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Asymptotic Approximations of Ratio Moments Based on Dependent Sequences

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Abstract: The widely orthant dependent (WOD) sequences are very weak dependent sequences of random variables. For the weighted sums of non-negative m-WOD random variables, we provide asymptotic expressions for their appropriate inverse moments which are easy to calculate. As applications, we also obtain asymptotic expressions for the moments of random ratios. It is pointed out that our random ratios can include some models such as change-point detection. Last, some simulations are illustrated to test our results.

Keywords: asymptotic approximation; inverse moments; WOD random variables; ratio moments

MSC: 60E15; 62E20

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1. Introduction

In this paper, we will study the asymptotic expressions for inverse moments of weighted sums based on dependent random variables. As applications, we obtain some asymptotic approximations to the random ratios which include some change-point models. In the following, let's introduce some inverse moment models and ratio models.

1.1. Inverse Moment Models and Ratio Models

First, we consider a weighted inverse moment model. Let $\{Z_n, n \geq 1\}$ be a non-negative and independent sequence of random variables and denote $\sigma_n^2 = \sum_{i=1}^n \mathrm{Var}(Z_i)$. For some $\eta > 0$, it is assumed that $\{Z_n, n \geq 1\}$ satisfies a Linderberg-type condition

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{n} E\{Z_i^2 I(Z_i > \eta\sigma_n)\} \to 0, \text{ as } n \to \infty.$$

Then, Wu et al. [1] obtained the asymptotic approximation of inverse moment that for all real numbers $a > 0$ and $\alpha > 0$,

$$E\Big(\frac{1}{(a + X_n)^\alpha}\Big) \sim \frac{1}{(a + EX_n)^\alpha}, \tag{1}$$

where $X_n = \frac{1}{\sigma_n} \sum_{i=1}^n Z_i$. Here $c_n \sim d_n$ means $c_n/d_n \to 1$ as $n \to \infty$. Usually, the left side formula in (1) is more difficult to calculate than the right side formula in (1). Under some regular conditions, the inverse moment can be approximated by the inverse of moment. The inverse moments can be used in many areas such as reliability life testing, evaluation of risks of estimators, insurance and financial mathematics, etc. (see [1–4] and references therein). Therefore, many authors have pay attention on the research of inverse moments. For example, [5–9] extended the results of Wu et al. [1] to some nonnegative dependent random variables.

Second, we consider a non-weighted inverse moment model. Shi et al. [10] establish the asymptotic approximation of inverse moment (1), where weighted case $X_n = \frac{1}{\sigma_n}\sum_{i=1}^{n} Z_i$ was replaced by the non-weighted case $X_n = \sum_{i=1}^{n} Z_i$. Yang et al. [11] extended Shi et al. [10] and obtained the convergence rates for inverse moments.

Third, let us consider a general weighted inverse moment model. Yang et al. [12] obtained the inverse moment result (1), where $X_n = \frac{1}{\sigma_n}\sum_{i=1}^{n} Z_i$ is replaced by a general weighted case $X_n = \sum_{i=1}^{n} w_{ni} Z_i$, and $\{w_{ni}, 1 \le i \le n, n \ge 1\}$ is a triangular array of non-negative weights. Li et al. [13] studied this general weighted case of inverse moment under nonnegative widely orthant dependent (WOD) random variables.

Fourth, let us recall some the ratio models. Shi et al. [14] used the inverse moment method to consider the ratio models such as $\frac{Z_i}{X_n}$ and $\frac{X_k}{X_n}$, where $X_k = \sum_{i=1}^{k} Z_i$ for $1 \le k \le n$. They obtained some asymptotic expressions for the means of $E\frac{Z_i}{X_n}$ and $E\frac{X_k}{X_n}$. For all $a > 0$, Yang et al. [12] investigated a general ratio $\frac{X_n}{a+Y_n}$, where $X_n = \sum_{i=1}^{n} w_{ni} Z_i$, $Y_n = \sum_{i=1}^{n} Z_i$ and $\{w_{ni}\}$ are non-negative weights. Some asymptotic expressions for ratios $E(\frac{X_n}{a+Y_n})$ and $E(\frac{X_n}{a+Y_n})^2$ were presented in Yang et al. [12].

To proceed the study of inverse moment models and ratio models, we give some definitions of dependent random variables in the next subsection.

1.2. Definitions of WOD and m-WOD

Definition 1. *For the random variables $\{Z_n, n \ge 1\}$, if there exists a finite sequence of real numbers $\{g_u(n), n \ge 1\}$ such that for each $n \ge 1$ and for all $z_i \in (-\infty, \infty), 1 \le i \le n$,*

$$P\Big(\bigcap_{i=1}^{n}(Z_i > z_i)\Big) \le g_u(n)\prod_{i=1}^{n} P(Z_i > z_i),$$

then we say that the random variables $\{Z_n, n \ge 1\}$ are widely upper orthant dependent (WUOD), if there exists a finite sequence of real numbers $\{g_l(n), n \ge 1\}$ such that for each $n \ge 1$ and for all $z_i \in (-\infty, \infty), 1 \le i \le n$,

$$P\Big(\bigcap_{i=1}^{n}(Z_i \le z_i)\Big) \le g_l(n)\prod_{i=1}^{n} P(Z_i \le z_i),$$

then we say that the random variables $\{Z_n, n \ge 1\}$ are widely lower orthant dependent (WLOD). If the random variables $\{Z_n, n \ge 1\}$ are both WUOD and WLOD, then we say that the random variables $\{Z_n, n \ge 1\}$ are widely orthant dependent (WOD).

Definition 2. *Let $m \ge 1$ be a fixed integer. A sequence of random variables $\{Z_n, n \ge 1\}$ is said to be m-WOD if for any $n \ge 2$ and any $i_1, i_2, \ldots, i_n$, in such that $|i_k - i_j| \ge m$ for all $1 \le k \ne j \le n$, we have that $Z_{i_1}, Z_{i_2}, \ldots, Z_{i_n}$ are WOD.*

On one hand, the notion of WOD random variables was introduced by Wang and Cheng [15] for risk models. On the other hand, Hu et al. [16] introduced m-negatively associated (m-NA) notion and gave its application to study the complete convergence. Inspired by WOD and m-NA, we give the notion of m-WOD and study the inverse moments and ratio moments based on these dependent sequences.

If $g_u(n) = g_l(n) = M \ge 1$, then WOD sequences become extended negatively dependent (END) sequences which were introduced by Liu [17]). In addition, END sequences contain several negative dependent sequences such as negatively orthant dependent (NOD, see Lehmann [18]), negatively superadditive dependent (NSD, see Hu [19]) and negatively associated (NA, see Joag-Dev and Proschan [20]). For $n \ge 2$, if joint distribution of $\{Z_1, \ldots, Z_n\}$ is a multivariate normal (Gaussian) distribution, then $\{Z_1, \ldots, Z_n\}$ is NA if and only if its components are non-positively correlated (see Joag-Dev and Proschan [20], Bulinski and Shaskin [21]). Obviously, Laplace distribution has heavier tails than Gaussian tails (see Davidian and Giltinan [22], Kozubowskia et al. [23]). So it is an interesting research how to use multivariate Laplace distribution to construct NA sequence. Likely, if joint distribution of $\{Z_1, \ldots, Z_n\}$ is a Frank copula

$$F(z_1, z_2, \ldots, z_n) = -\frac{1}{\theta} \ln \Big(1 + \frac{(e^{-\theta z_1} - 1) \ldots (e^{-\theta z_n} - 1)}{(e^{-\theta} - 1)^{n-1}}\Big), \ \theta < 0$$

where $0 < z_i < 1, 1 \le i \le n$ and $n \ge 2$, then $\{Z_1, \ldots, Z_n\}$ is END (see Ko and Tang [24], Yang et al. [25]). A lot of attention have been paid on the study of negative dependent sequences such WOD, END, NO, NA, m-NA, m-END, m-linearly negative quadrant dependent (m-LNQD), etc. We can refer to [26–35] the references therein.

1.3. Our Models

Let $\{w_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of non-negative and non-random weights. For all $a > 0$ and $\alpha > 0$, we proceed to study the general weighted inverse moment model

$$E\Big(\frac{1}{(a + X_n)^\alpha}\Big) \sim \frac{1}{(a + EX_n)^\alpha}, \tag{2}$$

where $X_n = \sum_{i=1}^n w_{ni} Z_i$ and $\{Z_n, n \ge 1\}$ is a sequence of non-negative m-WOD random variables. As an important application, we establish some asymptotic expressions for the means of ratios

$$\frac{X_n}{a + Y_n} \tag{3}$$

and

$$\frac{X_n}{Y_n}, \tag{4}$$

where $Y_n = \sum_{i=1}^n Z_i$. Yang et al. [12] studied the inverse moment model (2) and ratio model (3) based on the independent sequence $\{Z_n, n \ge 1\}$ and weighted condition $\max_{1 \le i \le n} w_{ni} = O(1)$. Li et al. [13] also studied this inverse moment model (2) based on WOD sequence $\{Z_n, n \ge 1\}$ and weighted condition $\max_{1 \le i \le n} w_{ni} = O(1)$. Models (2), (3) and (4) can be regarded as ratio models. So in this paper, we investigate the ratio models (2)–(4) based on m-WOD sequence $\{Z_n, n \ge 1\}$. But the weight w_{ni} does not require the condition $\max_{1 \le i \le n} w_{ni} = O(1)$. It is pointed that the ratio model (4) is an important statistic model, which can be used to detect change-points. For example, by taking $w_{ni} = \frac{i-1}{n-1}$ in (4), one can get the estimator

$$T_n = \frac{\sum_{i=1}^n \frac{i-1}{n-1} Z_i}{\sum_{i=1}^n Z_i} = \frac{X_n}{Y_n}$$

by Hsu [36]. Let $1 \le k \le n$. If $w_{ni} = 1$ for $1 \le i \le k$ and $w_{ni} = 0$ for $k < i \le n$, one can obtain the estimator

$$R_{nk} = \frac{Y_k}{Y_n} - \frac{k}{n}$$

by Inclán and Tiao [37]. Hsu [36] and Inclán and Tiao [37] used these estimators T_n and R_{nk} to do the research of change-point detection. For more details of ratio models, one can refer to [10,11,38,39], etc.

The rest of this paper is organized as follows. Some asymptotic approximation to inverse moments for (2) and ratio moments for (3) and (4) are presented in Section 2. We do some simulations in Section 3, which are agreed with the results obtained in this paper. Last, the conclusions and the proofs are given in Sections 4 and 5, respectively. Throughout the paper, let C_1, C_2 be some positive constants not depending on n and $C_1(m, q), C_2(m, q)$ be some positive constants depending only on m and q.

2. Results

In the following, let $\{Z_n, n \geq 1\}$ be a sequence of non-negative m-WOD random variables with the dominating coefficient $g(n) = \max\{g_u(n), g_l(n)\}$ and $\{w_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of non-negative and non-random weights. Denote $X_n = \sum_{i=1}^n w_{ni} Z_i$, $\mu_n = EX_n$ and $\mu_{n,s} = \sum_{i=1}^n w_{ni} E[Z_i I(Z_i \leq \mu_n^s)]$ for some $0 < s < 1$. In order to study the inverse moment model (2), we list some assumptions as follows.

Assumption 1. *(A.1) $g(n) = O(\mu_n^\beta)$ for some $\beta \geq 0$;*
(A.2) $\max\limits_{1 \leq i \leq n} w_{ni} = O(\mu_n^\gamma)$ for some γ such that $\gamma < \frac{1}{2}(1-s)$ and $0 < s < 1$;
(A.3) $\mu_n \to \infty$ as $n \to \infty$;
(A.4) $\mu_n \sim \mu_{n,s}$ as $n \to \infty$.

Theorem 1. *Let $EZ_n < \infty$ for all $n \geq 1$ and Assumptions (A.1)–(A.4) hold true. Then*

$$E\Big(\frac{1}{(a+X_n)^\alpha}\Big) \sim \frac{1}{(a+EX_n)^\alpha} \tag{5}$$

holds for all constants $a > 0$ and $\alpha > 0$.

Theorem 2. *For some $r > 2$, suppose that $EZ_n^r < \infty$ for all $n \geq 1$ and*

$$\sum_{i=1}^n E|Z_i - EZ_i|^r = O((\mu_n)^{r/2}) \quad and \quad \sum_{i=1}^n Var(Z_i) = O(\mu_n). \tag{6}$$

Let the Assumptions (A.1)–(A.4) be fulfilled and $\gamma + \beta/r < 1/2$. Then for all $a > 0$ and $\alpha > 0$,

$$\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}} - 1 = O\Big(\frac{1}{(a+EX_n)^{1-2\gamma-2\beta/r}}\Big) \tag{7}$$

and for all $a > 0$ and $\alpha > 1$,

$$E\Big(\frac{X_n}{(a+X_n)^\alpha}\Big) \Big/ \frac{EX_n}{(a+EX_n)^\alpha} - 1 = O\Big(\frac{1}{(a+EX_n)^{1-2\gamma-2\beta/r}}\Big). \tag{8}$$

Next, we apply Theorems 1 and 2 to evaluate the means ratios models (3) and (4). Let $X_n = \sum_{i=1}^n w_{ni} Z_i$, $Y_n = \sum_{i=1}^n Z_i$, $\nu_n = EY_n$ and $\nu_{n,s} = \sum_{i=1}^n E[Z_i I(Z_i \leq \nu_n^s)]$ for some $0 < s < 1$.

Assumption 2. *(B.1) $g(n) = O(\nu_n^\beta)$ for some $0 \leq \beta \leq 2$;*
(B.2) $\max\limits_{1 \leq i \leq n} w_{ni} = O(\nu_n^\lambda)$ for some λ satisfying $\lambda - \beta/2 < 1$.
(B.3) $\nu_n \to \infty$ as $n \to \infty$.
(B.4) $\nu_n \sim \nu_{n,s}$ as $n \to \infty$.
(B.5) Let $EZ_n^4 < \infty$ for all $n \geq 1$ and

$$\sum_{i=1}^n E|Z_i - EZ_i|^4 = O(\nu_n^2) \quad and \quad \sum_{i=1}^n Var(Z_i) = O(\nu_n). \tag{9}$$

Theorem 3. *Let Assumptions (B.1)–(B.5) hold true. Then for all $a > 0$,*

$$E\Big(\frac{X_n}{a+Y_n}\Big) = \frac{EX_n}{a+EY_n} + O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big), \tag{10}$$

where $\lambda - \beta/2 < 1$ and $0 \le \beta \le 2$. Moreover, if $(EY_n)^{\lambda+\beta/2} = o(EX_n)$, then for all $a > 0$,

$$E\Big(\frac{X_n}{a+Y_n}\Big)\Big/\frac{EX_n}{a+EY_n} - 1 = O\Big(\frac{(EY_n)^{\lambda+\beta/2}}{EX_n}\Big). \tag{11}$$

As an application of Theorems 2 and 3, we obtain the following Corollary 1 which does not contain the parameter a. The proof is complex and it is used (7) in Theorem 2 (with $a = 1$ and $\alpha = 1$) and (10) in Theorem 3 (with $a = 1$). Please see the details in Section 5. But the ratio model X_n/Y_n is important which can be used in change-point detection models (see details of (4) in Section 1).

Corollary 1. *Assume that Assumptions (B.1)–(B.5) be satisfied. Then*

$$E\Big(\frac{X_n}{Y_n}\Big) = \frac{EX_n}{EY_n} + O\Big(\frac{1}{(EY_n)^{1-\lambda-\beta/2}}\Big), \tag{12}$$

where $\lambda - \beta/2 < 1$ and $0 \le \beta \le 2$. In addition, if $(EY_n)^{\lambda+\beta/2} = o(EX_n)$, then

$$E\Big(\frac{X_n}{Y_n}\Big)\Big/\frac{EX_n}{EY_n} - 1 = O\Big(\frac{(EY_n)^{\lambda+\beta/2}}{EX_n}\Big). \tag{13}$$

Remark 1. *Since the dependent sequences of NA, NA, NOD, NSD, END and END, are m-WOD with $m = 1$ and $g(n) = O(1)$, all the results obtained in this paper hold true for all of them.*

3. Simulations

In this section, we will do some simulations for the ratio models (2) and (4) under different weighted sequences. For convenience, X and Y have the same distribution denoted by $X \overset{d}{=} Y$. Let $\sigma_{01}^2 > 0$ and $\sigma_{02}^2 > 0$. For $n \ge 1$, it is assumed that there is a change-point k^* such that

$$\xi_j \overset{d}{=} N(0,\sigma_{01}^2),\ \ j = 1,2,\ldots,k^*,\ \ \xi_j \overset{d}{=} N(0,\sigma_{02}^2),\ \ j = k^*+1,\ldots,n, \tag{14}$$

and for some $\rho \in (0,1)$,

$$\mathrm{Cov}(\xi_i,\xi_j) = -\rho^{-|i-j|},\ \ \forall\, i \ne j. \tag{15}$$

It can be seen that $\{\xi_1,\xi_2,\ldots,\xi_n\}$ is a NA sequence. Let $x^+ = \max(x,0)$ and $x^- = \max(-x,0)$. Then $\{\xi_1^-,\xi_2^-,\ldots,\xi_n^-\}$ and $\{\xi_1^+,\xi_2^+,\ldots,\xi_n^+\}$ are nonnegative NA sequences, which imply that they are m-WOD sequences with $m = 1$ and dominating coefficient $g_u(n) = g_l(n) = g(n) = 1$. Thus, in the following, we do the simulations for

$$\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}},\ \ \forall a,\alpha > 0 \tag{16}$$

and

$$E\Big(\frac{X_n}{Y_n}\Big)\Big/\frac{EX_n}{EY_n}, \tag{17}$$

where $X_n = \sum_{i=1}^n w_{ni}Z_i$, $Y_n = \sum_{i=1}^n Z_i$, $Z_i = \xi_i^+$ and ξ_i is defined in (14) and (15), $1 \le i \le n$. The weight sequences are listed as follows

(C.1) $w_{ni} = \frac{1}{\sigma_n}, 1 \le i \le n, \sigma_n^2 = \sum_{i=1}^n Var(Z_i), n \ge 2$;
(C.2) $w_{ni} = 1, 1 \le i \le n, n \ge 2$;
(C.3) $w_{ni} = \frac{i-1}{n-1}, 1 \le i \le n, n \ge 2$;
(C.4) $w_{ni} = i^{1/4}, 1 \le i \le n, n \ge 2$.

First, we do simulate (16). For all $a > 0$ and $\alpha > 0$, it has $(a+EX_n)^{-\alpha} \to 0$ as $n \to \infty$, by the fact that $EX_n \to \infty$ as $n \to \infty$. So we use MATLAB software to obtain the empirical values for the "ratio" $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ in (16) by repeating 10000 experiments and obtain the Figures 1–4 The label of y-axis "ratio" is empirical values of (16) and the label of x-axis "sample sizes" is the number of sample n. For $\rho = 0.3$ and $\sigma_{01}^2 = \sigma_{02}^2 = 1$ (or $k^* = n/2$, $\sigma_{01}^2 = 1, \sigma_{02}^2 = 2$), we take $n = 10, 20, 30, \ldots, 200$, $a = 0.2, 1$ and $\alpha = 0.5, 2$, and obtain the results of Figures 1–4 for the weighted cases (C.1)–(C.4), respectively.

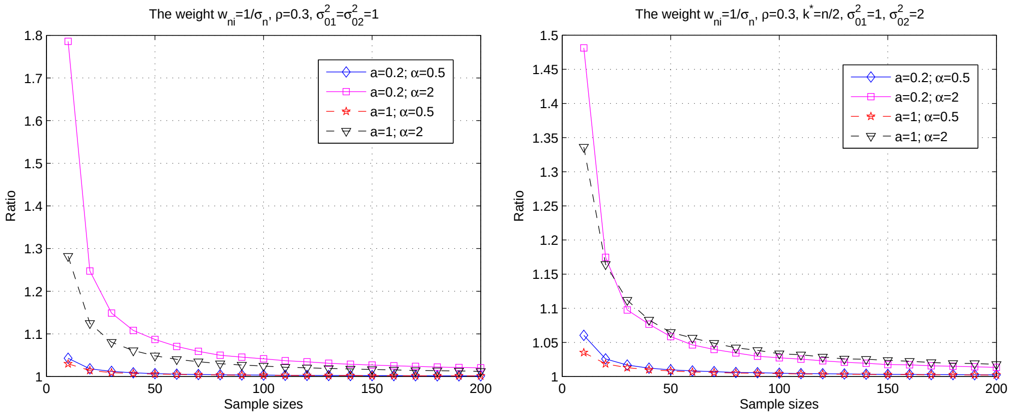

Figure 1. Empirical values of ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ for the weighted case $w_{ni} = \frac{1}{\sigma_n}$, where $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1, \sigma_{02}^2 = 2$.

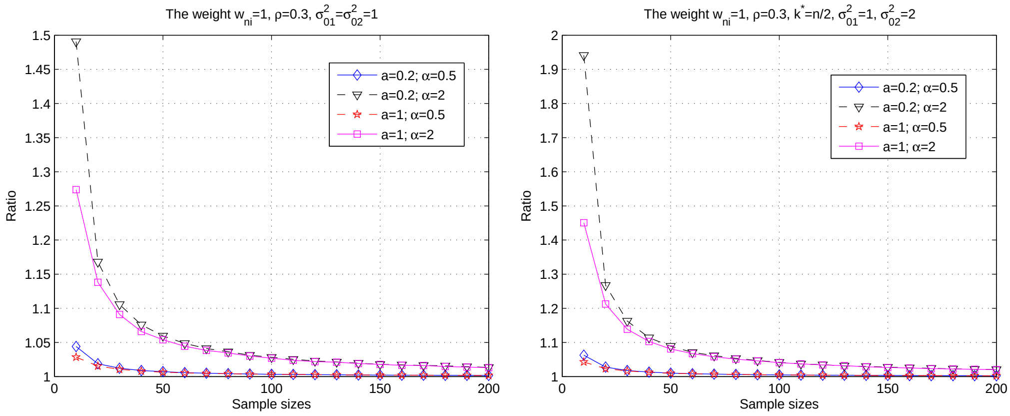

Figure 2. Empirical values of ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ for the weighted case $w_{ni} = 1$, where $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1, \sigma_{02}^2 = 2$.

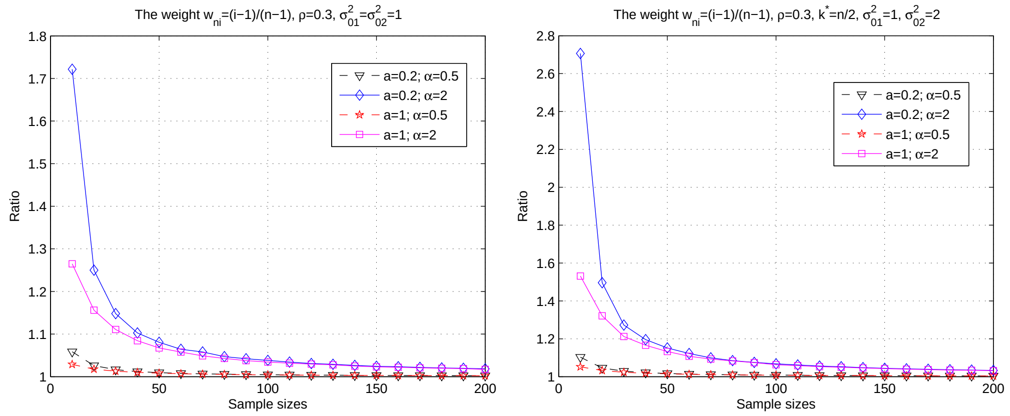

Figure 3. Empirical values of ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ for the weighted case $w_{ni} = \frac{i-1}{n-1}$, where $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1, \sigma_{02}^2 = 2$.

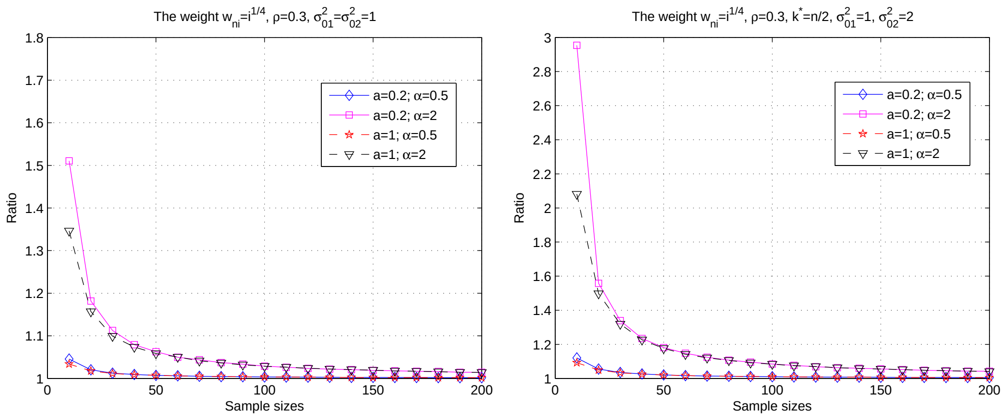

Figure 4. Empirical values of ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ for the weighted case $w_{ni} = i^{1/4}$, where $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1$, $\sigma_{02}^2 = 2$.

By Figures 1–4, it can be found that the ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}} \geq 1$ for all $a > 0$ and $\alpha > 0$, since it is guaranteed by Jensen's inequality. As the sample n increases, the ratio $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}}$ decreases to one. So, the results of Figures 1–4 are agreed with formulas of (5) and (7).

Second, we do the simulation for (17). It can be seen that $E(\frac{X_n}{Y_n})/\frac{EX_n}{EY_n} = 1$ under the equal weight cases of (C.1) and (C.2). So we do the simulation $E(\frac{X_n}{Y_n})/\frac{EX_n}{EY_n}$ under the unequal weight cases of (C.3) and (C.4), and obtain Figure 5.

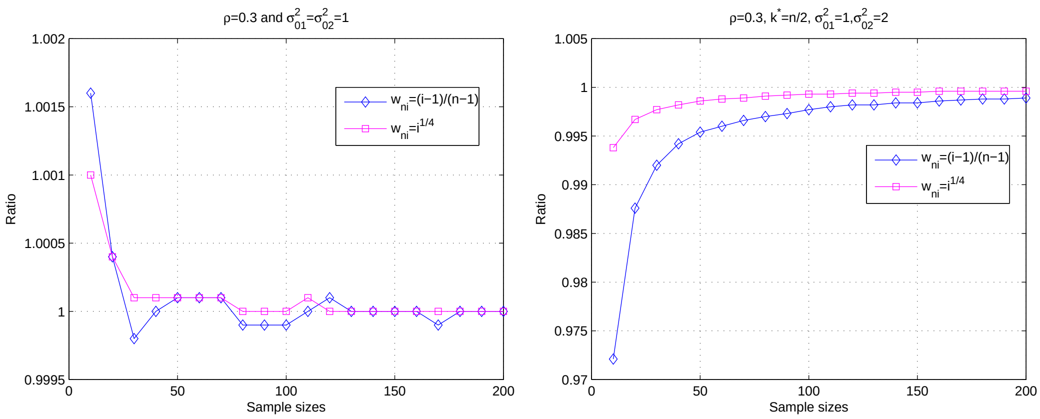

Figure 5. Empirical values of ratio $E(\frac{X_n}{Y_n})/\frac{EX_n}{EY_n}$ for the weighted cases $w_{ni} = \frac{i-1}{n-1}$ and $w_{ni} = i^{1/4}$, $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1$, $\sigma_{02}^2 = 2$.

In Figure 5, the label of y-axis "ratio" is empirical values of (17) and the label of x-axis "sample sizes" is the number of sample n taking values $n = 10, 20, 30, \ldots, 200$. By Figure 5, the ratio $E(\frac{X_n}{Y_n})/\frac{EX_n}{EY_n}$ goes to one as the sample n increases, which agrees with (13).

Third, we do some box plots of X_n/Y_n with the weight case of (C.3). If $\sigma_{01}^2 = \sigma_{02}^2 = 1$ in (14), then it is easy to check

$$\frac{EX_n}{EY_n} = \frac{\sum_{i=1}^n \frac{i-1}{n-1} EZ_i}{\sum_{i=1}^n EZ_i} = \frac{1}{2}. \tag{18}$$

Let $\lfloor x \rfloor$ denote the largest integer not exceeding x. Likewise, we take $k^* = \lfloor \frac{n}{2} \rfloor$, $\sigma_{01}^2 = 1$, $\sigma_{02}^2 = 2$ and obtain that

$$\frac{EX_n}{EY_n} \to \frac{5 - 2\sqrt{2}}{4} \approx 0.5429, \quad \text{as} \quad n \to \infty. \tag{19}$$

Thus, we take $\rho = 0.3$, $n = 200, 400, \ldots, 1000$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$ (or $k^* = n/2$ and $\sigma_{01}^2 = 1$, $\sigma_{02}^2 = 2$) and obtain the box plots in Figure 6, by repeating 10000 experiments.

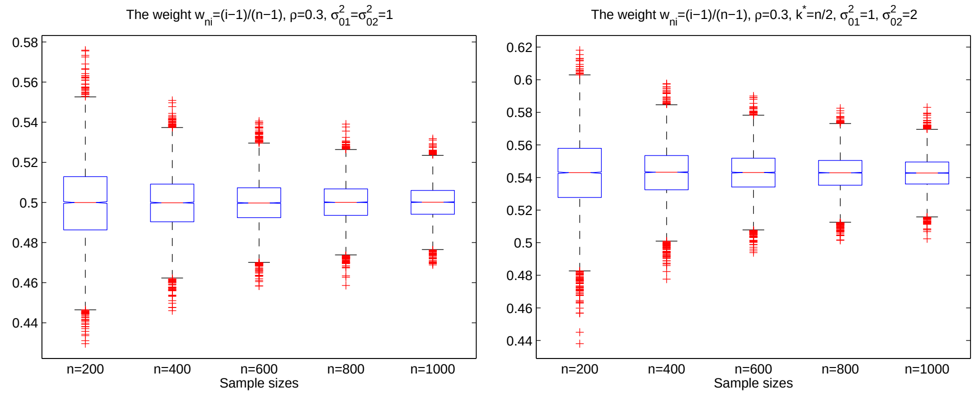

Figure 6. Box plots of $\frac{X_n}{Y_n}$ for the weighted case $w_{ni} = \frac{i-1}{n-1}$, where $\rho = 0.3$, $\sigma_{01}^2 = \sigma_{02}^2 = 1$, or $k^* = n/2$ and $\sigma_{01}^2 = 1$, $\sigma_{02}^2 = 2$.

By Figure 6, it is easy to see that the results are agreed with (18) and (19), respectively.

4. Conclusions

On one hand, by taking $\max\limits_{1\le i\le n} w_{ni} = O(1)$ (i.e., $\gamma = 0$) in Theorems 1 and 2, one can obtain the results of (5), (7) and (8) with $\gamma = 0$, which imply Theorems 2.1 and 2.2 of Li et al. [13] for nonnegative WOD sequences. Obviously, the condition (A.2) is weaker than the one of $\max\limits_{1\le i\le n} w_{ni} = O(1)$. So Theorems 1 and 2 generalize and improve the results of Li et al. [13]. On the other hand, independent sequence is a m-WOD sequence with $g(n) = 1$. So by taking $\max\limits_{1\le i\le n} w_{ni} = O(1)$ (i.e., $\lambda = 0$) and $\beta = 0$ in Theorem 3, we have (10) with $\lambda = 0$ and $\beta = 0$, which implies Theorem 2.3 of Yang et al. [12] for nonnegative independent sequences. Furthermore, by using Theorems 2 and 3, we obtain Corollary 1 which does not contain the parameter a. It can be easy to use in practice (for example change-point detection). In addition, we also do some simulations to check our results such as $\frac{E(a+X_n)^{-\alpha}}{(a+EX_n)^{-\alpha}} \to 1$ and $E(\frac{X_n}{Y_n})/\frac{EX_n}{EY_n} \to 1$ based on the different weight cases.

5. Proofs of Main Results

Lemma 1. *(Wang et al. [26] [Proposition 1.1]) Let $\{Z_n, n \ge 1\}$ be WUOD (WLOD) with dominating coefficients $g_u(n)$, $n \ge 1$ ($g_l(n), n \ge 1$). If $\{f_n(\cdot), n \ge 1\}$ are nondecreasing, then $\{f_n(Z_n), n \ge 1\}$ are still WUOD (WLOD) with dominating coefficients $g_u(n), n \ge 1$ ($g_l(n), n \ge 1$); if $\{f_n(\cdot), n \ge 1\}$ are nonincreasing, then $\{f_n(Z_n), n \ge 1\}$ are WLOD (WUOD) with dominating coefficients $g_l(n), n \ge 1$ ($g_u(n), n \ge 1$).*

Corollary 2. *Let $\{Z_n, n \ge 1\}$ be a sequence of m-WOD random variables. If $\{f_n(\cdot), n \ge 1\}$ are nondecreasing (nonincreading) functions, then $\{f(Z_n), n \ge 1\}$ are also m-WOD random variables with same dominating coefficients.*

Proof of Corollary 2. According to the definition of m-WOD, a sequence of m-WOD $\{Z_n, n \ge 1\}$ can decompose to m sequences of WOD, i.e. $\{Z_1, Z_{1+m}, Z_{1+2m}, \ldots\}$, $\{Z_2, Z_{2+m}, Z_{2+2m}, \ldots\}$, $\ldots$, $\{Z_m, Z_{2m}, Z_{3m}, \ldots\}$. Then, by Lemma 1, the sequences $\{f(Z_1), f(Z_{1+m}), f(Z_{1+2m}), \ldots\}$, $\{f(Z_2), f(Z_{2+m}), f(Z_{2+2m}), \ldots\}, \ldots, \{f(Z_m), f(Z_{2m}), f(Z_{3m}), \ldots\}$ are also WOD sequence with same dominating coefficients. Thus, by the definition of m-WOD again, $\{f(Z_n), n \ge 1\}$ are also m-WOD random variables with same dominating coefficients. □

Lemma 2. *(Wang et al. [28] [Corollary 2.3]) Let $q \geq 2$ and $\{Z_n, n \geq 1\}$ be a mean zero sequence of WOD random variables with dominating coefficient $g(n) = \max\{g_u(n), g_l(n)\}$ and $E|Z_n|^q < \infty$ for all $n \geq 1$. Then for all $n \geq 1$, there exist positive constants $C_1(q)$ and $C_2(q)$ depending only on q such that*

$$E\Big|\sum_{i=1}^{n} Z_i\Big|^q \leq C_1(q)\sum_{i=1}^{n} E|Z_i|^q + C_2(q)g(n)\Big(\sum_{i=1}^{n} EZ_i^2\Big)^{q/2}. \tag{20}$$

Corollary 3. *Let $q \geq 2$ and $\{Z_n, n \geq 1\}$ be a mean zero sequence of m-WOD random variables with dominating coefficient $g(n) = \max\{g_u(n), g_l(n)\}$ and $E|Z_n|^q < \infty$ for all $n \geq 1$. Then for all $n \geq 1$, there exist a positive constant $C(m,q)$ depending only on m and q such that*

$$E\Big|\sum_{i=1}^{n} Z_i\Big|^q \leq C(m,q)\Big\{\sum_{i=1}^{n} E|Z_i|^q + g(n)\Big(\sum_{i=1}^{n} EZ_i^2\Big)^{q/2}\Big\}. \tag{21}$$

Proof of Corollary 3. By the definition of m-WOD, the sums of m-WOD can be written as

$$S_n = \sum_{i=1}^{n} Z_i = \sum_{j=1}^{m}\sum_{i=1}^{i_j} Z_{t_i}^{(j)},$$

where $\sum_{i=1}^{i_j} Z_{t_i}^{(j)}$ are the sums of WOD, $1 \leq j \leq m$. Then, by C_r inequality and (20) in Lemma 2, it is easy to establish that

$$\begin{aligned} E|S_n|^q &\leq m^{q-1}\sum_{j=1}^{m} E\Big|\sum_{i=1}^{i_j} Z_{t_i}^{(j)}\Big|^q \\ &\leq m^{q-1}\sum_{j=1}^{m}\Big\{C_1(q)\sum_{i=1}^{i_j} E|Z_{t_i}^{(j)}|^q + C_2(q)g(n)\Big(\sum_{i=1}^{i_j} E(Z_{t_i}^{(j)})^2\Big)^{q/2}\Big\} \\ &\leq m^q\Big\{C_1(q)\sum_{i=1}^{n} E|Z_i|^q + C_2(q)g(n)\Big(\sum_{i=1}^{n} E(Z_i)^2\Big)^{q/2}\Big\} \\ &\leq C(m,q)\Big\{\sum_{i=1}^{n} E|Z_i|^q + g(n)\Big(\sum_{i=1}^{n} EZ_i^2\Big)^{q/2}\Big\}. \end{aligned}$$

So the proof of (21) is finished. □

Proof of Theorem 1. The proof is similar to the one of Theorem 2.1 in Li et al. [13], where Li et al. [13] consider WOD case with weight $\max_{1\leq i\leq n} w_{ni} = O(1)$. In this paper, we consider general case (A.2) and give the key parts of proofs. By Jensen's inequality, we have $E(a + X_n)^{-\alpha} \geq (a + EX_n)^{-\alpha}$ for all $a > 0$ and $\alpha > 0$. Thus, in order to prove (5), we only have to show that for $\forall\, \delta \in (0,1)$,

$$\limsup_{n\to\infty}\{(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha}\} \leq (1-\delta)^{-\alpha}, \tag{22}$$

(or see Li et al. [13]). By (A.4), there exist a $n(\delta) > 0$ such that for all $\delta \in (0,1)$ and

$$\sum_{i=1}^{n} w_{ni} E[Z_i I(Z_i > \mu_n^s)] \leq \frac{\delta}{4}\sum_{i=1}^{n} w_{ni} EZ_i, \quad n \geq n(\delta). \tag{23}$$

We can break $E(a + X_n)^{-\alpha}$ into two formulas:

$$E(a + X_n)^{-\alpha} := Q_1 + Q_2, \tag{24}$$

where

$$Q_1 = E[(a+X_n)^{-\alpha} I(U_n \le \mu_n - \delta\mu_n)],\ \ Q_2 = E[(a+X_n)^{-\alpha} I(U_n > \mu_n - \delta\mu_n)],$$
$$U_n = \sum_{i=1}^{n} w_{ni}[Z_i I(Z_i \le \mu_n^s) + \mu_n^s I(Z_i > \mu_n^s)].$$

Since $X_n \ge U_n$, we obtain $Q_2 \le E[(a+X_n)^{-\alpha} I(X_n > \mu_n - \delta\mu_n)] \le (a+\mu_n-\delta\mu_n)^{-\alpha}$. So, by (A.3), we establish

$$\limsup_{n\to\infty}\{(a+EX_n)^{\alpha} Q_2\} \le \limsup_{n\to\infty}\{(a+\mu_n)^{\alpha}(a+\mu_n-\delta\mu_n)^{-\alpha}\} = (1-\delta)^{-\alpha}. \tag{25}$$

It follows from (23) that $|\mu_n - EU_n| \le \delta\mu_n/2$ for all $n \ge n(\delta)$. Denote $Z_{n,i} = w_{ni}[Z_i I(Z_i \le \mu_n^s) + \mu_n^s I(Z_i > \mu_n^s)]$, $1 \le i \le n$. So, by Corollary 2, $\{Z_{n,i} - EZ_{n,i}, 1 \le i \le n\}$ are also mean zero m-WOD random variables with dominating coefficient $g(n)$. Thus, by Markov's inequality, Corollary 3 and C_r inequality, it has that for all $q > 2$ and $n \ge n(\delta)$,

$$\begin{aligned}
Q_1 &= E[(a+X_n)^{-\alpha} I(U_n \le \mu_n - \delta\mu_n)] \\
&\le a^{-\alpha} P(U_n \le \mu_n - \delta\mu_n) \\
&\le a^{-\alpha} P(|EU_n - U_n| \ge \delta\mu_n/2) \\
&\le \frac{2^q C_1(m,q)}{\delta^q}\mu_n^{-q}\Big\{\sum_{i=1}^{n} E|Z_{n,i}|^q + g(n)(\sum_{i=1}^{n} EZ_{n,i}^2)^{q/2}\Big\} \\
&\le \frac{C_2(m,q)}{\delta^q}\mu_n^{-q}\Big\{\sum_{i=1}^{n} w_{ni}^q[E(Z_i^q I(Z_i \le \mu_n^s)) + \mu_n^{sq} EI(Z_i > \mu_n^s)]\Big\} \\
&\quad + \frac{C_3(m,q)}{\delta^q}\mu_n^{-q} g(n)\Big\{\sum_{i=1}^{n} w_{ni}^2[E(Z_i^2 I(Z_i \le \mu_n^s)) + \mu_n^{2s} EI(Z_i > \mu_n^s)]\Big\}^{q/2} \\
&\le \frac{C_2(m,q)(\max\limits_{1\le i\le n} w_{ni})^{q-1}}{\delta^q}\mu_n^{-q}\Big\{\mu_n^{s(q-1)}\sum_{i=1}^{n} w_{ni}[E(Z_i I(Z_i \le \mu_n^s)) + E(Z_i I(Z_i > \mu_n^s))]\Big\} \\
&\quad + \frac{C_3(m,q)(\max\limits_{1\le i\le n} w_{ni})^{q/2}}{\delta^q}\mu_n^{-q} g(n)\Big\{\mu_n^{s}\sum_{i=1}^{n} w_{ni}[E(Z_i I(Z_i \le \mu_n^s)) + E(Z_i I(Z_i > \mu_n^s))]\Big\}^{q/2} \\
&=: I_{n1} + I_{n2}.
\end{aligned} \tag{26}$$

If $\gamma < 0$, then by the conditions (A.1)–(A.3) and (26) that

$$I_{n1} + I_{n2} \le \frac{C_4(m,q)}{\delta^q}\mu_n^{-q}[\mu_n^{s(q-1)}\mu_n + \mu_n^{\beta}(\mu_n^s\mu_n)^{q/2}] = \frac{C_4(m,q)}{\delta^q}[\mu_n^{-(q-1)(1-s)} + \mu_n^{\beta-\frac{q}{2}(1-s)}]. \tag{27}$$

Since $q > 2$, it has $q - 1 > \frac{q}{2}$. We take $q > \max\{2, 2(\alpha+\beta)/(1-s)\}$ in (27) and obtain that

$$\limsup_{n\to\infty}\{(a+EX_n)^{\alpha} Q_1\} \le \limsup_{n\to\infty}\{(a+\mu_n)^{\alpha}\frac{C_5(m,q)}{\delta^q}[\mu_n^{-(q-1)(1-s)} + \mu_n^{\beta-\frac{q}{2}(1-s)}]\} = 0. \tag{28}$$

Similarly, if $0 \le \gamma < \frac{1}{2}(1-s)$, then by the conditions (A.1)–(A.3) and (26) that

$$\begin{aligned}
I_{n1} + I_{n2} &\le \frac{C_6(m,q)}{\delta^q}\mu_n^{-q}[\mu_n^{\gamma(q-1)}\mu_n^{s(q-1)}\mu_n + \mu_n^{\gamma q/2}\mu_n^{\beta}(\mu_n^s\mu_n)^{q/2}] \\
&\le \frac{C_7(m,q)}{\delta^q}[\mu_n^{\gamma(q-1)-(q-1)(1-s)} + \mu_n^{\gamma(q-1)+\beta-\frac{q}{2}(1-s)}].
\end{aligned} \tag{29}$$

In view of $0 \le \gamma < \frac{1}{2}(1-s)$, $0 < s < 1$, $\alpha > 0$ and $\beta \ge 0$, if $q > \max\{2, \frac{\alpha+\beta}{\frac{1}{2}(1-s)-\gamma}\}$, then

$$q(\gamma - \frac{1}{2}(1-s)) < \frac{\alpha+\beta}{\frac{1}{2}(1-s)-\gamma}(\gamma - \frac{1}{2}(1-s)) = -\alpha - \beta \le -\alpha - \beta + \gamma,$$

which implies that

$$\gamma(q-1) + \beta - \frac{q}{2}(1-s) + \alpha < 0.$$

So we take $q > \max\{2, \frac{\alpha+\beta}{\frac{1}{2}(1-s)-\gamma}\}$ in (29) and obtain that

$$\begin{aligned} & \limsup_{n\to\infty}\{(a+EX_n)^\alpha Q_1\} \\ \le \; & \limsup_{n\to\infty}\{(a+\mu_n)^\alpha \frac{C_8(m,q)}{\delta^q}[\mu_n^{\gamma(q-1)-(q-1)(1-s)} + \mu_n^{\gamma(q-1)+\beta-\frac{q}{2}(1-s)}]\} = 0. \end{aligned} \tag{30}$$

So, by (24), (25), (28) and (30), (22) holds true. □

Proof of Theorem 2. By Taylor expansion, it can be checked that

$$E\Big(\frac{1}{(a+X_n)^\alpha}\Big) = \frac{1}{(a+EX_n)^\alpha} + \frac{\alpha(\alpha+1)}{2}E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}\Big), \tag{31}$$

where ξ_n lies between X_n and μ_n. Next, we will verify that

$$E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}\Big) = O\Big(\frac{1}{(a+EX_n)^{\alpha+1-2\gamma-2\beta/r}}\Big), \tag{32}$$

where $\gamma + 2\beta/r < 1$ and $r > 2$. It is easy to see that

$$E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}\Big) = E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}I(X_n > \mu_n)\Big) + E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}I(X_n \le \mu_n)\Big). \tag{33}$$

On the one hand, for some $r > 2$, it can be argued by Corollary 3, (6) and (A.2) that

$$\begin{aligned} & E\Big(\frac{(X_n-EX_n)^2}{(a+\xi_n)^{\alpha+2}}I(X_n > \mu_n)\Big) \le \frac{1}{(a+\mu_n)^{\alpha+2}}E(X_n-EX_n)^2 \\ \le \; & \frac{1}{(a+\mu_n)^{\alpha+2}}(E|X_n-EX_n|^r)^{2/r} = \frac{1}{(a+\mu_n)^{\alpha+2}}\Big(E\Big|\sum_{i=1}^n w_{ni}(Z_i-EZ_i)\Big|^r\Big)^{2/r} \\ \le \; & \frac{C_1(m,r)}{(a+\mu_n)^{\alpha+2}}\Big\{\sum_{i=1}^n w_{ni}^r E|Z_i-EZ_i|^r + g(n)\Big(\sum_{i=1}^n w_{ni}^2 \mathrm{Var}(Z_i)\Big)^{r/2}\Big\}^{2/r} \\ \le \; & \frac{C_1(m,r)(\max_{1\le i\le n} w_{ni})^2}{(a+\mu_n)^{\alpha+2}}\Big\{\sum_{i=1}^n E|Z_i-EZ_i|^r + g(n)\Big(\sum_{i=1}^n \mathrm{Var}(Z_i)\Big)^{r/2}\Big\}^{2/r} \\ \le \; & C_2(m,r)\Big(\frac{(EX_n)^{2\gamma+1+2\beta/r}}{(a+EX_n)^{\alpha+2}}\Big) = O\Big(\frac{1}{(a+EX_n)^{\alpha+1-2\gamma-2\beta/r}}\Big), \end{aligned} \tag{34}$$

where $2\gamma + 2\beta/r < 1$.

On the other hand, for some $r > 2$, by the proof of (34), it can be found that

$$\begin{aligned} & E\Big(\frac{(X_n - EX_n)^2}{(a+\xi_n)^{\alpha+2}} I(X_n \le \mu_n)\Big) \le E\Big(\frac{(X_n - EX_n)^2}{(a+X_n)^{\alpha+2}}\Big) \\ \le\ & [E|X_n - EX_n|^r]^{2/r}[E(a+X_n)^{\frac{(-\alpha-2)r}{r-2}}]^{\frac{r-2}{r}} \quad \text{(by Hölder inequality)} \\ \le\ & C_1(m,r)\Big(E\Big|\sum_{i=1}^n w_{ni}(Z_i - EZ_i)\Big|^r\Big)^{2/r}[(a+EX_n)^{\frac{(-\alpha-2)r}{r-2}}]^{\frac{r-2}{r}} \quad \text{(by Theorem 2.1)} \\ =\ & O\Big(\frac{1}{(a+EX_n)^{\alpha+1-2\gamma-2\beta/r}}\Big), \end{aligned} \tag{35}$$

where $2\gamma + 2\beta/r < 1$. Thus, (32) follows from (33) to (35). Combining (31) with (32), we obtain the result of (7) with $\gamma + 2\beta/r < 1$.

Last, similar to the proof of (2.3) in Li et al. [13], by (31) and (32), one can obtain that for all $a > 0$ and $\alpha > 1$,

$$\begin{aligned} E\Big(\frac{X_n}{(a+X_n)^\alpha}\Big) &= \frac{1}{(a+EX_n)^{\alpha-1}} + O\Big(\frac{1}{(a+EX_n)^{\alpha-2\gamma-2\beta/r}}\Big) \\ &\quad -\Big\{\frac{a}{(a+EX_n)^\alpha} + O\Big(\frac{1}{(a+EX_n)^{\alpha+1-2\gamma-2\beta/r}}\Big)\Big\} \\ &= \frac{EX_n}{(a+EX_n)^\alpha} + O\Big(\frac{1}{(a+EX_n)^{\alpha-2\gamma-2\beta/r}}\Big). \end{aligned} \tag{36}$$

Hence, by (36), (8) holds true. □

Proof of Theorem 3. The proof is similar to the one of Theorem 2.3 in Yang et al. [12], where Yang et al. [12] consider the independent case and weight condition $\max_{1\le i\le n} w_{ni} = O(1)$. In this paper, weight condition (B.2) is very weak. So we give the complete proofs here. By bivariate Taylor expansion, one can establish that

$$\begin{aligned} E\Big(\frac{X_n}{a+Y_n}\Big) &= \frac{EX_n}{a+EY_n} - E\Big(\frac{(X_n - EX_n)(Y_n - EY_n)}{\eta_n^2}\Big) + E\Big(\frac{\xi_n(Y_n - EY_n)^2}{\eta_n^3}\Big) \\ &:= \frac{EX_n}{a+EY_n} + EH_{n1} + EH_{n2}, \end{aligned} \tag{37}$$

where ξ_n lies between X_n and EX_n, η_n lies between $a+Y_n$ and $a+EY_n$, and $H_{n1} := -\frac{(X_n-EX_n)(Y_n-EY_n)}{\eta_n^2}$, $H_{n2} := \frac{\xi_n(Y_n-EY_n)^2}{\eta_n^3}$. Next, we will verify that

$$E|H_{n1}| = O\big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\big) \text{ and } E|H_{n2}| = O\big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\big), \tag{38}$$

where $\lambda - \beta/2 < 1$. It follows from (B.2) that

$$EX_n = \sum_{i=1}^n w_{ni}EZ_i \le \max_{1\le i\le n} w_{ni} \sum_{i=1}^n EZ_i \le C_1(EY_n)^{1+\lambda}. \tag{39}$$

By Corollary 3 and conditions (B.1) and (B.5), one can check that

$$\begin{aligned} E(X_n-EX_n)^4 &\le C_1(m,4)(\max_{1\le i\le n} w_{ni})^4\Big(\sum_{i=1}^n E(Z_i-EZ_i)^4+g(n)\Big(\sum_{i=1}^n \mathrm{Var}(Z_i)\Big)^2\Big) \\ &\le C_2(m)(EY_n)^{2+\beta+4\lambda}. \quad (40) \\ E(Y_n-EY_n)^4 &\le C_3(m,4)\Big(\sum_{i=1}^n E(Z_i-EZ_i)^4+g(n)\Big(\sum_{i=1}^n \mathrm{Var}(Z_i)\Big)^2\Big) \\ &\le C_4(m)(EY_n)^{2+\beta}, \quad (41) \end{aligned}$$

which implies

$$\begin{aligned} E(X_n-EX_n)^2 &\le [E(X_n-EX_n)^4]^{1/2}\le C_5(m)(EY_n)^{1+2\lambda+\beta/2}, \quad (42) \\ E(Y_n-EY_n)^2 &\le [E(Y_n-EY_n)^4]^{1/2}\le C_6(m)(EY_n)^{1+\beta/2}. \quad (43) \end{aligned}$$

Combining Hölder inequality with (42) and (43), we obtain that

$$\begin{aligned} & E|H_{n1}I(a+Y_n>a+EY_n)| \\ =\ & E\Big|\frac{(X_n-EX_n)(Y_n-EY_n)}{\eta_n^2}I(a+Y_n>a+EY_n)\Big| \\ \le\ & \frac{1}{(a+EY_n)^2}E|(X_n-EX_n)(Y_n-EY_n)| \\ \le\ & \frac{1}{(a+EY_n)^2}[E(X_n-EX_n)^2]^{1/2}[E(Y_n-EY_n)^2]^{1/2} \\ \le\ & \frac{C_1(m)(EY_n)^{1+\lambda+\beta/2}}{(a+EY_n)^2}=O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \quad (44) \end{aligned}$$

By Hölder inequality, (40), (41), (B.1), (B.3), (B.4), we have that

$$\begin{aligned} & E|H_{n1}I(a+Y_n\le a+EY_n)| \\ =\ & E\Big|\frac{(X_n-EX_n)(Y_n-EY_n)}{\eta_n^2}I(a+Y_n\le a+EY_n)\Big| \\ \le\ & E\Big|\frac{(X_n-EX_n)(Y_n-EY_n)}{(a+Y_n)^2}\Big| \\ \le\ & \{E[(X_n-EX_n)^2(Y_n-EY_n)^2]\}^{1/2}\Big[E\Big(\frac{1}{(a+Y_n)^4}\Big)\Big]^{1/2} \\ \le\ & \{[E(X_n-EX_n)^4]^{1/2}[E(Y_n-EY_n)^4]^{1/2}\}^{1/2}\Big[E\Big(\frac{1}{(a+Y_n)^4}\Big)\Big]^{1/2} \\ \le\ & C_1(m)(EY_n)^{1+\lambda+\beta/2}\Big(\frac{1}{(a+EY_n)^4}\Big)^{1/2} \quad \text{(by Theorem 2.1 with } \gamma=0) \\ =\ & O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \quad (45) \end{aligned}$$

Similarly, we apply Hölder inequality, (39) and (41), then obtain that

$$\begin{aligned} & E|H_{n2}I(X_n \le EX_n, a+Y_n \le a+EY_n)| \\ = \;& E\Big|\frac{\xi_n(Y_n-EY_n)^2}{\eta_n^3}I(X_n \le EX_n, a+Y_n \le a+EY_n)\Big| \\ \le \;& EX_nE\Big(\frac{(Y_n-EY_n)^2}{(a+Y_n)^3}\Big) \\ \le \;& EX_n[E(Y_n-EY_n)^4]^{1/2}\Big[E\Big(\frac{1}{(a+Y_n)^6}\Big)\Big]^{1/2} \\ \le \;& C_1(m)(EY_n)^{2+\lambda+\beta/2}\Big(\frac{1}{(a+EY_n)^6}\Big)^{1/2} \quad \text{(by Theorem 2.1)} \\ = \;& O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \end{aligned} \tag{46}$$

It follows from (39) and (43) that

$$\begin{aligned} & E|H_{n2}I(X_n \le EX_n, a+Y_n > a+EY_n)| \\ = \;& E\Big|\frac{\xi_n(Y_n-EY_n)^2}{\eta_n^3}I(X_n \le EX_n, a+Y_n > a+EY_n)\Big| \\ \le \;& \frac{EX_n}{(a+EY_n)^3}E(Y_n-EY_n)^2 \le \frac{C_1(m)(EY_n)^{2+\lambda+\beta/2}}{(a+EY_n)^3} \\ = \;& O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \end{aligned} \tag{47}$$

It can be seen that

$$\begin{aligned} & E|H_{n2}I(X_n > EX_n, a+Y_n \le a+EY_n)| \\ = \;& E\Big|\frac{\xi_n(Y_n-EY_n)^2}{\eta_n^3}I(X_n > EX_n, a+Y_n \le a+EY_n)\Big| \\ \le \;& E\Big(\frac{X_n(Y_n-EY_n)^2}{(a+Y_n)^3}\Big) \\ \le \;& E\Big|\frac{(X_n-EX_n)(Y_n-EY_n)^2}{(a+Y_n)^3}\Big| + EX_nE\Big(\frac{(Y_n-EY_n)^2}{(a+Y_n)^3}\Big) \\ := \;& K_{n1}+K_{n2}. \end{aligned} \tag{48}$$

For K_{n1}, by Hölder inequality, (40) and (41),

$$\begin{aligned} |K_{n1}| \;\le\;& \{E[|X_n-EX_n|^{4/3}|Y_n-EY_n|^{8/3}]\}^{3/4}\Big[E\Big(\frac{1}{(a+Y_n)^{12}}\Big)\Big]^{1/4} \\ \le\;& C_1\{[E(X_n-EX_n)^4]^{1/3}[E(Y_n-EY_n)^4]^{2/3}\}^{3/4}\Big[\frac{1}{(a+EY_n)^3}\Big] \quad \text{(by Theorem 2.1)} \\ \le\;& C_2(m)\{[(EY_n)^{2+4\lambda+\beta}]^{1/3}[(EY_n)^{2+\beta}]^{2/3}\}^{3/4}\Big[\frac{1}{(a+EY_n)^3}\Big] \\ =\;& O\Big(\frac{1}{(a+EY_n)^{\frac{3}{2}-\lambda-\frac{3}{4}\beta}}\Big) = O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\frac{\beta}{2}+\frac{1}{2}-\frac{\beta}{4}}}\Big) \\ =\;& O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\frac{\beta}{2}}}\Big), \end{aligned} \tag{49}$$

since $\beta \leq 2$. Similarly, for K_{n2}, by Hölder inequality, (39) and (41),

$$\begin{aligned} |K_{n2}| &\leq EX_n[E(Y_n - EY_n)^4]^{1/2}\Big[E\Big(\frac{1}{(a+Y_n)^6}\Big)\Big]^{1/2} \\ &\leq C_2(m)(EY_n)^{2+\lambda+\beta/2}\Big(\frac{1}{(a+EY_n)^3}\Big) \quad \text{(by Theorem 2.1)} \\ &= O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \end{aligned} \tag{50}$$

In addition, we have by Hölder inequality, (39), (41)–(43) and $\beta \leq 4$ that

$$\begin{aligned} & E|H_{n2}I(X_n > EX_n, a+Y_n > a+EY_n)| \\ = \; & E\Big|\frac{\xi_n(Y_n - EY_n)^2}{\eta_n^3} I(X_n > EX_n, a+Y_n > a+EY_n)\Big| \\ \leq \; & \frac{1}{(a+EY_n)^3}E[X_n(Y_n-EY_n)^2] \\ \leq \; & \frac{1}{(a+EY_n)^3}E|(X_n-EX_n)(Y_n-EY_n)^2| \\ & +\frac{EX_n}{(a+EY_n)^3}E(Y_n-EY_n)^2 \\ \leq \; & \frac{1}{(a+EY_n)^3}[E(X_n-EX_n)^2]^{1/2}[E(Y_n-EY_n)^4]^{1/2} \\ & +\frac{EX_n}{(a+EY_n)^3}E(Y_n-EY_n)^2 \\ \leq \; & \frac{C_1(m)}{(a+EY_n)^{\frac{3}{2}-\lambda-\frac{3}{4}\beta}} + \frac{C_2(m)}{(a+EY_n)^{1-\lambda-\beta/2}} \\ = \; & O\Big(\frac{1}{(a+EY_n)^{1-\lambda-\beta/2}}\Big). \end{aligned} \tag{51}$$

Therefore, (38) follows from (44) to (51) immediately. Combining (37) with (38), the proof of (10) is completed. □

Proof of Corollary 1. In view of (B.2), there exists a positive constant C such that

$$X_n = \sum_{i=1}^n w_{ni}Z_i \leq \max_{1\leq i\leq n} w_{ni}\sum_{i=1}^n Z_i \leq C(EY_n)^\lambda Y_n := C_nY_n. \tag{52}$$

Then

$$\frac{X_n}{1+Y_n} \leq \frac{X_n}{Y_n} \leq \frac{C_n+X_n}{1+Y_n}, \tag{53}$$

which implies

$$E\Big(\frac{X_n}{1+Y_n}\Big) \leq E\Big(\frac{X_n}{Y_n}\Big) \leq E\Big(\frac{C_n+X_n}{1+Y_n}\Big). \tag{54}$$

By (10) in Theorem 3 with $a=1$, we establish that

$$E\Big(\frac{X_n}{1+Y_n}\Big) = \frac{EX_n}{1+EY_n} + O\Big(\frac{1}{(EY_n)^{1-\lambda-\beta/2}}\Big), \tag{55}$$

where $\lambda - \beta/2 < 1$. In addition, by (7) in Theorem 2 with $a=1$, $\alpha=1$, $\gamma=0$ and $r=4$, we obtain that

$$E\Big(\frac{1}{1+Y_n}\Big) = \frac{1}{1+EY_n} + O\Big(\frac{1}{(EY_n)^{2-\beta/2}}\Big). \tag{56}$$

Thus, by (52), (55) and (56), it can be checked that

$$
\begin{aligned}
E\Big(\frac{C_n+X_n}{1+Y_n}\Big) &= C_n E\Big(\frac{1}{1+Y_n}\Big)+E\Big(\frac{X_n}{1+Y_n}\Big)\\
&= C_n\Big(\frac{1}{1+EY_n}+O\Big(\frac{1}{(EY_n)^{2-\beta/2}}\Big)\Big)+\frac{EX_n}{1+EY_n}+O\Big(\frac{1}{(EY_n)^{1-\lambda-\beta/2}}\Big)\\
&= \frac{EX_n}{1+EY_n}+\frac{C_n}{1+EY_n}+O\Big(\frac{C_n}{(EY_n)^{2-\beta/2}}\Big)+O\Big(\frac{1}{(EY_n)^{1-\lambda-\beta/2}}\Big),\\
&= \frac{EX_n}{1+EY_n}+O\Big(\frac{1}{(EY_n)^{1-\lambda-\beta/2}}\Big),
\end{aligned}
\tag{57}
$$

where $\lambda+\beta/2<1$. Furthermore, in view of $\lim\limits_{n\to\infty}\left(\frac{EX_n}{1+EY_n}\times\frac{EY_n}{EX_n}\right)=1$ and (54)-(57), the proof of (12) is finished to prove. Last, by $(EY_n)^{\lambda+\beta/2}=o(EX_n)$ and (54)-(57), we have that

$$
E\Big(\frac{X_n}{Y_n}\Big)\Big/\frac{EX_n}{EY_n}=1+O\Big(\frac{(EY_n)^{\lambda+\beta/2}}{EX_n}\Big).
$$

Thus, (13) is completely proved. □

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