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Symmetric Conformable Fractional Derivative of Complex Variables

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Abstract: It is well known that the conformable and the symmetric differential operators have formulas in terms of the first derivative. In this document, we combine the two definitions to get the symmetric conformable derivative operator (SCDO). The purpose of this effort is to provide a study of SCDO connected with the geometric function theory. These differential operators indicate a generalization of well known differential operator including the Sàlàgean differential operator. Our contribution is to impose two classes of symmetric differential operators in the open unit disk and to describe the further development of these operators by introducing convex linear symmetric operators. In addition, by acting these SCDOs on the class of univalent functions, we display a set of sub-classes of analytic functions having geometric representation, such as starlikeness and convexity properties. Investigations in this direction lead to some applications in the univalent function theory of well known formulas, by defining and studying some sub-classes of analytic functions type Janowski function and convolution structures. Moreover, by using the SCDO, we introduce a generalized class of Briot–Bouquet differential equations to introduce, what is called the symmetric conformable Briot–Bouquet differential equations. We shall show that the upper bound of this class is symmetric in the open unit disk.

Keywords: univalent function; conformable fractional derivative; subordination and superordination; analytic function; open unit disk

MSC: 30C45

1. Introduction

The term Symmetry from Greek means arrangement and organization in measurements. In free language, it mentions a concept of harmonious and attractive proportion and equilibrium. In mathematics, it discusses an object that is invariant via certain transformation or rotation or scaling. In geometry, the object has symmetry if there is an operator or transformation that maps the object onto itself [1,2].

Sàlàgean (1983) presented a differential operator for a class of analytic functions (see [3]). Many sub-classes of analytic functions are studied using this operator. Al-Oboudi [4] generalized this operator. These operators are studied widely in the last decade (see [5–10] for recent works). Our investigation is to study classes of analytic functions by using the symmetric differential operator in a complex domain. Recently, Ibrahim and Jahangiri [7] defined a special type of differential operators, which is called a complex conformable differential operator. This operator is an extension of the Anderson–Ulness operator [11].

A conformable calculus (CC) is a branch of the fractional calculus. It develops the term $\chi^{1-\wp} f'(\chi)$. While the complex conformable calculus (CCC) indicates the term $\xi\varphi'(\xi)$, where ξ is a complex variable and φ is a complex valued analytic function. In this work, we present a new SCDO in the open unit disk. We formulate it in some sub-classes of univalent functions. As applications, we generalize a class of Briot–Bouquet differential equations by using SCDO.

2. Methodology

This section deals with the mathematical processing to study the SCDO for some classes of analytic functions in the open unit disk $\cup = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Let $\bigwedge$ be the following class of analytic functions

$$\Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} \Upsilon_n \xi^n, \quad \xi \in \cup. \tag{1}$$

A function $\Upsilon \in \bigwedge$ is starlike via the (0,0) (origin in $\cup$) if the linear segment joining the origin to every other point of Υ lies entirely in $\Upsilon(\xi : |\xi| < 1)$. A univalent function ($\Upsilon \in ⓢ$) is convex in $\cup$ if the linear segment joining any two points of $\Upsilon(\xi : |\xi| < 1)$ lies entirely in $\Upsilon(\xi : |\xi| < 1)$. We denote these classes by $\mathcal{S}^*$ and $\mathcal{C}$ for starlike and convex respectively. In addition, suppose that the class $\mathcal{P}$ involves all functions Υ analytic in $\cup$ with a positive real part in $\cup$ achieving $\Upsilon(0) = 1$. Mathematically, $\Upsilon \in \mathcal{S}^*$ if and only if $\xi\,\Upsilon'(\xi)/\Upsilon(\xi) \in \mathcal{P}$ and $\Upsilon \in \mathcal{C}$ if and only if $1 + \xi\,\Upsilon''(\xi)/\Upsilon'(\xi) \in \mathcal{P}$. Equivalently, $\Re(\xi\,\Upsilon'(\xi)/\Upsilon(\xi)) > 0$ for the starlikeness and $1 + \Re(\xi\,\Upsilon''(\xi)/\Upsilon'(\xi)) > 0$ for the convexity.

For two functions Υ_1 and Υ_2 belong to the class $\bigwedge$, are said to be subordinate, noting by $\Upsilon_1 \prec \Upsilon_2$, if we can find a Schwarz function $\top$ with $\top(0) = 0$ and $|\top(\xi)| < 1$ achieving $\Upsilon_1(\xi) = \Upsilon_2(\top(\xi))$, $\xi \in \cup$ (the detail can be located in [12]). Obviously, $\Upsilon_1(\xi) \prec \Upsilon_2(\xi)$ if $\Upsilon_1(0) = \Upsilon_2(0)$ and $\Upsilon_1(\cup) \subset \Upsilon_2(\cup)$.

Lemma 1 ([12]). *Suppose that $a \in \mathbb{C}$, n is a positive integer and $\aleph[a,n] = \{\Upsilon : \Upsilon(\xi) = a + a_n\xi^n + a_{n+1}\xi^{n+1} + ...\}$ is a set of analytic functions.*

i. If $\ell \in \mathbb{R}$ then $\Re\Big(\Upsilon(\xi) + \ell\,\xi\,\Upsilon'(\xi)\Big) > 0 \Longrightarrow \Re\big(\Upsilon(\xi)\big) > 0$. In addition, if $\ell > 0$ and $\Upsilon \in \aleph[1,n]$, then there occurs some constants $a > 0$ and $b > 0$ with $b = b(\ell, a, n)$ where

$$\Upsilon(\xi) + \ell\xi\,\Upsilon'(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b \Rightarrow \Upsilon(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^a.$$

ii. If $\eth \in [0,1)$ and $\Upsilon \in \aleph[1,n]$ then a constant $k > 0$ exists satisfying $k = k(a,n)$ so that

$$\Re\Big(\Upsilon^2(\xi) + 2\Upsilon(\xi).\xi\Upsilon'(\xi)\Big) > \eth \Rightarrow \Re(\Upsilon(\xi)) > k.$$

iii. If $\Upsilon \in \aleph[a,n]$ with $\Re(a) > 0$ then $\Re\Big(\Upsilon(\xi) + \xi\,\Upsilon'(\xi) + \xi^2\,\Upsilon''(\xi)\Big) > 0$ or for $\imath : \cup \to \mathbb{R}$ with $\Re\Big(\Upsilon(\xi) + \imath(\xi)\dfrac{\xi\,\Upsilon'(\xi)}{\Upsilon(\xi)}\Big) > 0$ then $\Re(\Upsilon(\xi)) > 0$.

Lemma 2 ([12]). *Assume that $\hbar$ is a convex function satisfying $\hbar(0) = a$, and let $\Bbbk \in \mathbb{C} \setminus \{0\}$ be a complex number with $\Re(\Bbbk) \geq 0$. If $\Upsilon \in \aleph[a,n]$, and*

$$\Upsilon(\xi) + (1/\Bbbk)\xi\,\Upsilon'(\xi) \prec \hbar(\xi), \quad \xi \in \cup,$$

then $\Upsilon(\xi) \prec \iota(z) \prec \hbar(z)$, *where*

$$\iota(z) = \frac{\Bbbk}{n\xi^{\Bbbk/n}} \int_0^{\xi} \hbar(\tau)\tau^{\overline{(n-1)}^{\Bbbk}} d\tau, \quad \xi \in \cup.$$

Lemma 3 ([13]). *Suppose that* $\Upsilon \in \bigwedge$ *and there occurs a positive constant* $0 < \upsilon \leq 1$. *If*

$$\frac{\xi\,\Upsilon'(\xi) - \xi}{\Upsilon(\xi)} \prec \frac{2\upsilon\xi}{1+\xi}$$

then

$$\frac{\Upsilon(\xi)}{\xi} \prec 1 + \upsilon\,\xi, \quad \xi \in \cup.$$

And the result is sharp.

The Operator SCDO

This sections deals with definition of the SCDO as follows:

Definition 1. *Let* $\Upsilon(\xi) \in \bigwedge$, *and let* $\nu \in [0,1]$ *be a constant then the SCDO keeps the following operating*

$$\begin{aligned}
\mathcal{S}_\nu^0\,\Upsilon(\xi) &= \Upsilon(\xi)\\
\mathcal{S}_\nu^1\,\Upsilon(\xi) &= \left(\frac{\kappa_1(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)\xi\,\Upsilon'(\xi) - \left(\frac{\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)\xi\,\Upsilon'(-\xi)\\
&= \left(\frac{\kappa_1(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)\left(\xi + \sum_{n=2}^{\infty} n\,\Upsilon_n\,\xi^n\right) - \left(\frac{\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)\left(-\xi + \sum_{n=2}^{\infty} n(-1)^n\,\Upsilon_n\,\xi^n\right)\\
&= \xi + \sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\nu,\xi)+(-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)\Upsilon_n\,\xi^n\\
\mathcal{S}_\nu^2\,\Upsilon(\xi) &= \mathcal{S}_\nu^1[\mathcal{S}_\nu^1\,\Upsilon(\xi)] \qquad (2)\\
&= \xi + \sum_{n=2}^{\infty} n^2\left(\frac{\kappa_1(\nu,\xi)+(-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)^2\Upsilon_n\,\xi^n\\
&\vdots\\
\mathcal{S}_\nu^k\,\Upsilon(\xi) &= \mathcal{S}_\nu^1[\mathcal{S}_\nu^{k-1}\,\Upsilon(\xi)]\\
&= \xi + \sum_{n=2}^{\infty} n^k\left(\frac{\kappa_1(\nu,\xi)+(-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)^k\Upsilon_n\,\xi^n.
\end{aligned}$$

so that $\kappa_1(\nu,\xi) \neq -\kappa_0(\nu,\xi)$,

$$\lim_{\nu\to 0}\kappa_1(\nu,\xi) = 1, \quad \lim_{\nu\to 1}\kappa_1(\nu,\xi) = 0, \quad \kappa_1(\nu,\xi) \neq 0,\ \forall\xi \in \cup,\ \nu \in (0,1),$$

and

$$\lim_{\nu\to 0}\kappa_0(\nu,\xi) = 0, \quad \lim_{\nu\to 1}\kappa_0(\nu,\xi) = 1, \quad \kappa_0(\nu,\xi) \neq 0,\ \forall\xi \in \cup\ \nu \in (0,1).$$

The value $\nu = 0$ indicates the Sàlàgean operator $\mathcal{S}^k\,\Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} n^k\,\Upsilon_n\,\xi^n$. We proceed to impose a linear differential operator having the SCDO and the Ruscheweyh derivative. For $\Upsilon \in \bigwedge$, the Ruscheweyh derivative is defined as follows:

$$\mathcal{R}^k\,\Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} \complement_{k+n-1}^{k}\,\Upsilon_n\,\xi^n,$$

where $\complement^k_{k+n-1}$ are the combination terms.

Definition 2. *Let $\Upsilon \in \bigwedge, \nu \in [0,1]$ and $0 \leq \alpha \leq 1$. The linear combination operator joining $\mathcal{R}^k \Upsilon(\xi)$ and $\mathcal{S}^k_\nu \Upsilon(\xi)$ is given by the formal*

$$\begin{aligned}\mathbf{C}^k_{\nu,\alpha} \Upsilon(\xi) &= (1-\alpha)\mathcal{R}^k \Upsilon(\xi) + \alpha \mathcal{S}^k_\nu \Upsilon(\xi) \\ &= \xi + \sum_{n=2}^{\infty} (1-\alpha)\complement^k_{k+n-1} + \alpha \left[n \left(\frac{\kappa_1(\nu,\xi) + (-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)} \right) \right]^k \Upsilon_n \xi^n.\end{aligned} \tag{3}$$

Remark 1.

- $k = 0 \Longrightarrow \mathbf{C}^0_{\nu,\alpha} \Upsilon(\xi) = \Upsilon(\xi)$;
- $\nu = 0 \Longrightarrow \mathbf{C}^k_{1,\alpha} \Upsilon(\xi) = \mathcal{L}^k_\kappa \Upsilon(\xi)$; *[14] (Lupas operator)*
- $\alpha = 0 \Longrightarrow \mathbf{C}^k_{\nu,\alpha} \Upsilon(\xi) = \mathcal{R}^k \Upsilon(\xi)$;
- $\alpha = 1, \kappa = 1 \Longrightarrow \mathbf{C}^k_{0,1} \Upsilon(\xi) = \mathcal{S}^k \Upsilon(\xi)$;
- $\alpha = 1 \Longrightarrow \mathbf{C}^k_{\nu,1} \Upsilon(\xi) = \mathcal{S}^k_\nu \Upsilon(\xi)$.

Definition 3. *Let $\epsilon \in [0,1), \nu, \alpha \in [0,1$ and $k \in \mathbb{N}$. A function $\Upsilon \in \bigwedge$ belongs to the set $\mathcal{B}_k(\nu,\alpha,\epsilon)$ if and only if*

$$\Re\Big((\mathbf{C}^k_{\nu,\alpha} \Upsilon(\xi))' \Big) > \epsilon, \quad \xi \in \cup.$$

Definition 4. *The function $\Upsilon \in \bigwedge$ is specified to be in $\mathbb{J}^\flat_\nu(A,B,k)$ if it satisfies the inequality*

$$1 + \frac{1}{\flat}\Big(\frac{2\mathcal{S}^{k+1}_\nu \Upsilon(\xi)}{\mathcal{S}^k_\nu \Upsilon(\xi) - \mathcal{S}^k_\nu \Upsilon(-\xi)} \Big) \prec \frac{1+A\xi}{1+B\xi},$$

$$\Big(\xi \in \cup,\ -1 \leq B < A \leq 1,\ k = 1,2,...,\ \flat \in \mathbb{C} \setminus \{0\},\ \nu \in [0,1] \Big).$$

- $\nu = 0 \Longrightarrow$ [6];
- $\nu = 0, B = 0 \Longrightarrow$ [7];
- $\nu = 0, A = 1, B = -1, \flat = 2 \Longrightarrow$ [8].

The class $\mathbb{J}^\flat_\nu(A, B, k)$ is a generalization of the class of the Janowski starlike functions [15]

$$\rho(\xi) \prec \frac{1+A\xi}{1+B\xi}, \quad \xi \in \cup,$$

where $\rho(0) = 1$, $\rho(\cup) \subset \Omega[A, B]$. The domain $\Omega[A, B]$ is a circular domain and it is referring to an open circular disk with center on the real axis and diameter end points $\frac{1-A}{1-B}$, provide that $B \neq -1$. Functions in the class $\mathbb{J}^\flat_\nu(A,B,k)$ have a circular domain with respect to symmetrical points.

Definition 5. *Let $\epsilon \in [0,1), \nu, \alpha \in [0,1$ and $k \in \mathbb{N}_0$. A function $\Upsilon \in \bigwedge$ is in the set $\mathbb{S}_k(\nu,\epsilon)$ if it achieves the real inequality*

$$\Re\Big(\frac{\mathcal{S}^{k+1}_\nu \Upsilon(\xi)}{\mathcal{S}^k_\nu \Upsilon(\xi)} \Big) > \epsilon, \quad \xi \in \cup.$$

Note that $\mathbb{S}_0(\nu,\epsilon) = \mathcal{S}^*,\ \mathbb{S}_1(0,\epsilon) = \mathcal{C}$

3. The Outcomes

In this section, we study some properties of the SCDO.

Theorem 1. *For $\Upsilon \in \bigwedge$ and $\alpha \in \mathbb{C}\backslash\{0\}$, if one of the sequencing subordination valid*

- *The operator* $\mathcal{S}_\nu^k \curlyvee(\xi)$ *is of bounded turning type;*
- $\curlyvee$ *satisfies the relation*

$$(\mathcal{S}_\nu^k \curlyvee(\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^b, \quad b > 0,\ \xi \in \cup;$$

- $\curlyvee$ *fulfilled the inequality*

$$\Re\Big((\mathcal{S}_\nu^k \curlyvee(\xi))' \frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big) > \frac{\delta}{2}, \quad \delta \in [0,1),\ \xi \in \cup,$$

- $\curlyvee$ *admits the inequality*

$$\Re\Big(\xi \mathcal{S}_\nu^k \curlyvee(\xi))'' - \mathcal{S}_\nu^k \curlyvee(\xi))' + 2\frac{\mathcal{S}_\nu^k \curlyvee(\xi))}{\xi}\Big) > 0,$$

- $\curlyvee$ *confesses the inequality*

$$\Re\Big(\frac{\xi \mathcal{S}_\nu^k \curlyvee(\xi))'}{\mathcal{S}_\nu^k \curlyvee(\xi))} + 2\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big) > 1,$$

then $\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} \in \mathcal{P}(\epsilon),\ \epsilon \in [0,1).$

Proof. Formulate a function σ as pursues:

$$\sigma(\xi) = \frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} \Rightarrow \xi\sigma'(\xi) + \sigma(\xi) = (\mathcal{S}_\nu^k \curlyvee(\xi))'. \tag{4}$$

By the first relation, $\mathcal{S}_\nu^k \curlyvee(\xi)$ is of bounded turning, this indicates that

$$\Re(\xi\,\sigma'(\xi) + \sigma(\xi)) > 0.$$

Therefore, according to Lemma 1—i, we attain $\Re(\sigma(\xi)) > 0$ which gets the first term of the theorem. According to second inequality, we indicate the pursuing subordination inequality

$$(\mathcal{S}_\nu^k \curlyvee(\xi))' = \xi\,\sigma'(\xi) + \sigma(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b.$$

Now, by employing Lemma 1—i, there occurs a fixed constant $a > 0$ with $b = b(a)$ with the pursuing property

$$\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^a.$$

Consequently, we indicate that $\Re(\mathcal{S}_\nu^k \curlyvee(\xi)/\xi) > \epsilon$, for values of $\epsilon \in [0,1)$. Lastly, agree with the third relation to get

$$\Re\Big(\sigma^2(\xi) + 2\sigma(\xi).\xi\,\sigma'(\xi)\Big) = 2\Re\Big((\mathcal{S}_\nu^k \curlyvee(\xi))' \frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big) > \delta. \tag{5}$$

According to Lemma 1—ii, there occurs a positive fixed number $\lambda > 0$ achieving the real inequality $\Re(\sigma(\xi)) > \lambda$, and yielding

$$\sigma(\xi) = \frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} \in \mathcal{P}(\epsilon)$$

for a few value in $\epsilon \in [0,1)$. It indicates from (5) that $\Re\Big(\mathcal{S}_\nu^k \curlyvee(\xi))'\Big) > 0$; thus, according to Noshiro-Warschawski and Kaplan Lemmas, this leads to $\mathcal{S}_\nu^k \curlyvee(\xi)$ is univalent and of bounded turning in $\cup$. Now, via the differentiating (4) and concluding the real case, we indicate that

$$\begin{aligned}
&\Re\Big(\sigma(\xi)+\xi\,\sigma'(\xi)+\xi^2\sigma''(\xi)\Big)\\
&=\Re\Big(\xi(\mathcal{S}_\nu^k \curlyvee(\xi))''-(\mathcal{S}_\nu^k \curlyvee(\xi))'+2\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big)\\
&>0.
\end{aligned}$$

Thus, by the conclusion of Lemma 1—ii, we have

$$\Re(\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi})>0.$$

Taking the logarithmic differentiation (4) and indicating the real, we arrive at the following conclusion:

$$\begin{aligned}
&\Re\Big(\sigma(\xi)+\frac{\xi\,\sigma'(\xi)}{\sigma(\xi)}+\xi^2\sigma''(\xi)\Big)\\
&=\Re\Big(\frac{\xi(\mathcal{S}_\nu^k \curlyvee(\xi))'}{\mathcal{S}_\nu^k \curlyvee(\xi)}+2\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi}-1\Big)\\
&>0.
\end{aligned}$$

A direct application of Lemma 1—iii, we get the positive real i.e., $\Re(\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi})>0$. This completes the proof. □

Theorem 2. *Suppose that $\curlyvee\in\mathbb{J}_\alpha^\flat(A,B,m)$ then for every function of the form*

$$\mathfrak{X}(\xi)=\frac{1}{2}[\curlyvee(\xi)-\curlyvee(-\xi)],\quad \xi\in\cup$$

agrees with the pursuing relation

$$1+\frac{1}{\flat}\Big(\frac{\mathcal{S}_\nu^{k+1}\mathfrak{X}(\xi)}{\mathcal{S}_\nu^{k}\mathfrak{X}(\xi)}-1\Big)\prec\frac{1+A\xi}{1+B\,\xi},$$

and

$$\Re\Big(\frac{\xi\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)}\Big)\geq\frac{1-\curlywedge^2}{1+\curlywedge^2},\quad |\xi|=\curlywedge<1,$$

$$\Big(\xi\in\cup,\ -1\leq B\ <\ A\leq 1,\ m=1,2,...,\ \flat\in\mathbb{C}\setminus\{0\},\ \nu\in[0,1]\Big).$$

Proof. Because the function $\curlyvee\in\mathbb{J}_\alpha^\flat(A,B,m)$ then there occurs a function $\wp\in\mathbb{J}(A,B)$, where

$$\flat(\wp(\xi)-1)=\Big(\frac{2\mathcal{S}_\nu^{k+1}\curlyvee(\xi)}{\mathcal{S}_\nu^{k}\curlyvee(\xi)-\mathcal{S}_\nu^{k}\curlyvee(-\xi)}\Big)$$

and

$$\flat(\wp(-\xi)-1)=\Big(\frac{-2\mathcal{S}_\nu^{k+1}\curlyvee(-\xi)}{\mathcal{S}_\nu^{k}\curlyvee(\xi)-\mathcal{S}_\nu^{k}\curlyvee(-\xi)}\Big).$$

This implies that

$$1+\frac{1}{\flat}\Big(\frac{\mathcal{S}_\nu^{k+1}\mathfrak{X}(\xi)}{\mathcal{S}_\nu^{k}\mathfrak{X}(\xi)}-1\Big)=\frac{\wp(\xi)+\wp(-\xi)}{2}.$$

Also, since $\wp(\xi) \prec \frac{1+A\xi}{1+B\xi}$, where $\frac{1+A\xi}{1+B\xi}$ is univalent then by the concept of the subordination, we have

$$1 + \frac{1}{\flat}\Big(\frac{\mathcal{S}_{\nu}^{k+1}\mathfrak{X}(\xi)}{\Delta_{\alpha}^{m}\mathfrak{X}(\xi)} - 1\Big) \prec \frac{1+A\xi}{1+B\xi}.$$

But the function $\mathfrak{X}(\xi)$ is starlike in $\cup$, which means that

$$\frac{\xi\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)} \prec \frac{1-\xi^2}{1+\xi^2}$$

and there occurs a Schwarz function $\mathsf{T} \in \cup, |\,\mathsf{T}\,(\xi)| \leq |\xi| < 1, \mathsf{T}(0) = 0$ such that

$$\Psi(\xi) := \frac{\xi\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)} \prec \frac{1-\mathsf{T}(\xi)^2}{1+\mathsf{T}(\xi)^2}.$$

This implies that there exists $\zeta, |\zeta| = \curlywedge < 1$ achieving

$$\mathsf{T}^2(\zeta) = \frac{1-\Psi(\zeta)}{1+\Psi(\zeta)}, \quad \zeta \in \cup.$$

A computation yields

$$\Big|\frac{1-\Psi(\zeta)}{1+\Psi(\zeta)}\Big| = |\,\mathsf{T}\,(\zeta)|^2 \leq |\zeta|^2.$$

Thus, we conclude that

$$\Big|\Psi(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\Big|^2 \leq \frac{4|\zeta|^4}{(1-|\zeta|^4)^2}$$

or

$$\Big|\Psi(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\Big| \leq \frac{2|\zeta|^2}{(1-|\zeta|^4)}.$$

Consequently, we obtain

$$\Re(\Psi(\zeta)) \geq \frac{1-\curlywedge^2}{1+\curlywedge^2}, \quad |\zeta| = \curlywedge < 1.$$

□

Theorem 3. *Suppose that $\Upsilon \in \mathcal{B}_k(\nu, \alpha, \epsilon)$, and the convex analytic function g satisfies the integral equation*

$$F(\xi) = \frac{2+c}{\xi^{1+c}} \int_0^{\xi} \tau^c\, \Upsilon\,(\tau) d\tau, \quad \xi \in \cup$$

then the subordination

$$\Big(\mathbf{C}_{\nu,\alpha}^{k}\,\Upsilon\,(\xi)\Big)' \prec g(\xi) + \frac{(\xi g'(\xi))}{2+c}, \quad c > 0,$$

implies the subordination

$$\Big(\mathbf{C}_{\nu,\alpha}^{k} F(\xi)\Big)' \prec g(\xi),$$

and the outcome is sharp.

Proof. Here, we aim to utilize the result of Lemma 2. By the conclusion of $F(\xi)$, we acquire

$$\Big(\mathbf{C}_{\nu,\alpha}^{k} F(\xi)\Big)' + \frac{\Big(\mathbf{C}_{\nu,\alpha}^{k} F(\xi)\Big)''}{2+c} = \Big(\mathbf{C}_{\nu,\alpha}^{k}\,\Upsilon\,(\xi)\Big)'.$$

Following the conditions of the theorem, we get

$$\left(\mathbf{C}^k_{\nu,\alpha}F(\xi)\right)' + \frac{\left(\mathbf{C}^k_{\nu,\alpha}F(\xi)\right)''}{2+c} \prec g(\xi) + \frac{(\xi\, g'(\xi))}{2+c}.$$

By assuming

$$\varrho(\xi) := \left(\mathbf{C}^k_{\nu,\alpha}F(\xi)\right)',$$

We have

$$\varrho(\xi) + \frac{(\xi\varrho'(\xi))}{2+c} \prec g(\xi) + \frac{(\xi g'(\xi))}{2+c}.$$

According to Lemma 2, we obtain

$$\left(\mathbf{C}^k_{\nu,\alpha}F(\xi)\right)' \prec g(\xi),$$

and g is the best dominant. □

Theorem 4. *Let g be convex such that $g(0) = 1$. If*

$$\left(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)\right)' \prec g(\xi) + \xi g'(\xi), \quad \xi \in \cup,$$

then $\dfrac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi} \prec g(\xi)$*, and this result is sharp.*

Proof. Define the following function

$$\varrho(\xi) := \frac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi} \in \aleph[1,1]. \tag{6}$$

A direct application of Lemma 1 yields

$$\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi) = \xi\varrho(\xi) \Longrightarrow \left(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)\right)' = \varrho(\xi) + \xi\varrho'(\xi).$$

Thus, we introduce the following subordination:

$$\varrho(\xi) + \xi\varrho'(\xi) \prec g(\xi) + \xi g'(\xi).$$

Hence, we conclude that $\dfrac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi} \prec g(\xi)$, and g is the best dominant. □

Theorem 5. *If $\curlyvee \in \bigwedge$ fulfills the subordination*

$$(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^b, \quad \xi \in \cup,\ b > 0,$$

then

$$\Re\left(\frac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi}\right) > \epsilon, \quad \epsilon \in [0,1).$$

Proof. Construct ϱ as in (6). Thus, by subordination possessions, we indicate that

$$(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi))' = \xi\varrho'(\xi) + \varrho(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b.$$

With the help of Lemma 1—i, there occurs a fixed number $a > 0$ with $b = b(a)$ where

$$\frac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^a.$$

This leads to real conclusion $\Re(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)/\xi) > \epsilon$, $\epsilon \in [0,1)$. □

Theorem 6. *If $\curlyvee \in \bigwedge$ fulfills the real inequality*

$$\Re\Big((\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi))' \frac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi}\Big) > \Re(\frac{\alpha}{2}), \quad \xi \in \cup, \alpha \in \mathbb{C}$$

then $\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi) \in \mathcal{B}_k(\nu, \alpha, \epsilon)$.

Proof. Formulate ϱ as in (6). A clear evaluation gives

$$\Re\Big(\varrho^2(\xi) + 2\varrho(\xi).\xi\varrho'(\xi)\Big) = 2\Re\Big(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi))' \frac{\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)}{\xi}\Big) > \Re(\alpha). \tag{7}$$

By the advantage of Lemma 1—ii, there occurs a constant κ concerning on $\Re(\alpha)$ where $\Re(\varrho(\xi)) > \kappa$, this gives $\Re(\varrho(\xi)\ ) > \epsilon$, $\epsilon \in [0,1)$. By virtue of (7), it implies that $\Re\Big(\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi))'\Big) > \epsilon$ and hence based on the idea of Noshiro-Warschawski and Kaplan Theorems, $\mathbf{C}^k_{\nu,\alpha} \curlyvee (\xi)$ is univalent and of bounded boundary rotation in $\cup$. □

Theorem 7. *The set $\mathcal{B}_k(\nu, \alpha, \epsilon)$ is convex.*

Proof. Suppose that $\curlyvee_i \in \mathcal{B}_k(\nu, \alpha, \epsilon)$, $i = 1, 2$ achieve the formulas $\curlyvee_1(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n$ and $\curlyvee_2(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n$ respectively. It is adequate to show that the linear combination function

$$G(\xi) = w_1 \curlyvee_1 (\xi) + w_2 \curlyvee_2 (\xi), \quad \xi \in \cup$$

belongs to $\mathcal{B}_k(\nu, \alpha, \epsilon)$, where $w_1 > 0, w_2 > 0$ and $w_1 + w_2 = 1$.

By the definition of $G(\xi)$, a computation yields that

$$G(\xi) = \xi + \sum_{n=2}^{\infty} (w_1 a_n + w_2 b_n) \xi^n$$

then under the formal $\mathbf{C}^k_{\nu,\alpha}$, we obtain

$$\begin{aligned}\mathbf{C}^k_{\nu,\alpha} G(\xi) = \xi + &\sum_{n=2}^{\infty} (w_1 a_n + w_2 b_n) \\ &\times \Big[(1-\alpha) C^m_{m+n-1} + \alpha \left(\frac{\kappa_1(\nu,\xi) + (-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi) + \kappa_0(\nu,\xi)}\right)^k \Big] \xi^n.\end{aligned}$$

By considering the derivative, we have

$$\begin{aligned}
&\Re\Big\{(\mathbf{C}^k_{\nu,\alpha}G(\xi))'\Big\}\\
&=1+w_1\Re\Big\{\sum_{n=2}^{\infty}n\Big[(1-\alpha)C^m_{m+n-1}+\alpha\left(\frac{\kappa_1(\nu,\xi)+(-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)^k\Big]a_n\xi^{n-1}\Big\}\\
&+w_2\Re\Big\{\sum_{n=2}^{\infty}n\Big[(1-\alpha)C^m_{m+n-1}+\alpha\left(\frac{\kappa_1(\nu,\xi)+(-1)^{n+1}\kappa_0(\nu,\xi)}{\kappa_1(\nu,\xi)+\kappa_0(\nu,\xi)}\right)^k\Big]b_n\xi^{n-1}\Big\}\\
&>1+w_1(\epsilon-1)+w_2(\epsilon-1)=\epsilon.
\end{aligned}$$

□

4. Applications

A set of complex differential equations is an assembly of differential equations with complex variables. The most important study in this direction is to establish the existence and uniqueness results. There are diffident types of techniques including the utility of majors and minors (or subordination and superordination concepts) (see [12]). Investigation of ODEs in the complex domain suggests the detection of novel transcendental special functions, which currently called a Briot–Bouquet differential equation (BBDE)

$$\omega\,\Upsilon(\xi)+(1-\omega)\frac{\xi(\Upsilon(\xi))'}{\Upsilon(\xi)}=\hbar(\xi),$$

$$\Big(\hbar(0)=\Upsilon(0),\,\omega\in[0,1],\ \xi\in\cup,\,\Upsilon\in\bigwedge\Big).$$

In this place, we shall generalize the BBDE into a symmetric BBDE by using SCDO. Numerous presentations of these comparisons in the geometric function model have recently achieved in [12].

Needham and McAllister [16] presented a two-dimensional complex holomorphic dynamical system, pleasing the 2-D form

$$\xi_t=\Theta(\xi,\omega);\quad \omega_t=\Theta(\xi,w),\quad \xi,\omega\in\cup$$

and t is in any real interval. Development application of the BBDE seemed newly, with different approaches (see [17]) to solve the equation of electronic nano-shells (see [18]). Controlled by the situation effort of traditional shell theory, the transposition fields of the nano-shell take the dynamic system

$$\xi_t=\Theta(\xi,\omega)+\Theta_\theta(\xi,\omega);\quad \omega_t=\Theta(\xi,\omega)+\Theta_\theta(\bar{\xi},\bar{\omega}),\quad \xi,\omega\in\cup,$$

where θ is the angles between ξ and ω and their conjugates.

Our purpose is to generalize this class of equation by utilizing the SCDO and establish its properties by applying the subordination concept. In view of (2), we have the generalized BBDE

$$\omega\,\Upsilon(\xi)+(1-\omega)\Big(\frac{\xi(\mathcal{S}^k_\nu\Upsilon(\xi))'}{\mathcal{S}^k_\nu\Upsilon(\xi)}\Big)=\hbar(\xi),\quad \hbar(0)=\Upsilon(0),\,\xi\in\cup. \tag{8}$$

The subordination settings and alteration bounds for a session of SCDO specified in the following formula. A trivial resolution of (8) is given when $\omega=1$. Consequently, our vision is to carry out the situation, $\Upsilon\in\bigwedge$ and $\omega=0$. We proceed to present the behavior of the solution of (8).

Theorem 8. *For $\Upsilon \in \bigwedge$, $\alpha \in [0,\infty)$ and $\hbar$ is univalent convex in $\cup$ if*

$$\Big(\frac{\xi(\mathcal{S}_\nu^k)'}{\mathcal{S}_\nu^k \Upsilon(\xi)}\Big) \prec \hbar(\xi), \quad \xi \in \cup \tag{9}$$

then

$$\mathcal{S}_\nu^k \Upsilon(\xi) \prec \xi \exp\Big(\int_0^\xi \frac{\hbar(\top(\xi))-1}{\ell} d\ell\Big),$$

where $\top$ is a Schwarz function in $\cup$. In addition, we have

$$|\xi| \exp\left(\int_0^1 \frac{\hbar(\top(-\sigma))-1}{\sigma} d\sigma\right) \leq \left|\Delta_\alpha^m \Upsilon(\xi)\right| \leq |\xi| \exp\left(\int_0^1 \frac{\hbar(\top(\sigma))-1}{\sigma} d\sigma\right).$$

Proof. The subordination in (9) implies that there occurs a Schwarz function $\top$ such that

$$\Big(\frac{\xi(\mathcal{S}_\nu^k \Upsilon(\xi))'}{\mathcal{S}_\nu^k \Upsilon(\xi)}\Big) = \hbar(\top(\xi)), \quad \xi \in \cup.$$

This yields the inequality

$$\Big(\frac{\xi(\mathcal{S}_\nu^k \Upsilon(\xi))'}{\mathcal{S}_\nu^k \Upsilon(\xi)}\Big) - \frac{1}{\xi} = \frac{\hbar(\top(\xi))-1}{\xi}.$$

By making the integrated operating, we have

$$\log\left(\frac{\mathcal{S}_\nu^k \Upsilon(\xi)}{\xi}\right) = \int_0^\xi \frac{\hbar(\top(\ell))-1}{\ell} d\ell. \tag{10}$$

Consequently, we have

$$\log \mathcal{S}_\nu^k \Upsilon(\xi) = \left(\int_0^\xi \frac{\hbar(\top(\ell))-1}{\ell} d\ell\right) - \log(\xi). \tag{11}$$

A calculation brings the next subordination relation

$$\mathcal{S}_\nu^k \Upsilon(\xi) \prec \xi \exp\Big(\int_0^\xi \frac{\hbar(\top(\ell))-1}{\ell} d\ell\Big).$$

Moreover, the function $\hbar$ translates the disk $0|\xi|\sigma \leq 1$ into a convex symmetric domain toward the x-axis; in other words, we have

$$\hbar(-\sigma|\xi|) \leq \Re(\hbar(\top(\sigma\xi))) \leq \hbar(\sigma|\xi|), \quad \sigma \in (0,1], |\xi| \neq \sigma,$$

which implies the inequalities:

$$\hbar(-\sigma) \leq \hbar(-\sigma|\xi|), \quad \hbar(\sigma|\xi|) \leq \hbar(\sigma)$$

and

$$\int_0^1 \frac{\hbar(\top(-\sigma|\xi|))-1}{\sigma} d\sigma \leq \Re\Big(\int_0^1 \frac{\hbar(\top(\sigma))-1}{\sigma} d\sigma\Big) \leq \int_0^1 \frac{\hbar(\top(\sigma|\xi|))-1}{\eta} d\sigma.$$

By employing (10) and the last inequality, we arrive at

$$\int_0^1 \frac{\hbar(\top(-\sigma|\xi|))-1}{\sigma} d\sigma \leq \log\Big|\frac{\mathcal{S}_\nu^k \Upsilon(\xi)}{\xi}\Big| \leq \int_0^1 \frac{\hbar(\top(\sigma|\xi|))-1}{\sigma} d\sigma.$$

This equivalence to the fact

$$\exp\Big(\int_0^1 \frac{\hbar(\top(-\sigma|\xi|))-1}{\sigma}d\sigma\Big) \leq \Big|\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big| \leq \exp\Big(\int_0^1 \frac{\hbar(\top(\sigma|\xi|))-1}{\sigma}d\sigma\Big).$$

□

We note that the condition of Theorem 8, which the BB formula subordinates by a convex univalent function $\hbar$ can be replaced by a general condition as follows:

Theorem 9. *Suppose that* $\curlyvee \in \bigwedge$, $\alpha \in [0,\infty)$ *and* $0 < v \leq 1$. *If*

$$\Big(\frac{\xi(\mathcal{S}_\nu^k \curlyvee(\xi))' - \xi}{\mathcal{S}_\nu^k \curlyvee(\xi)}\Big) \prec \frac{2v\xi}{1+\xi}, \quad \xi \in \cup \tag{12}$$

then

$$\Big|\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} - 1\Big| \leq v. \tag{13}$$

Moreover, define the term

$$v := \frac{1}{(1-r)^{\mathfrak{v}}}, \quad 0 < r < 1,$$

for some positive constant $\mathfrak{v}$, *then*

$$\Big|\Big(\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi}\Big)'\Big| \leq \frac{\mathfrak{v}+1}{(1-r)^{\mathfrak{v}+1}}. \tag{14}$$

Proof. In view of Lemma 3, we have the subordination inequality

$$\frac{\mathcal{S}_\nu^k \curlyvee(\xi)}{\xi} \prec 1 + v\xi.$$

Since the result is sharp, then directly, we obtain the inequality (13). Consequently, by ([19], lemma 5.1.3), we have the inequality (14). □

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