H(t,s)(1-\varepsilon)\, |\nu(s) - \omega(s)]|\, ds \\
&\leq (1-\varepsilon) \sup_{t\in[0,1]} \int_0^1 H(t,s)ds|\nu(s) - \omega(s)| \\
&\leq \tfrac{1}{8}(1-\varepsilon)|\nu(s) - \omega(s)| \\
&\leq (1-\varepsilon)\max\left\{d(\nu,\omega), d(\nu,g\nu), d(y,g\omega), \tfrac{1}{2}\left[d(\nu,g\omega) + d(y,g\nu)\right]\right\} \\
&\quad +\Lambda\varepsilon^\alpha\psi(\varepsilon)\left[1 + \|\nu\| + \|\omega\| + \|g\nu\| + \|g\omega\|\right]^\beta \quad \lambda \geq 0 \quad \alpha \geq 1 \text{ and} \beta \in [0,\alpha] \\
&= C_{g,f}(\nu,\omega)
\end{aligned}$$

Note that for all $t \in [0,1]$, $\int_0^1 H(t,s)ds = -\frac{t^2}{2} - \frac{t}{2}$, which implies that $\sup_{t\in[0,1]} \int_0^1 H(t,s)ds = \frac{1}{8}$.
Let $\zeta(t,s) = s - t$ for all $s,t \in [0,\infty)$
Now

$$\zeta(d(g\nu,g\omega), C_{g,f}(\nu,\omega)) = C_{g,f}(\nu,\omega) - d(g\nu,g\omega) \tag{26}$$

Then from (26), we have $\zeta(d(g\nu,g\omega), C_{g,f}(\nu,\omega)) \geq 0$. Therefore the mapping g is Pata—Suzuki—Z contraction.

Applying Corollary 1, we obtain that g has a unique fixed point in $C[0,1]$, which is a solution of integral equation. □

4. Conclusions

In this paper, we combine and extend Pata type contractions and Suzuki type contraction via simulation function. The success of V. Pata [9] is to define an auxiliary distance function $\|u\| = d(u,a)$ where a is an arbitrary but fixed point. This is based on the fact that most of the proofs in metric fixed point theory are established on the Picard sequence:

For a self-mapping f on a metric space X and arbitrary point "a" (renamed as "a_0"). Then, $a_1 = Ta_0$,

$$a_n = fa_{n-1} \text{ for all positive integers.}$$

In Banach's proof (and also, in many other metric fixed point theorems) for any point "a", this sequence converges to the fixed point of T. Under this setting, V.Pata, suggest such auxiliary distance function (initiated from an arbitrary point "a") to refine Banach's fixed point theorem, like the construction of Picard operator.

In this short note, we employ the approach of Pata in a more general case to generalize and unify several existing results in the literature. For this purpose, we have use simulation functions. We also emphasize that the simulation functions are very wide, see, e.g., [2–6]. Thus, several consequences of our results can be listed by using the examples that have been introduced in [2–6]. Similarly, we can generalize more inequalities on metric and normed spaces.

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