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Article

Integral Representation for the Solutions of Autonomous Linear Neutral Fractional Systems with Distributed Delay

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Abstract: The aim of this work is to obtain an integral representation formula for the solutions of initial value problems for autonomous linear fractional neutral systems with Caputo type derivatives and distributed delays. The results obtained improve and extend the corresponding results in the particular case of fractional systems with constant delays and will be a useful tool for studying different kinds of stability properties. The proposed results coincide with the corresponding ones for first order neutral linear differential systems with integer order derivatives.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; integral representation

MSC: 34A08; 34A12

1. Introduction and Notations

Fractional Calculus has a long history, but it has attracted considerable attention recently as an important tool for modeling of various real problems, such as viscoelastic systems, diffusion processes, signal and control processing, and seismic processes. Detailed information about the fractional calculus theory and its applications can be found in the monographs [1–4]. Some results for fractional linear systems with delays are in given in the book [5]. The monograph [6] is devoted to the impulsive differential and functional differential equations with fractional derivatives, as well as to some of their applications.

It is well known that the study of linear fractional equations (integral representation, several types of stability, etc.) is an evergreen theme for research. Concerning these fields of fundamental and qualitative investigations for linear fractional ordinary differential equations and systems we refer to [2,4,7] and the references therein. Using the Laplace transform method, several interesting results in this direction are obtained in [8,9] as well. Regarding works concerning fractional differential systems with constant delays, we point out [10–13]. Concerning the retarded differential systems with variable or distributed delays—fundamental theory and application (stability properties)—we refer to [11,14–18]. Neutral fractional systems with distributed delays are essentially studied less (see [19–21]). Stability properties of retarded fractional systems with derivatives of distributed order are studied in [22]. One of the existing best applications of fractional order equations with delays is modeling human manual control, in which perceptual and neuromuscular delays introduce a delay term. As interesting studies, we refer to [23,24].

The problem of establishing an integral representation for the solutions for neutral or delayed linear fractional differential equations and/or systems needs a theorem for the existence of a fundamental matrix, i.e., theorem for existence and uniqueness of the solution to the initial value problem (IVP) in the case of discontinuous initial functions. As far as we know, there are only a few results concerning the IVP for delayed and neutral systems with discontinuous initial function, for the delayed case [14,15,25–27] and for the neutral case [28].

The aim of the work is to prove an integral representation formula for the general solution of an autonomous linear fractional neutral system with Caputo type derivatives and distributed delays. Note that our results extend and improve the results obtained in [10,12,15]. The proposed results coincide with the corresponding ones for a first order neutral linear differential system with integer order derivatives.

The paper is organized as follows. In Section 2, we recall some necessary definitions of Riemann-Liouville and Caputo fractional derivatives, as well as part of their properties. In this section, we also present the linear neutral fractional system under consideration together with some conditions. In Section 3, as a main result, integral representations of the solutions of the IVP for autonomous linear fractional neutral system with Caputo type derivatives and distributed delays are obtained for the homogeneous and inhomogeneous case. In Section 4, we present an illustrative example. In Section 5 we explain the practical benefits and application options of the obtained theoretical results.

In what follows, we use the notations: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ – the sets of natural, real and complex numbers, respectively; $\langle m, n\rangle$ – the set of integers $m, m+1, \ldots, n$ $(m \leq n)$; $\mathbb{R}^{n\times n}$ – the space of real $n \times n$ matrices A with elements A_{pq}; $\mathbb{R}^n = \mathbb{R}^{n\times 1}$; $A^\top$ – the transposed matrix A with elements $(A^\top)_{pq} = A_{qp}$. The elements of $\mathbb{R}^n$ are the real column n-vectors $x = [x_1; x_2; \ldots; x_n]$ with elements x_k. The row n-vectors are denoted as $\xi = [\xi_1, \xi_2, \ldots, \xi_n]$(note that the elements of a vector column and a vector row are separated by ";" and ",", respectively). The identity and the zero matrices are denoted by E and Θ, respectively.

We also denote $\mathbb{C}_+ = \{p \in \mathbb{C} | \mathrm{Re}(p) > 0\}$, $\overline{\mathbb{C}}_+ = \{p \in \mathbb{C} | \mathrm{Re}\, p \geq 0\}$, $\mathbb{C}_- = \mathbb{C} \setminus \overline{\mathbb{C}}_+$, $\mathbb{R}_+ = (0, \infty)$, and $J_s = [s, \infty)$. For $p \in \mathbb{C}$, $y = [y_1; y_2; \ldots; y_n] \in \mathbb{C}^n$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\beta_k \in [-1, 1]$ we set $I_\beta(p) = \mathrm{diag}\left(p^{\beta_1}, p^{\beta_2}, \ldots, p^{\beta_n}\right)$ and $I_\beta(y) = \mathrm{diag}\left(y_1^{\beta_1}, y_2^{\beta_2}, \ldots, y_n^{\beta_n}\right)$. The linear space of locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$ is denoted by $L_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$.

2. Preliminaries and Problem Statement

Below, the definitions of Riemann–Liouville and Caputo fractional derivatives and some of their properties necessary for our exposition are described in order to avoid possible misunderstandings. For more details and other properties, we refer to [2–4].

Let $\alpha \in (0, 1)$ be an arbitrary number. Then for $a \in \mathbb{R}$, each $t > a$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ the left-sided fractional integral operator, the left side Riemann-Liouville and Caputo fractional derivatives of order α are defined by

$$\begin{aligned} (D_{a+}^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \ {}_{RL}D_{a+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(D_{a+}^{-(1-\alpha)} f(t)\right), \\ {}_C D_{a+}^{\alpha} f(t) &= {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t) = {}_{RL}D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \end{aligned}$$

respectively. The following relations [4] involving fractional derivatives will be used

$$(D_{a+}^{0} f)(t) = f(t), \ {}_C D_{a+}^{\alpha} D_{a+}^{-\alpha} f(t) = f(t), \ D_{a+}^{-\alpha}\, {}_C D_{a+}^{\alpha} f(t) = f(t) - f(a).$$

Concerning the Laplace transform $\mathfrak{L}$,

$$\mathfrak{L}f(p) = \int_0^\infty \exp(-pt)f(t)\mathrm{d}t,\ p \in \mathbb{C},$$

we shall need the relations

$$\mathfrak{L}D_{0+}^{-\alpha}f(p) = p^{-\alpha}(\mathfrak{L}f)(p),\ \mathfrak{L}_{RL}D_{0+}^{\alpha}f(p) = p^{\alpha}(\mathfrak{L}f)(p) - [_{RL}D_{0+}^{\alpha-1}f(t)]_{t=0},$$
$$\mathfrak{L}_C D_{0+}^{\alpha}f(p) = p^{\alpha}(\mathfrak{L}f)(p) - p^{\alpha-1}f(0).$$

In what follows, we consider the autonomous linear neutral fractional system with distributed delay

$$D^{\alpha}\left(X(t) - \sum_{l=1}^{r}\int_{-\tau}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right]X(t+\theta)\right) = \sum_{i=0}^{m}\int_{-\sigma}^{0}\left[\mathrm{d}_\theta U^i(\theta)\right]X(t+\theta) + F(t), \tag{1}$$

as well as the corresponding homogeneous system

$$D^{\alpha}\left(X(t) - \sum_{l=1}^{r}\int_{-\tau}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right]X(t+\theta)\right) = \sum_{i=0}^{m}\int_{-\sigma}^{0}\left[\mathrm{d}_\theta U^i(\theta)\right]X(t+\theta), \tag{2}$$

where

$$X, F : J_0 \to \mathbb{R}^n,\ U^i, V^l : \mathbb{R} \to \mathbb{R}^{n\times n},\ U^i(\theta) = \left[u_{kj}^i(\theta)\right],\ V^l(\theta) = \left[v_{kj}^l(\theta)\right],$$
$$\tau, \sigma > 0,\ \tau_r \in (0,\tau],\ l \in \langle 1, r\rangle,\ \sigma_i \in (0,\sigma],\ i \in \langle 1, m\rangle,\ h = \max(\sigma,\tau),\ \sigma_0 = 0,$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ \alpha_k \in (0,1),\ k \in \langle 1, n\rangle,\ J_s = [s,\infty).$$

For simplicity, D^{α_k} denotes the left side Caputo fractional derivative $_C D_{0+}^{\alpha_k}$ in (1) and (2), and we use the notations

$$D^{\alpha}X(t) = [D^{\alpha_1}x_1(t); D^{\alpha_2}x_2(t); \ldots; D^{\alpha_n}x_n(t)],\ D^{\alpha} = \mathrm{diag}(D^{\alpha_1}, D^{\alpha_2}, \ldots, D^{\alpha_n}),$$
$$X(t) = [x_1(t); x_2(t); \ldots; x_n(t)],\ F(t) = [f_1(t); f_2(t); \ldots; f_n(t)].$$

Denote by $BV[-h, 0]$ the linear space of matrix valued functions

$$W : \mathbb{R} \to \mathbb{R}^{n\times n},\ W(\theta) = \left[\omega_{kj}(\theta)\right]$$

with bounded variation in θ on $[-h, 0]$,

$$\mathrm{Var}_{[-h,0]}W(.) = \sum_{k,j=1}^{n}\mathrm{Var}_{[-h,0]}w_{kj}(.),\ |W(\theta)| = \sum_{k,j=1}^{n}|w_{kj}(\theta)|.$$

As a space of initial functions, we use the Banach space $\widetilde{C} = PC([-h,0],\mathbb{R}^n)$ of the piecewise continuous on $[-h, 0]$ vector functions $\Phi = [\phi_1; \phi_2; \ldots; \phi_n] : [-h, 0] \to \mathbb{R}^n$ with norm

$$\|\Phi\| = \sum_{k=1}^{n}\sup_{s\in[-h,0]}|\phi_k(s)| < \infty.$$

The initial condition for the system (1) or (2) is

$$X(t) = \Phi(t),\ t \in [-h, 0]. \tag{3}$$

Definition 1. *The vector function X is a solution of the IVP* (1), (3) *in the interval J_{-h} if $X|_{J_0} \in C(J_0, \mathbb{R}^n)$ and if it satisfies the system* (1) *for $t \in \mathbb{R}_+$ and the initial condition* (3) *for $t \in [-h, 0]$.*

We say that for the kernels $U^i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $V^l : \mathbb{R} \to \mathbb{R}^{n\times n}$ the assumptions (SA) are fulfilled, if for each $i \in \langle 0, m\rangle$ and $l \in \langle 1, r\rangle$ the following conditions hold.

(SA1) The matrix valued functions $\theta \mapsto U^i(\theta)$ and $\theta \mapsto V^l(\theta)$ are measurable in $\theta \in \mathbb{R}$ and normalized so that $U^i(\theta) = 0$ and $V^l(\theta) = 0$ for $\theta \geq 0$, $U^i(\theta) = U^i(-\sigma_i)$ for $\theta \leq -\sigma_i$ and $V^l(\theta) = V^l(-\tau_l)$ for $\theta \leq -\tau_l$.

(SA2) The kernels $U^i(\theta)$ and $V^l(\theta)$ are left continuous for $\theta \in (-\sigma, 0)$ and $\theta \in (-\tau, 0]$ and $U^i(\cdot)$, $V^l(\cdot) \in BV[-h, 0]$.

(SA3) The Lebesgue decomposition of the kernels $U^i(\theta)$ and $V^l(\theta)$ for $\theta \in [-h, 0]$ is

$$\begin{aligned} U^i(\theta) &= \aleph^i(\theta) + \int_{-h}^{\theta} B^i(s)\mathrm{d}s + \Upsilon^i(\theta), \\ V^l(\theta) &= \widetilde{\aleph}^l(\theta) + \int_{-h}^{\theta} \widetilde{B}^l(s)\mathrm{d}s + \widetilde{\Upsilon}^l(\theta), \end{aligned}$$

where $A^i = \left[a^i_{kj}\right]$, $\widetilde{A}^l = \left[\widetilde{a}^l_{kj}\right] \in \mathbb{R}^{n\times n}$ and

$$\begin{aligned} &\aleph^i(\theta) = \left[a^i_{kj} H(\theta + \sigma_i)\right], \; \widetilde{\aleph}^l(\theta) = \left[\widetilde{a}^l_{kj} H(\theta + \tau_l)\right], \\ &\Upsilon^i(\theta) = \left[g^i_{kj}(\theta)\right], \; \widetilde{\Upsilon}^l(\theta) = \left[\widetilde{g}^l_{kj}(\theta)\right] \in C(\mathbb{R}, \mathbb{R}^{n\times n}), \\ &B^i(\theta) = \left[b^i_{kj}(\theta)\right], \; \widetilde{B}^l(\theta) = \left[\widetilde{b}^l_{kj}(\theta)\right] \in L_1^{\mathrm{loc}}(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}). \end{aligned}$$

Remark 1. *The conditions (SA) are used essentially in the work [21] to establish an apriory estimate of all solutions of the IP* (1), (3)*, which estimate guaranties that the Laplace transform can be correct applied to System* (2) *and to System* (1) *too, when the function F is exponentially bounded.*

Let $s \geq 0$ be an arbitrary number, $J_s = [s, \infty)$ and consider the matrix IVP

$$D^\alpha\left(Q(t,s) - \sum_{l=1}^{r}\int_{-\tau}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right] Q(t+\theta, s)\right) = \sum_{i=0}^{m}\int_{-\sigma}^{0}\left[\mathrm{d}_\theta U^i(t,\theta)\right] Q(t+\theta, s) \tag{4}$$

with initial condition

$$Q(t,t) = I;\; Q(t,s) = 0,\; t < s. \tag{5}$$

Definition 2. *For each $s \geq 0$ the matrix valued function*

$$t \mapsto Q(t,s) = \left[\gamma_{kj}(t,s)\right],\; Q(\cdot, s) : J_s \to \mathbb{R}^{n\times n},$$

is called a solution of the IVP (4), (5) *for $t \in J_s$, if $Q(\cdot, s)$ is continuous in t on J_s and satisfies the matrix Equation* (4) *for $t \in (s, \infty)$ and the initial condition* (5).

It is well known that the problem of existence of a fundamental matrix for a linear homogeneous fractional system (delayed or neutral) leads to establishing that the corresponding IVP (4), (5) with discontinuous initial function has a unique solution. In the case when $s = 0$, the matrix $Q(t) = Q(t, 0)$ will be called fundamental (or Cauchy) matrix of system (2).

Following [20,21], we introduce the characteristic matrix of System (2)

$$G(p) = I_\alpha(p) - W(p), \tag{6}$$

where

$$
\begin{aligned}
W(p) &= \sum_{i=0}^{m} U_i(p) + I_\alpha(p) \sum_{l=1}^{r} V_l(p),\ i \in \langle 0, m\rangle,\ l \in \langle 1, r\rangle, \\
U_i(p) &= \left[\int_{-h}^{0} \exp(p\theta)\mathrm{d}u_{kj}^{i}(\theta)\right],\ V_l(p) = \left[\int_{-h}^{0} \exp(p\theta)\mathrm{d}v_{kj}^{l}(\theta)\right].
\end{aligned}
$$

3. Main Results

The results in this section are a generalization of the results concerning the autonomous case obtained in [10,15,16,25].

Theorem 1. *Let us assume the conditions (SA) are satisfied. Then the IVP* (4), (5) *has a unique solution* $Q(t,s)$ *in* J_s *for every* $s \geq 0$ *and the fundamental matrix* $Q(t,0) = Q(t)$ *of Equation* (2) *is*

$$
Q(t) = \mathfrak{L}^{-1}\left(I_{\alpha-1}(p)G^{-1}(p)\right)(t). \tag{7}
$$

Proof. Using the results from [28], we obtain that the IVP (4), (5) has a unique solution $Q(t,s)$ in J_s for every $s \geq 0$, and hence, a fundamental matrix $Q(t,0) = Q(t)$. In virtue of Theorem 3 [21], we can conclude that the Laplace transform can be applied to both sides of Equation (4). Substituting $t + \theta = \eta$ we obtain

$$
\begin{aligned}
&\int_0^\infty \exp(-pt) \sum_{i=0}^{m} \int_{-h}^{0} \left[\mathrm{d}_\theta U^i(\theta)\right] Q(t+\theta)\mathrm{d}t = \sum_{i=0}^{m} \int_{-h}^{0} \left[\mathrm{d}_\theta U^i(\theta)\right] \left(\exp(p\theta)\int_\theta^0 \exp(-p\eta)Q(\eta)\mathrm{d}\eta\right) \\
&+ \int_0^\infty \exp(-p\eta)Q(\eta)\mathrm{d}\eta \sum_{i=0}^{m} \int_{-h}^{0} \exp(p\theta)\mathrm{d}_\theta U^i(\theta) = \mathfrak{L}Q(p) \sum_{i=0}^{m} \int_{-h}^{0} \exp(p\theta)\mathrm{d}_\theta U^i(\theta).
\end{aligned}
$$

In a similar way for the left-hand side of Equation (4), we have that

$$
\begin{aligned}
&\mathfrak{L}D^\alpha\left(Q(t) - \sum_{l=1}^{r} \int_{-\tau}^{0} \left[\mathrm{d}_\theta V^l(\theta)\right] Q(t+\theta)\right)(p) = I_\alpha(p)\mathfrak{L}Q(p)\left[E - \sum_{l=1}^{r} \int_{-h}^{0} \exp(p\theta)\mathrm{d}_\theta V^l(\theta)\right] \\
&- I_{\alpha-1}(p)\left[E - \sum_{l=1}^{r} \int_{-\tau}^{0} \left[\mathrm{d}_\theta V^l(\theta)\right] Q(\theta)\right].
\end{aligned} \tag{8}
$$

From Equation (8), it follows that

$$
\mathfrak{L}Q(p)\left[I_\alpha - I_\alpha(p)\sum_{l=1}^{r} \int_{-h}^{0} \exp(p\theta)\mathrm{d}_\theta V^l(\theta) - \sum_{l=1}^{m} \int_{-h}^{0} \exp(p\theta)\mathrm{d}_\theta U^i(\theta)\right] = I_{\alpha-1}(p)
$$

and hence, $\mathfrak{L}Q(p) = I_{\alpha-1}(p)G^{-1}(p)$, which completes the proof. □

Let us introduce the following functions:

$$
\Phi_l(t) = \Phi(t),\ t \in [-\tau_l, 0],\ \ \Phi_l(t) = 0,\ t \in \mathbb{R} \setminus [-\tau_l, 0],\ \ l \in \langle 1, r\rangle,
$$

and

$$
\Phi_i(t) = \Phi(t),\ t \in [-\sigma_i, 0],\ \ \Phi_i(t) = 0,\ t \in \mathbb{R} \setminus [-\sigma_i, 0],\ \ i \in \langle 1, m\rangle.
$$

Then, applying the Laplace transform second shifting theorem, we obtain

$$\begin{aligned}\int_{\theta}^{0}\exp(-p(\eta-\theta))\Phi_l(\eta)\mathrm{d}\eta &= \exp(p\theta)\mathfrak{L}\Phi_l(t)(p),\\ \int_{\theta}^{0}\exp(-p(\eta-\theta))\Phi_i(\eta)\mathrm{d}\eta &= \exp(p\theta)\mathfrak{L}\Phi_i(t)(p).\end{aligned} \tag{9}$$

Now we are in position to prove the following theorem.

Theorem 2. *Let us assume the conditions (SA) are satisfied. Then for each $\Phi \in \widetilde{C}$ the IVP (2), (3) has a unique solution $X_{\Phi}(t)$ with the integral representation:*

$$\begin{aligned}X_{\Phi}(t) &= Q(t)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi_l(\theta)\right) + \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi_l(t+\theta)\\ &+ \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]D^{\frac{1}{2}}Q(t) * D^{\frac{1}{2}}\Phi_l(t+\theta) + \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)]D^{1-\alpha}Q(t) * \Phi_i(t+\theta)\\ &+ \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)]D^{-\alpha}\Phi_i(t+\theta).\end{aligned} \tag{10}$$

Proof. Let $\Phi \in \widetilde{C}$. Then using the results from [26], we can conclude that the IVP (2), (3) has a unique solution $X_{\Phi}(t)$. In virtue of Theorem 3 from [21], we can conclude that the Laplace transform can be applied to both sides of Equation (2). Then, substituting $X_{\Phi}(t)$ in Equation (2), applying the Laplace transform to Equation (2) and substituting $t+\theta = \eta$, we obtain for the right-hand side of Equation (2)

$$\begin{aligned}&\mathfrak{L}\left(\sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)]X_{\Phi}(t+\theta)\right)(p) = \int_{0}^{\infty}\exp(-p\eta)X_{\Phi}(\eta)\mathrm{d}\eta\sum_{i=0}^{m}\int_{-h}^{0}\exp(p\theta)\mathrm{d}_{\theta}U^{i}(\theta)\\ &+ \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)](\int_{\theta}^{0}\exp(p(\theta-\eta))X_{\Phi}(\eta)\mathrm{d}\eta) = \mathfrak{L}X_{\Phi}(t)(p)\sum_{i=0}^{m}\int_{-h}^{0}\exp(p\theta)\mathrm{d}_{\theta}U^{i}(\theta)\\ &+ \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_i(\eta)\mathrm{d}\eta.\end{aligned} \tag{11}$$

Similarly, for the left-hand side of Equation (2), one obtains that

$$\begin{aligned}&\mathfrak{L}D^{\alpha}\left(X_{\Phi}(t) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]X_{\Phi}(t+\theta)\right)(p) = -I_{\alpha-1}(p)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi_l(\theta)\right)\\ &+ \mathfrak{L}X_{\Phi}(t)(p)\left(I_{\alpha}(p) - I_{\alpha}(p)\sum_{l=1}^{r}\int_{-h}^{0}\exp(p\theta)\mathrm{d}_{\theta}V^{l}(\theta)\right)\\ &- I_{\alpha}(p)\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_{\theta}V^{l}(\theta)]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_l(\eta)\mathrm{d}\eta.\end{aligned} \tag{12}$$

From Equations (11) and (12), it follows

$$\begin{aligned}&\mathfrak{L}X_{\Phi}(p)\left(I_{\alpha}(p) - I_{\alpha}(p)\sum_{l=1}^{r}\int_{-h}^{0}\exp(p\theta)d_{\theta}V^{l}(\theta) - \sum_{i=0}^{m}\int_{-h}^{0}\exp(p\theta)d_{\theta}U^{i}(\theta)\right)\\ &= I_{\alpha-1}(p)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi_l(\theta)\right) + I_{\alpha}(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_l(\eta)\mathrm{d}\eta\\ &+ \sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_i(\eta)\mathrm{d}\eta\end{aligned}$$

and hence,

$$\mathfrak{L}X_{\Phi}(p) = G^{-1}(p)I_{\alpha-1}(p)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi_{l}(\theta)\right)$$
$$+ G^{-1}(p)I_{\alpha}(p) + \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{l}(\eta)\mathrm{d}\eta \tag{13}$$
$$+ G^{-1}(p)\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{i}(\eta)\mathrm{d}\eta.$$

The representations of Equations (7) and (13) imply that

$$\mathfrak{L}X_{\Phi}(p) = \mathfrak{L}Q(t)(p)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_{\theta}V^{l}(\theta)]\Phi_{l}(\theta)\right)$$
$$+ I_{1}(p)\mathfrak{L}Q(t)(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{l}(\eta)\mathrm{d}\eta \tag{14}$$
$$+ I_{1-\alpha}(p)\mathfrak{L}Q(t)(p)\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{i}(\eta)\mathrm{d}\eta.$$

In view of Equation (9), we obtain for the second term in the right-hand side of Equation (14) that

$$I_{1}(p)\mathfrak{L}Q(t)(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{l}(\eta)\mathrm{d}\eta$$
$$= I_{\frac{1}{2}}(p)\mathfrak{L}Q(t)(p)I_{\frac{1}{2}}(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}\Phi_{l}(t+\theta)(p)$$
$$= \mathfrak{L}D^{\frac{1}{2}}Q(t)(p) + I_{-\frac{1}{2}}(p)I_{\frac{1}{2}}(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}\Phi_{l}(t+\theta)(p) \tag{15}$$
$$= \mathfrak{L}D^{\frac{1}{2}}Q(t)(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]I_{\frac{1}{2}}(p)\mathfrak{L}\Phi_{l}(t+\theta)(p)$$
$$+ \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}\Phi_{l}(t+\theta)(p).$$

For the first term in the right-hand side of Equation (15) we have

$$\mathfrak{L}D^{\frac{1}{2}}Q(t)(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]I_{\frac{1}{2}}(p)\mathfrak{L}\Phi_{l}(t+\theta)(p)$$
$$= \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}D^{\frac{1}{2}}Q(t)(p)\mathfrak{L}D^{\frac{1}{2}}\Phi_{l}(t+\theta)(p)$$

and hence, from Equation (15) it follows

$$I_{1}(p)\mathfrak{L}Q(t)(p)\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{l}(\eta)\mathrm{d}\eta$$
$$= \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right](\mathfrak{L}D^{\frac{1}{2}}Q(t)(p)\mathfrak{L}D^{\frac{1}{2}}\Phi_{l}(t+\theta)(p) \tag{16}$$
$$+ \sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_{\theta}V^{l}(\theta)]\mathfrak{L}\Phi_{l}(t+\theta)(p).$$

Analogously, for the third term in the right-hand side of Equation (14), we have

$$
\begin{aligned}
&I_{1-\alpha}(p)\mathfrak{L}Q(t)(p)\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\int_{\theta}^{0}\exp(p(\theta-\eta))\Phi_{i}(\eta)\mathrm{d}\eta \\
&=I_{1-\alpha}(p)\mathfrak{L}Q(t)(p)\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\mathfrak{L}\Phi_{i}(t+\theta)(p) \\
&=\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\mathfrak{L}D^{1-\alpha}Q(t)(p)\mathfrak{L}\Phi_{i}(t+\theta)(p) \\
&+\sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_{\theta}U^{i}(\theta)]\mathfrak{L}D^{-\alpha}\Phi_{i}(t+\theta)(p)
\end{aligned}
\tag{17}
$$

From Equations (14), (16) and (17), it follows that

$$
\begin{aligned}
&\mathfrak{L}X_{\Phi}(p)=\mathfrak{L}Q(t)(p)\left(\Phi(0)-\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\Phi(\theta)\right)+\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}\Phi_{l}(t+\theta)(p) \\
&+\sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_{\theta}V^{l}(\theta)\right]\mathfrak{L}D^{\frac{1}{2}}Q(t)(p)\mathfrak{L}D^{\frac{1}{2}}\Phi_{l}(t+\theta)(p) \\
&+\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\mathfrak{L}D^{1-\alpha}Q(t)(p)\mathfrak{L}\Phi_{i}(t+\theta)(p)+\sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_{\theta}U^{i}(\theta)\right]\mathfrak{L}D^{-\alpha}\Phi_{i}(t+\theta)(p)
\end{aligned}
\tag{18}
$$

Applying the inverse Laplace transform to both sides of Equation (18), we obtain Equation (10). □

Theorem 3. *Let the following conditions be satisfied:*

(i) *The conditions (SA) hold.*
(ii) *The function $F \in L_1^{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathbb{R}^n)$ is exponentially bounded.*

Then the solution $X^F(t)$ of the IVP (1), (3) with initial function $\Phi(t) \equiv 0$, $t \in [-h, 0]$ has the following representation:

$$
X^{F}(t)=\int_{0}^{t}D^{1-\alpha}Q(t-s)F(s)\mathrm{d}s+D^{-\alpha}F(t),
\tag{19}
$$

where $Q(t)$ is the fundamental matrix of System (2).

Proof. First we substitute $X_\Phi(t)$ in Equation (1) and use the fact that $X^F(t) = 0$, $t \in [-h, 0]$. Since the function F is exponentially bounded, then we can apply to both sides the Laplace transform in order to get

$$
\begin{aligned}
&\mathfrak{L}X^{F}(t)(p)\left(I_{\alpha}(p)-I_{\alpha}(p)\sum_{l=1}^{r}\int_{-h}^{0}\exp(p\theta)\mathrm{d}_{\theta}V^{l}(\theta)-\sum_{i=0}^{m}\int_{-h}^{0}\exp(p\theta)\mathrm{d}_{\theta}U^{i}(\theta)\right) \\
&=\mathfrak{L}X^{F}(t)(p)G(p)=\mathfrak{L}F(t)(p).
\end{aligned}
\tag{20}
$$

Now it follows from the equality $G^{-1}(p) = I_{1-\alpha}(p)\mathfrak{L}Q(t)(p)$ that

$$
\begin{aligned}
&\mathfrak{L}X^{F}(t)(p)=I_{1-\alpha}(p)I_{\alpha-1}(p)G^{-1}(p)\mathfrak{L}F(t)(p) \\
&=I_{1-\alpha}(p)\mathfrak{L}Q(t)(p)\mathfrak{L}F(t)(p)=\left(\mathfrak{L}D^{1-\alpha}Q(t)(p)+I_{-\alpha}(p)\right)\mathfrak{L}F(t)(p) \\
&=\mathfrak{L}D^{1-\alpha}Q(t)(p)\mathfrak{L}F(t)(p)+\mathfrak{L}D^{-\alpha}F(t)(p).
\end{aligned}
\tag{21}
$$

Finally, we apply the inverse Laplace transform to Equation (21) and the representation Equation (19) follows. □

Corollary 1. *Let the conditions of Theorem 3 hold. Then for every initial function $\Phi \in \widetilde{C}$, the corresponding unique solution $X^F_\Phi(t)$ of the IVP (1), (3) has the integral representation*

$$X^F_\phi(t) = \int_0^t D^{1-\alpha}Q(t-s)F(s)\mathrm{d}s + D^{-\alpha}F(t) + Q(t)\left(\Phi(0) - \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right]\Phi(\theta)\right)$$
$$+ \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right]\Phi(t+\theta) + \sum_{l=1}^{r}\int_{-h}^{0}\left[\mathrm{d}_\theta V^l(\theta)\right] D^{\frac{1}{2}}Q(t) * D^{\frac{1}{2}}\Phi_l(t+\theta)$$
$$+ \sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_\theta U^i(\theta)\right] D^{1-\alpha}Q(t) * \Phi_i(t+\theta) + \sum_{i=0}^{m}\int_{-h}^{0}\left[\mathrm{d}_\theta U^i(\theta)\right] D^{-\alpha}\Phi_i(t+\theta),$$

where $Q(t)$ is the fundamental matrix of System (2).

Proof. Let $\Phi \in \widetilde{C}$ be an arbitrary initial function and let the functions $X_\Phi(t)$ and $X^F(t)$ be defined by the Equalities (9) and (19), respectively. Then, according to the superposition principle, the function $X_\Phi(t) + X^F(t)$ is the unique solution of the IVP (1), (3). Now the statement of Corollary 1 follows immediately from Theorems 2 and 3. □

4. Example

First, we give some results needed for the illustrative example presented below:

The delayed Mittag-Leffler type matrix function $\mathbf{E}^{B,\tau}_{\alpha,1} : \mathbb{R} \to \mathbb{R}^{n\times n}$ for every matrix $B \in \mathbb{R}^{n\times n}$ and for $\tau \in \mathbb{R}_+$ is defined by

$$\mathbf{E}^{Bt^\alpha}_\tau(t) := I + \sum_{k=1}^{\infty}\frac{B^k(t-(k-1)\tau)^{\alpha k}}{\Gamma(\alpha k+1)}H(k\tau - t), \quad t \geq 0 \tag{22}$$

with $\mathbf{E}^{Bt^\alpha}_\tau(0) := I$, $\mathbf{E}^{Bt^\alpha}_\tau(t) := \Theta$ for $t < 0$ and $H(t)$ is the Heavyside function with $H(0) = 1$. This is a slight modification of the original definition in [29], and note that for each $t \geq 0$, the sum in Equation (22) is finite and for $\tau = 0$ we have

$$\mathbf{E}^{Bt^\alpha}_0(t) := E_\alpha(Bt^\alpha) = \sum_{k=0}^{\infty}\frac{B^k t^{\alpha k}}{\Gamma(\alpha k+1)}, \quad t \geq 0, \tag{23}$$

where the right side is the standard Mittag-Leffler type matrix function.

Example 1. *Consider the nonhomogeneous system for $t > 0$:*

$$\begin{aligned} &D^{0.5}_{0_+}x_1(t) = x_1(t-1) + 1 \\ &D^{0.5}_{0_+}(x_2(t) + x_1(t-1) + x_2(t-1)) = x_2(t) + x_2(t-1) + x_1(t-2) \end{aligned} \tag{24}$$

with the initial conditions

$$\Phi(t) = (0,2)^T, t \in [-2,0] \quad i.e. \quad x_1(t) = 0,\ x_2(t) = 2 \quad for \quad t \in [-2,0]. \tag{25}$$

The homogenious system has the form

$$\begin{aligned} &D^{0.5}_{0_+}\overline{x}_1(t) = \overline{x}_1(t-1) \\ &D^{0.5}_{0_+}(\overline{x}_2(t) + \overline{x}_1(t-1) + \overline{x}_2(t-1)) = \overline{x}_2(t) + \overline{x}_2(t-1) + \overline{x}_1(t-2) \end{aligned} \tag{26}$$

and introduce the following initial conditions necessary for the calculating the fundamental matrix $Q(t)$:

$$1.\ x_1(0) = 1, x_2(0) = 0 \quad and \quad x_1(t) = 0, x_2(t) = 0 \quad for \quad t \in [-2,0); \tag{27}$$

$$2.\ x_1(0) = 0, x_2(0) = 1 \quad \textit{and} \quad x_1(t) = 0, x_2(t) = 0 \quad \textit{for} \quad t \in [-2, 0). \tag{28}$$

Let consider the IP (26), (27). Then the first Equation of (26) in virtue of Theorem 3.1 in [29] has the solution $\overline{x}_1^{\tau}(t) = \mathbf{E}_1^{t^{0.5}}$ $(\tau = 1, \alpha = 0.5)$*. Taking into account Equation (27), it is simple to check that* $(D^{0.5}_{0_+}\overline{x}^1_1(s-1))(t) = (D^{0.5}_{0_+}\overline{x}^1_1(s))(t-1)$*, and then in virtue of Theorem 3.1 in [29] we have that* $(D^{0.5}_{0_+}\overline{x}^1_1)(t-1) = \overline{x}^1_1(t-2)$*, and hense, from the second equation and Equation (27), we obtain that* $\overline{x}^1_2(t) \equiv 0$ *for* $t \in [-2, \infty)$*. Thus the IP (26), (27) have the following solution* $\overline{x}^1_1(t) = \mathbf{E}_1^{t^{0.5}}$, $\overline{x}^1_2(t) \equiv 0$ *for* $t \in [-1, \infty)$*.*

Consider the IP (26), (28) . Then obviously $\overline{x}^2_1(t) \equiv 0$ *for* $t \in [-2, \infty)$ *and the second equation become the form:* $D^{0.5}_{0_+}(\overline{x}_2(t) + \overline{x}_2(t-1)) = \overline{x}_2(t) + \overline{x}_2(t-1)$ *and by making the substitutuon* $y(t) = \overline{x}_2(t) + \overline{x}_2(t-1)$ *we obtain the equations* $D^{0.5}_{0_+}y(t) = y(t)$ *with initial codition* $y(0) = 1$*, i.e., the following IP*

$$D^{0.5}_{0_+}y(t) = y(t), \quad t > 0; \quad y(0) = 1. \tag{29}$$

Applying Lemma 2.23 in [2] for the case when $\lambda = 1$, $\tau = 1$, $\alpha = 0.5$ *we obtain that the solution of the IP (29) is the fuction* $y(t) = \mathbf{E}_1^{t^{0.5}}(t) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}$*. Then, using the step method, we obtain for each* $k \in \mathbb{N}$ *and* $t \in [k-1, k)]$ *that* $\overline{x}^2_2(t) = \sum_{k=1}^{\infty}(-1)^{k-1}\mathbf{E}_1^{t^{0.5}}(t-(k-1))H(k-t)$ *for* $t > 0$*. Thus, we obtain that the fundamental matrix have the form:*

$$Q(t) = \begin{pmatrix} \mathbf{E}_1^{t^{0.5}}(t) & 0 \\ 0 & \sum_{k=1}^{\infty}(-1)^{k-1}\mathbf{E}_1^{t^{0.5}}(t-(k-1))H(k-t) \end{pmatrix}. \tag{30}$$

In the IP (24), (25) we have that: $\Phi(t) = (0, 2)^T, t \in [-2, 0]$; $F(t) = (1, 0)^T$*. Then from Equation (19), we have*

$$x^F_1(t) = \frac{1}{\Gamma(0.5)} \int_0^t \left(\int_0^{t-s} (t-s-\eta)^{-0.5} \left(\mathbf{E}_1^{t^{0.5}}\right)' (\eta) \mathrm{d}\eta \right) \mathrm{d}s + \frac{\sqrt{t}}{\Gamma(1.5)},$$
$$x^F_2(t) = 0.$$

From Equation (10), it follows

$$x^{\Phi}_1(t) = 0,$$
$$x^{\Phi}_2(t) = 2 + 2\int_0^t \left(\int_0^s (s-\eta)^{-0.5} \left(\sum_{k=1}^{\infty} (-1)^{k-1}\mathbf{E}_1^{t^{0.5}}(\eta-(k-1))H(k-\eta) \right)' \mathrm{d}\eta \right) \mathrm{d}s + \frac{2\sqrt{t}}{\Gamma(1.5)}.$$

Then, the solution of the IP (24), (25), according Corollary 1, is

$$x_1(t) = x^{\Phi}_1(t) + x^F_1(t) = \frac{1}{\Gamma(0.5)} \int_0^t \left(\int_0^{t-s} (t-s-\eta)^{-0.5} \left(\mathbf{E}_1^{t^{0.5}}\right)' (\eta) \mathrm{d}\eta \right) \mathrm{d}s + \frac{\sqrt{t}}{\Gamma(1.5)},$$

$$x_2(t) = x^{\Phi}_2(t) + x^F_2(t) = 2 + 2\int_0^t \left(\int_0^s (s-\eta)^{-0.5} \left(\sum_{k=1}^{\infty} (-1)^{k-1}\mathbf{E}_1^{t^{0.5}}(\eta-(k-1))H(k-\eta) \right)' \mathrm{d}\eta \right) \mathrm{d}s + \frac{2\sqrt{t}}{\Gamma(1.5)}.$$

5. Conclusions

Following the investigations way in the case of functional differential systems with integer order derivatives, we proved a formula for integral representation of the solutions of Cauchy problem for fractional neutral systems, which improves and extends the corresponding former results obtained in the particular case of fractional systems with constant delays. However, the main idea is not only to make a standard generalization of existing results, but as in the case of systems with integer derivatives, the proved formula to be an useful tool for further study of different kinds stability properties of linear neutral fractional systems, which have a lot of practical applications.

As examples in this direction, we refer to the works [29,30], where finite time stability is studied by this approach, i.e., in the partial case of one constant delay. In the mentioned articles, first a formula for integral representation of the solutions of Cauchy problem is proved, and then, using the obtained result, sufficient conditions for finite time stability of the considered fractional delayed system are established. Furthermore, applying the same approach, in [16], the asymptotic stability properties of nonlinear perturbed linear fractional delayed systems are studied .

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References

1. Diethelm, K. *The Analysis of Fractional Differential Equations, an Application–Oriented Exposition Using Differential Operators of Caputo Type*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2010; Volume 2004.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science B.V: Amsterdam, The Netherlands, 2006.
3. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Longman Scientific & Technical: Harlow, UK; John Wiley & Sons, Inc.: New York, NY, USA, 1994.
4. Podlubny, I. *Fractional Differential Equation*; Academic Press: San Diego, CA, USA, 1999.
5. Kaczorek, T. *Selected Problems of Fractional Systems Theory*; Lecture Notes in Control and Information Sciences; Springer: Berlin/Heidelberg, Germany, 2011.
6. Stamova, I.; Stamov, G. *Functional and Impulsive Differential Equations of Fractional Order*; Qualitative analysis and applications; CRC Press: Boca Raton, FL, USA, 2017.
7. Bonilla, B.; Rivero, M.; Trujillo, J. On systems of linear fractional differential equations with constant coefficients. *Appl. Comput. Math.* **2007**, *187*, 68–78. [CrossRef]
8. Li, K.; Peng, J. Laplace transform and fractional differential equations, *Appl. Math. Let.* **2011**, *24*, 2019–2023. [CrossRef]
9. Lin, S.D.; Lu, C.H. Laplace transform for solving some families of fractional differential equations and its applications. *Adv. Differ. Equ.* **2013**, *137*, 1–9. [CrossRef]
10. Golev, A.; Milev, M. Integral representation of the solution of the Cauchy problem for autonomous linear neutral fractional system. *Int. J. Pure Appl. Math.* **2018**, *119*, 235–247.
11. Kiskinov H.; Zahariev A. Asymptotic stability of delayed fractional systems with nonlinear perturbation. *AIP Conf. Proc.* **2018**, *2048*, 050014.
12. Zhang, H.; Cao, J.; Jiang, W. General solution of linear fractional neutral differential difference equations. *Discret. Dyn. Nat. Soc.* **2013**. [CrossRef]
13. Zhang, H.; Wu, D. Variation of constant formulae for time invariant and time varying Caputo fractional delay differential systems. *J. Math. Res. Appl.* **2014**, *34*, 549-560.
14. Kiskinov, H.; Zahariev A. On fractional systems with Riemann-Liouville derivatives and distributed delays—Choice of initial conditions, existence and uniqueness of the solutions. *Eur. Phys. Jo. Spec. Top.* **2017**, *26*, 3473-3487. [CrossRef]
15. Krol, K. Asymptotic properties of fractional delay differential equations, *Appl. Math. Comput.* **2011**, *218*, 1515–1532.
16. Milev, M.; Zlatev, S. A note about stability of fractional retarded linear systems with distributed delays. *Int. J. Pure Appl. Math.* **2017**, *115*, 873–881. [CrossRef]

17. Veselinova, M.; Kiskinov, H.; Zahariev, A. Stability analysis of linear fractional differential system with distributed delays. *AIP Conf. Proc.* **2015**, *1690*, 040013.
18. Veselinova, M.; Kiskinov, H.; Zahariev, A. About stability conditions for retarded fractional differential systems with distributed delays. *Commun. Appl. Anal.* **2016**, *20*, 325–334.
19. Kiskinov H.; Milev N.; Zahariev A. A comparison type theorem for linear neutral fractional systems with distributed delays. *AIP Conf. Proc.* **2017**, *1910*, 050009.
20. Veselinova, M.; Kiskinov, H.; Zahariev, A. Stability analysis of neutral linear fractional system with distributed delays. *Filomat* **2016**, *30*, 841–851. [CrossRef]
21. Veselinova, M.; Kiskinov, H.; Zahariev, A. Explicit conditions for stability of neutral linear fractional system with distributed delays. *AIP Conf. Proc.* **2016**, *1789*, 040005.
22. Boyadzhiev, D.; Kiskinov, H.; Veselinova, M.; Zahariev, A. Stability analysis of linear distributed order fractional systems with distributed delays. *Fract. Calc. Appl. Anal.* **2017**, *20*, 914–935. [CrossRef]
23. Martinez-Garcia, M.; Gordon, T.; Shu, L. Extended crossover model for human-control of fractional order plants. *IEEE Access* **2017**, 5, 27622-27635. [CrossRef]
24. Martinez-Garcia, M.; Zhang, Yu; Gordon T. Memory pattern identification for feedback tracking control in human-machine systems. *Hum. Factors* **2019**. [CrossRef]
25. Boyadzhiev, D.; Kiskinov, H.; Zahariev, A. Integral representation of solutions of fractional system with distributed delays. *Integral Transforms Spec. Funct.* **2018**, *29*, 725–744. [CrossRef]
26. Zahariev, A.; Kiskinov, H.; Angelova E. Linear fractional system of incommensurate type with distributed delay and bounded Lebesgue measurable initial conditions. *Dyn. Syst. Appl.* **2019**, *28*, 491–506.
27. Zahariev, A.; Kiskinov, H.; Angelova, E. Smoothness of the fundamental matrix of linear fractional system with variable delays. *Neural Parallel Sci. Comput.* **2019**, *27*, 71–83.
28. Zahariev, A.; Kiskinov, H. Existence of fundamental matrix for neutral linear fractional system with distributed delays. *Int. J. Pure and Appl. Math.* **2018**, *119*, 31–51.
29. Li, M.; Wang, J. Finite time stability of fractional delay differential equations. *Appl. Math. Lett.* **2017**, *64*, 170–176. [CrossRef]
30. Li, M.; Wang, J. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl. Mathe. Comput.* **2018**, *324*, 254–265. [CrossRef]