mathematics

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Competition-Independence Game and Domination Game

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Received: 31 January 2020; Accepted: 17 February 2020; Published: 5 March 2020

Abstract: The domination game is played on a graph by two players, Dominator and Staller, who alternately choose a vertex of G. Dominator aims to finish the game in as few turns as possible while Staller aims to finish the game in as many turns as possible. The game ends when all vertices are dominated. The game domination number, denoted by $\gamma_g(G)$ (respectively $\gamma'_g(G)$), is the total number of turns when both players play optimally and when Dominator (respectively Staller) starts the game. In this paper, we study a version of this game where the set of chosen vertices is always independent. This version turns out to be another game known as the competition-independence game. The competition-independence game is played on a graph by two players, Diminisher and Sweller. They take turns in constructing maximal independent set M, where Diminisher tries to minimize $|M|$ and Sweller tries to maximize $|M|$. Note that, actually, it is the domination game in which the set of played vertices is independent. The competition-independence number, denoted by $I_d(G)$ (respectively $I_s(G)$) is the optimal size of the final independent set in the competition-independence game if Diminisher (respectively Sweller) starts the game. In this paper, we check whether some well-known results in the domination game hold for the competition-independence game. We compare the competition-independence numbers to the game domination numbers. Moreover, we provide a family of graphs such that many parameters are equal. Finally, we present a realization result on the competition-independence numbers.

Keywords: domination game; competition-independence game

1. Introduction

A *dominating set* of a graph G is a set S of vertices of G such that every vertex in G is an element in S or is adjacent to an element in S. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. A set S is *independent* if no two vertices in S are adjacent. The *independence number* of a graph G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. An *independent dominating set* of a graph G is a dominating set of G which is independent. The *independent domination number* of G, denoted by $i(G)$, is the cardinality of a minimum independent dominating set of G.

In 2010, Brešar, Klavžar, and Rall [1] first introduced the domination game. The *domination game* is played on a graph G by two players, Dominator and Staller, who alternately choose a vertex of G in such a way that at least one new vertex is dominated. The game ends when all vertices are dominated. Dominator aims to finish the game in as few turns as possible while Staller aims to finish the game in as many turns as possible. The *game domination number*, denoted by $\gamma_g(G)$ (respectively $\gamma'_g(G)$), is the total number of turns when both players play optimally and when Dominator (respectively Staller) starts the game.

Also, they investigated the relationship between domination number and game domination numbers of a graph. They proved that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ for any graph G. In References [1,2], the authors showed that the two game domination numbers of a graph can differ by at most one. The domination game is being studied extensively since it was introduced. Kinnersley, West, and Zamani [2] posted the 3/5 conjecture: for any isolate-free graph G, $\gamma_g(G) \leq 3|V(G)|/5$, and it has attracted several researches (See References [3–7]). In addition, there are researchers studying the domination game numbers on several classes of graphs such as paths, cycles, forests, disjoint union of graphs, split graphs, etc. (See References [4,8–11] for examples).

In 2015, a variation of the domination game called *total domination game* was introduced by Henning, Klavžar, and Rall [12]. In this version of the game, domination is replaced by total domination where a chosen vertex dominates its neighbors but not itself (See References [13–15]). The concept of total domination naturally gives rise to a variety of domination games such as Z-domination game, L-domination game, and LL-domination game (See Reference [16]). Many results in the domination game also hold for any of the above variations. For example, the difference between the number of moves in Dominator-start game and in Staller-start game is at most 1 for any of the above games.

In this paper, we are interested in studying the domination game such that the set of played vertices must be independent. In other words, a player can only play on an undominated vertex. Since an independent dominating set is a maximal independent set and vice versa, this independent version is the same as the competition-independence game which was introduced by Philips and Slater [17] in 2001.

The *competition-independence game* is played on a graph by two players, Diminisher and Sweller. They take turns in constructing maximal independent set M, where Diminisher tries to minimize $|M|$ and Sweller tries to maximize $|M|$. The *competition-independence number*, denoted by $I_d(G)$ (respectively $I_s(G)$), is the optimal size of the final independent set in the competition-independence game if Diminisher (respectively Sweller) starts the game. For the rest of the paper, we will use Dominator instead of Diminisher and Staller instead of Sweller.

Philips and Slater [18] provided the competition-independence numbers of a path. Consequently, the competition-independence numbers of a cycle of n vertices established since the first move in C_n produces P_{n-3}. In 2018, the competition-independence game in trees was studied by Goddard and Henning [19]. They provided the maximum and minimum values of the competition-independence game for trees of maximum degree 3.

In this paper, we check whether some well-known results in the domination game hold for the competition-independence game. We compare the competition-independence numbers to the game domination numbers. Also, we give some classes of graphs in which many parameters are equal. Finally, we establish a realization result of the competition-independence numbers.

2. Relationship between the Competition-Independence Numbers and Other Parameters

A fundamental tool for analyzing the original domination game is the Continuation Principle, which is proved in Reference [2]. We start this section by showing that the Continuation Principle for the domination game does not hold for the competition-independence game. Next, we compare the competition-independence numbers to the game domination numbers. Also, we provide a family of graphs such that many parameters are equal.

Theorem 1 ([2] (Continuation Principle))**.** *Let G be a (partially dominated) graph and let A and B be subsets of $V(G)$. Let G_A and G_B be partially dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma'_g(G_A) \leq \gamma'_g(G_B)$.*

This result is very intuitive and natural. The more coverage the chosen vertices have, the fewer vertices are needed to be chosen to dominate the remaining undominated vertices. This result also holds for total domination game, Z-domination game, L-domination game, and LL-domination game [16].

However, it does not hold for the competition-independence game as the restriction of independence sometimes prevents players from making good moves. We present two simple counterexamples here.

Example 2. *Consider the complete bipartite graph $K_{1,3}$ with vertex set $\{u, v_1, v_2, v_3\}$ and edge set $\{uv_1, uv_2, uv_3\}$. If $A = \{u, v_1\}$ and $B = \emptyset$, then $I_d(G_A) = 2$ and $I_d(G_B) = 1$.*

Example 3. *Let G be the graph obtained from the complete bipartite graph $K_{3,3}$ with partite sets V_1 and V_2 by deleting a perfect matching. If $A = V_1$ and $B = \emptyset$, then $I_s(G_A) = 3$ and $I_s(G_B) = 2$.*

Now, we compute the competition-independence numbers of complete multipartite graphs and complete bipartite graphs minus perfect matchings. These families of graphs will appear as examples throughout this paper.

Lemma 4. *Let $n_1, n_2, ..., n_m$ be positive integers. Then, $I_d(K_{n_1,n_2,...,n_m}) = \min\{n_1, n_2, ..., n_m\}$ and $I_s(K_{n_1,n_2,...,n_m}) = \max\{n_1, n_2, ..., n_m\}$.*

Proof. Let $V_{n_1}, V_{n_2}, ..., V_{n_m}$ be the partite sets of $K_{n_1,n_2,...,n_m}$ with $|V_{n_i}| = n_i$ for $1 \le i \le m$. If a player starts the game by playing a vertex in V_{n_i} for some $1 \le i \le m$, then both players must alternately choose vertices in the set V_{n_i} so the game will end in n_i moves. Hence, $I_d(K_{n_1,n_2,...,n_m}) = \min\{n_1, n_2, ..., n_m\}$ and $I_s(K_{n_1,n_2,...,n_m}) = \max\{n_1, n_2, ..., n_m\}$. □

Lemma 5. *Let G be the graph obtained from $K_{n,n}$ by deleting a perfect matching where $n \ge 2$. Then, $I_d(G) = n$ and $I_s(G) = 2$.*

Proof. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be the partite sets of the $K_{n,n}$, where a_i and b_i are not adjacent in G for $i \in \{1, 2, ..., n\}$.

We first show that $I_d(G) = n$. Without loss of generality, assume that Dominator starts the game by playing vertex $a_1 \in A$. Then, each vertex in $B - \{b_1\}$ is dominated. Staller plays another vertex in A. Now all vertices in B are dominated. Then, Dominator and Staller must alternately play all vertices in A. Consequently, $I_d(G) = n$.
We next show that $I_s(G) = 2$. Without loss of generality, Staller starts the game by playing vertex $a_1 \in A$. Then, Dominator finishes the game by playing b_1. Hence, $I_s(G) = 2$. □

The following four theorems show that the difference between the competition-independence numbers and the game domination numbers of a graph can be arbitrarily large.

Theorem 6. *For a nonnegative integer n, there is a graph G such that $I_d(G) - \gamma_g(G) = n$.*

Proof. Consider $K_{n+3,t}$, where $t \ge n+3$. By Lemma 4, $I_d(K_{n+3,t}) = n+3$. Note that $\gamma_g(K_{n+3,t}) = 3$. Thus, $I_d(K_{n+3,t}) - \gamma_g(K_{n+3,t}) = n$. □

Theorem 7. *For a positive integer n, there is a graph G such that $\gamma_g(G) - I_d(G) = n$.*

Proof. Let u be a vertex and $P_{9n+1} = x_1y_1z_1x_2y_2z_2...x_{3n}y_{3n}z_{3n}x_{3n+1}$ be a path. Define a graph G by $V(G) = \{u\} \cup V(P_{9n+1})$ and $E(G) = \{ux_j | 1 \le j \le 3n+1\} \cup E(P_{9n+1})$. See Figure 1 for example.
We first show that $I_d(G) = 3n+1$. To show that $I_d(G) \le 3n+1$, we present a strategy for Dominator. Dominator starts the game by playing the vertex u first. Since $ux_j \in E(G)$ for all $1 \le j \le 3n+1$, Staller and Dominator cannot play any vertex x_j for all $1 \le j \le 3n+1$. Thus, after the game ends, for each j, exactly one of y_j or z_j is played. Therefore, $3n+1$ vertices are played and so $I_d(G) \le 3n+1$.
To show that $I_d(G) \ge 3n+1$, we present a strategy for Staller.

Case 1: Dominator starts the game by playing the vertex u.

Then, Staller plays y_1, and thus by similar arguments as above, after the game ends, $3n+1$ vertices are played. Therefore, the number of moves in this case is at least $3n+1$.

Case 2: Dominator starts the game by playing the vertex y_i or z_i for some $1 \le i \le 3n$.

Then, Staller plays the vertex u, and thus by similar arguments as above, after the game ends, $3n+1$ vertices are played. Therefore, the number of moves in this case is at least $3n+1$.

Case 3: Dominator starts the game by playing the vertex x_i for some $1 \le i \le 3n+1$.

Without loss of generality, we may assume that $1 \le i \le \frac{3n+1}{2}$. Then, Staller plays his first move on vertex y_{i+1}. Consider the set $X := \{x_1, x_2, x_3, ..., x_{3n+1}\} - \{x_i, x_{i+1}\}$. Notice that each $x \in X$ is not dominated. In order to dominate X, an additional $3n-1$ vertices are played. Thus, after the game ends, at least $3n+1$ vertices are played.

From the above cases, $I_d(G) \ge 3n+1$. Hence, $I_d(G) = 3n+1$.

We next show that $\gamma_g(G) = 4n+1$. To show that $\gamma_g(G) \ge 4n+1$, we present a strategy for Staller.

Case 1: Dominator starts the game by playing the vertex u.

In each turn of Staller, he plays a vertex which dominates only one more undominated vertex. Since the vertex x_j is dominated for all $1 \le j \le 3n+1$, it follows that in each turn of Dominator, he can dominate at most two more undominted vertices. Consequently, in each round of the game played by Staller and then Dominator, there are at most three new dominated vertices. Since after the first move of Dominator at u there are $9n+2-(3n+1)-1 = 6n$ undominated vertices, it follows that the number of moves in this case is at least $\frac{2}{3} \cdot 6n + 1 = 4n+1$.

Case 2: Dominator starts the game by playing a vertex v where $v \ne u$.

Then, v dominates at most four vertices including v itself. Staller plays his first move on the vertex u, and after that, he plays to dominate one new undominated vertex at a time. Thus, for each $1 \le j \le 3n+1$, x_j is dominated. After that, for each round of the game, Dominator can dominate at most two more undominated vertices. Consequently, in each round of the game played by Dominator and then Staller, there are at most three new dominated vertices. After u and v are played, there are at least $(9n+2)-4-(3n+1)+1 = 6n-2$ undominated vertices. Note that $(6n-2) = (6n-3)+1$. Therefore, the number of moves in this case is at least $\frac{2}{3} \cdot (6n-3)+1+2 = 4n+1$.

From the above cases, $\gamma_g(G) \ge 4n+1$.

To show that $\gamma_g(G) \le 4n+1$, we present a strategy for Dominator. Dominator starts the game by playing the vertex u. For each turn of Dominator, he plays a vertex to dominate two new leftmost vertices if possible. Note that if Dominator cannot dominate two new vertices in his turn, that is he can dominate only one new vertex, then it means that Staller played a vertex to dominate two new vertices earlier. Therefore, on average, a move by Staller and a move by Dominator dominates at least three new vertices. Thus, $\gamma_g(G) \le \frac{2}{3}(6n)+1 = 4n+1$. Consequently, $\gamma_g(G) = 4n+1$.

Therefore, $\gamma_g(G) - I_d(G) = (4n+1)-(3n+1) = n$, as required. □

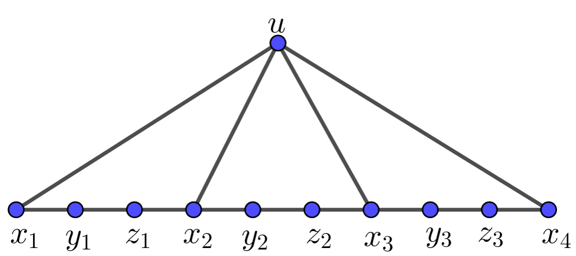

Figure 1. The graph G in the proof of Theorem 7, where $n = 1$.

Theorem 8. *For a nonnegative integer n, there is a graph G such that $I_s(G) - \gamma'_g(G) = n$.*

Proof. Consider $K_{1,n+2}$. By Lemma 4, $I_s(K_{1,n+2}) = n+2$. Note that $\gamma'_g(K_{1,n+2}) = 2$. Thus, $I_s(K_{1,n+2}) - \gamma'_g(K_{1,n+2}) = n$. □

Theorem 9. *For a positive integer n, there is a graph G such that $\gamma'_g(G) - I_s(G) = n$.*

Proof. Let $P_{9n+1} = x_1y_1z_1x_2y_2z_2...x_{3n}y_{3n}z_{3n}x_{3n+1}$ be a path and K_{3n+1} be a complete graph with $V(K_{3n+1}) = \{u_1, u_2, ..., u_{3n+1}\}$.

Define a graph G by $V(G) = V(P_{9n+1}) \cup V(K_{3n+1})$ and $E(G) = E(P_{9n+1}) \cup E(K_{3n+1}) \cup \{u_jx_k | j \neq k \text{ and } 1 \leq j, k \leq 3n+1\}$. See Figure 2 for example.

We first show that $\gamma'_g(G) = 4n+2$. To show that $\gamma'_g(G) \leq 4n+2$, we present a strategy for Dominator.

Case 1: Staller starts the game by playing the vertex u_i for some $1 \leq i \leq 3n+1$.

Without loss of generality, we may assume that $1 \leq i \leq \frac{3n+1}{2}$. Dominator responds by playing the vertex y_i. At this point, y_i, z_i and all u_k's ($1 \leq k \leq 3n+1$) are dominated. After that, in each move of Dominator, he plays the leftmost vertex that can dominate two new vertices if possible. Note that, if Dominator cannot dominate two new vertices in his turn, that is he can dominate only one new vertex, then it means that Staller played a vertex to dominate two new vertices earlier. Therefore, excluding the first two moves, on average, a move by Staller and a move by Dominator dominate at least three new vertices. After Dominator plays y_i, there are $(12n+2) - 2(3n+1) - 2 = 6n-2$ undominated vertices. Notice that $6n-2 = (6n-3)+1 = 3(2n-1)+1$. Hence, the number of moves in this case is at most $\frac{2}{3}(6n-3)+1+2 = 4n+1 < 4n+2$.

Case 2: Staller starts the game by playing the vertex y_i or z_i for some $1 \leq i \leq 3n$.

Without loss of generality, we may assume that Staller starts the game by playing the vertex y_i for some $1 \leq i \leq 3n$. Dominator responds by playing the vertex u_i and so each vertex x_j is dominated for $1 \leq j \leq 3n+1$ and that each vertex u_k is dominated for $1 \leq k \leq 3n+1$. By similar arguments as in case 1, after Dominator plays u_i, there are $(12n+2) - 2(3n+1) - 2 = 6n-2$ undominated vertices and so the number of moves in this case is at most $4n+2$.

Case 3: Staller starts the game by playing the vertex x_i for some $1 \leq i \leq 3n$.

Without loss of generality, we may assume that $1 \leq i \leq \frac{3n+1}{2}$. Dominator responds by playing the vertex u_i and so each vertex x_j is dominated for $1 \leq j \leq 3n+1$. By similar arguments as in case 1, after Dominator plays u_i, there are at most $(12n+2) - 2(3n+1) - 1 = 6n-1$ undominated vertices. Note that $6n-1 = (6n-3)+2 = 3(2n-1)+2$. Hence, the number of moves in this case is at most $\frac{2}{3}(6n-3)+2+2 = 4n+2$.

From the above cases, $\gamma'_g(G) \leq 4n+2$.

To show that $\gamma'_g(G) \geq 4n+2$, we present a strategy for Staller. Staller starts the game by playing the vertex x_1. Then, each vertex u_j is dominated for $2 \leq j \leq 3n+1$ and y_1 is also dominated.

Case 1: Dominator responds by playing the vertex u_1.

In each round of Staller's turn, he plays the leftmost vertex that can dominate one new vertex. Note that, in each Dominator's turn, he can dominate at most two new vertices. Therefore, excluding the first two moves, on average, a move by Staller and a move by Dominator dominate at most three new vertices. After Dominator plays u_1, there are $(12n+2) - 2(3n+1) - 1 = 6n-1$ undominated vertices. Notice that $6n-1 = (6n-3)+2 = 3(2n-1)+2$. Therefore, the number of moves in this case is at least $\frac{2}{3}(6n-3)+2+2 = 4n+2$.

Case 2: Dominator responds by playing the vertex w where $w \neq u_1$.

Then, Staller plays the vertex u_1 in his next turn, and after that, he will play the leftmost vertex that can dominate one new vertex. Note that, in each Dominator's turn, he can dominate at most two new vertices. Therefore, excluding the first three moves, on average, a move by Dominator and a move by Staller dominate at most three new vertices. After Staller plays u_1, there are at least

$(12n+2) - 2(3n+1) - 2 = 6n - 2$ undominated vertices. Notice that $6n - 2 = (6n-3) + 1 = 3(2n-1) + 1$. Hence, the number of moves in this case is at least $\frac{2}{3}(6n-3) + 1 + 3 = 4n + 2$.
From the above cases, $\gamma'_g(G) \geq 4n + 2$. Consequently, $\gamma'_g(G) = 4n + 2$.

We next show that $I_s(G) = 3n + 2$. To show that $I_s(G) \leq 3n + 2$, we present a strategy for Dominator.

Case 1: Staller starts the game by playing a vertex u_i for some $1 \leq i \leq 3n + 1$.

Dominator responds by playing the vertex x_i. After that, Staller and Dominator cannot play any vertex u_j or x_j for all $1 \leq j \leq 3n + 1$. For the remainder of the game, Staller and Dominator can only play either y_j or z_j for some $1 \leq j \leq 3n$. After the game ends, for each j, exactly one of y_j or z_j is played for $1 \leq j \leq 3n$. Therefore, $3n + 2$ vertices are played and so the number of moves in this case is at most $3n + 2$.

Case 2: Staller starts the game by playing a vertex x_i for some $1 \leq i \leq 3n + 1$.

Dominator responses by playing the vertex u_i. By similar arguments as in case 1, we have that $3n + 2$ vertices are played and so the number of moves in this case is at most $3n + 2$.

Case 3: Staller starts the game by playing either y_i or z_i for some $1 \leq i \leq 3n$.

Dominator responds by playing the vertex u_i if Staller played y_i; otherwise, he plays u_{i+1}. By similar arguments as in case 1, we have that $(3n-1) + 2 = 3n + 1$ vertices are played. Therefore, the number of moves in this case is at most $3n + 1$.
From the above cases, $I_s(G) \leq 3n + 2$.

To show that $I_s(G) \geq 3n + 2$, we present a strategy for Staller. Staller starts the game by playing the vertex x_1.

Case 1: Dominator responds by playing the vertex u_1.

Then, the vertices x_j and u_j are dominated for $1 \leq j \leq 3n + 1$. Thus, after the game ends, for each $1 \leq j \leq 3n$, exactly one of y_j or z_j is played. Therefore, $3n + 2$ vertices are played. Consequently, the number of moves in this case is at least $3n + 2$.

Case 2: Dominator responds by playing the vertex y_i or z_i for some $1 \leq i \leq 3n$.

Without loss of generality, we may assume that Dominator responds by playing the vertex y_i for some $1 \leq i \leq 3n$. Then, Staller plays his next turn at the vertex u_1. At this point, the vertices x_j and u_j are dominated for $1 \leq j \leq 3n + 1$, and y_i and z_i are also dominated. Thus, after the game ends, for each $1 \leq k \leq 3n$, exactly one of y_k or z_k is played. Hence, $3n + 2$ vertices are played. Consequently, the number of moves in this case is at least $3n + 2$.

Case 3: Dominator responds by playing the vertex x_i for some $2 \leq i \leq 3n + 1$.

Then, each vertex u_j is dominated for $1 \leq j \leq 3n + 1$. Note that, after Dominator plays x_i, there are at least $(12n+2) - 3n - 2 - 4 = 9n - 4$ undominated vertices. In each of Staller's turns, he plays to dominate at most two new undominated vertices. Notice that, in each turn of Dominator, he can dominate at most three new undominated vertices. Therefore, the number of moves is at least $\frac{2}{5}(9n-4) + 2 = 3n + \frac{3}{5}n + \frac{2}{5}$. If $n \geq 2$, then the number of moves is at least $3n + 2$. For $n = 1$, it can be checked that the number of moves is at least 5.
From the above cases, $I_s(G) \geq 3n + 2$. Consequently, $I_s(G) = 3n + 2$.
Therefore, $\gamma'_g(G) - I_s(G) = 4n + 2 - (3n + 2) = n$. $\square$

By the definitions, $i(G) \leq I_d(G) \leq \alpha(G)$ and $i(G) \leq I_s(G) \leq \alpha(G)$. We show that the difference between the competition-independence numbers and independent domination numbers can be arbitrarily large.

Theorem 10. *For a nonnegative integer n, there is a graph G such that $I_d(G) - i(G) = n$.*

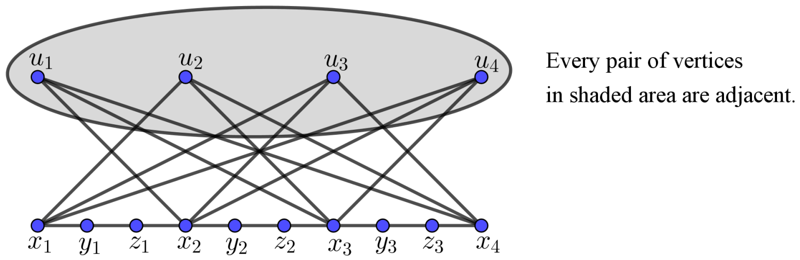

Figure 2. The graph G in the proof of Theorem 9, where $n = 1$.

Proof. Let G be the graph obtained from $K_{n+2,n+2}$ by deleting a perfect matching. By Lemma 5, we have $I_d(G) = n + 2$. Note that $i(G) = 2$. Thus $I_d(G) - i(G) = n$. □

Theorem 11. *For a nonnegative integer n, there is a graph G such that $\alpha(G) - I_d(G) = n$.*

Proof. Consider $K_{1,n+1}$. Note that $\alpha(K_{1,n+1}) = n+1$ and $I_d(K_{1,n+1}) = 1$. Thus $\alpha(K_{1,n+1}) - I_d(K_{1,n+1}) = n$. □

Theorem 12. *For a nonnegative integer n, there is a graph G such that $I_s(G) - i(G) = n$.*

Proof. Consider $K_{1,n+1}$. Note that $I_s(K_{1,n+1}) = n+1$ and $i(K_{1,n+1}) = 1$. Thus $I_s(K_{1,n+1}) - i(K_{1,n+1}) = n$. □

Theorem 13. *For a nonnegative integer n, there is a graph G such that $\alpha(G) - I_s(G) = n$.*

Proof. Let G be the graph obtained from $K_{n+2,n+2}$ by deleting a perfect matching. Note that $\alpha(G) = n + 2$ and $I_s(G) = 2$. Thus, $\alpha(G) - I_s(G) = n$. □

Next, we show a family of graphs such that many parameters are equal.

The *corona product* of graphs G and H, denoted by $G \circ H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the ith copy of H to the ith vertices of G, where $1 \leq i \leq |V(G)|$.

Theorem 14. *There exists a graph G such that $I_d(G) = I_s(G) = \gamma_g(G) = \gamma'_g(G) = \alpha(G) = i(G)$.*

Proof. Let G be a graph $K_n \circ K_1$. Note that $I_d(G) = I_s(G) = \gamma_g(G) = \gamma'_g(G) = \alpha(G) = i(G) = n$. □

3. Realization of the Competition-Independence Numbers

For a pair (a, b) of positive integers, we say G *realizes* (a, b) if $I_d(G) = a$ and $I_s(G) = b$.

In this section, we show that any pair (a, b) of positive integers can be realized.

Theorem 15. *For positive integers a and b, there is a connected graph G such that $I_d(G) = a$ and $I_s(G) = b$.*

Proof. Let a and b be positive integers.

Case 1: $a \leq b$.

Let G be the complete bipartite graph $K_{a,b}$. By Lemma 4, we have $I_d(G) = \min\{a, b\} = a$ and $I_s(G) = \max\{a, b\} = b$.

Case 2: $a > b$.

Let H be the complete b-partite graph such that the size of each partite set is a i.e., $H = K_{a,a,\ldots,a}$. Suppose that $V_1, V_2, \ldots, V_b$ are the partite sets of H and that $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,a}\}$ for $1 \leq i \leq b$. Let G be the graph $H - \{v_{i,j}v_{k,j} | 1 \leq i < k \leq b \text{ and } 1 \leq j \leq a\}$.

Let $T := \{v_{k,1} | 2 \leq k \leq b\}$. We first show that $I_d(G) = a$.

Without loss of generality, we may assume that Dominator starts the game by playing vertex $v_{1,1}$. Then, the set $(V_1 - \{v_{1,1}\}) \cup T$ is the set of all undominated vertices. Since $a > b$, Staller responds by playing a vertex in $V_1 - \{v_{1,1}\}$, and then, both players must alternately choose undominated vertices in the set V_1. Therefore, the game will end in a moves. Hence, $I_d(G) = a$.

We next show that $I_s(G) = b$.

Without loss of generality, we may assume that Staller starts the game by playing vertex $v_{1,1}$. Then, the set $(V_1 - \{v_{1,1}\}) \cup T$ is the set of all undominated vertices. Since $a > b$, Dominator responds by playing a vertex in the set T, and then, both players must alternately choose undominated vertices in the set T. Therefore, the game will end in b moves. Hence, $I_s(G) = b$. $\square$

In particular, Theorem 15 shows that the difference between the two competition-independence numbers of a graph can be arbitrarily large. This is very different from the domination game and the other variations where the difference is at most one [1,2,16].

4. Conclusions

In this paper, we studied the domination game such that the set of played vertices is independent. This game is known as the competition-independence game. We showed that the Continuation Principle for the domination game does not hold for the competition-independence game. We proved that the difference between the competition-independence numbers and the game domination numbers of a graph can be arbitrarily large. Also, we gave a family of graphs such that many parameters are equal. Furthermore, we showed that any pair of positive integers can be realized as the competition-independence numbers of some graph.

Author Contributions: Investigation, W.R.; writing—original draft preparation, W.R.; writing—review and editing, C.W.; supervision, C.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by The Thailand Research Fund (grant number MRG6180301).

Acknowledgments: The authors would like to thank the reviewers for their suggestions and comments. The second author was supported by The Thailand Research Fund (grant number MRG6180301). The author takes this oppotunity to thank The Thailand Research Fund for the support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Brešar, B.; Klavžar, S.; Rall, D.F. Domination game and an imagination strategy. *SIAM J. Discret. Math.* **2010**, *24*, 979–991. [CrossRef]
2. Kinnersley, W.B.; West, D.B.; Zamani, R. Extremal problems for game domination number. *SIAM J. Discret. Math.* **2013**, *27*, 2090–2107. [CrossRef]
3. Brešar, B.; Klavžar, S.; Košmrlj G.; Rall, D.F. Domination game: Extremal families of graphs for the 3/5-conjectures. *Discret. Appl. Math.* **2013**, *161*, 1308–1316. [CrossRef]
4. Bujtas, C. Domination game on forests. *Discret. Math.* **2015**, *338*, 2220–2228. [CrossRef]
5. Bujtas, C. On the game domination number of graphs with given minimum degree. *Electron. J. Combin.* **2015**, *22*, 3.29. [CrossRef]
6. Henning, M.A.; Kinnersley, W.B. Domination game: A proof of the 3/5-conjecture for graphs with minimum degree at least two. *SIAM J. Discret. Math.* **2016**, *30*, 20–35. [CrossRef]
7. Henning, M.A.; Löwenstein, C. Domination game: Extremal families for the 3/5-conjecture for forests. *Discuss. Math. Graph Theory* **2017**, *37*, 369–381. [CrossRef]
8. Dorbec, P.; Košmrlj, G.; Renault, G. The domination game played on unions of graphs. *Discret. Math.* **2015**, *338*, 71–79. [CrossRef]
9. James, T.; Klavžar, S.; Vijayakumar, A. The domination game on split graphs, *Bull. Aust. Math. Soc.* **2019**, *99*, 327–337. [CrossRef]
10. Ruksasakchai, W.; Onphaeng, K.; Worawannotai, C. Game domination numbers of a disjoint union of paths and cycles. *Quaest. Math.* **2019**, *42*, 1357–1372. [CrossRef]
11. Zamani, R. Hamiltonian cycles through specified edges in bipartite graphs, domination game, and the game of revolutionaries and spies. Ph.D. Thesis, University of Illinois at Urbana-Champaign, IL, USA.

12. Henning, M.A.; Klavžar, S.; Rall, D.F. Total version of the domination game. *Graphs Combin.* **2015**, *31*, 1453–1462. [CrossRef]
13. Bujtas, C. On the game total domination number. *Graphs Comb.* **2018**, *34*, 415–425. [CrossRef]
14. Dorbec, P.; Henning, M.A. Game total domination for cycles and paths. *Discret. Appl. Math.* **2016**, *208*, 7–18. [CrossRef]
15. Henning, M.A.; Klavžar, S.; Rall, D.F. The 4/5 upper bound on the game total domination number. *Combinatorica* **2017**, *37*, 223–251. [CrossRef]
16. Brešar, B.; Bujtas, C.; Gologranc, T.; Klavžar, S.; Košmrlj, G.; Marc, T.; Patkos, B.; Tuza, Z.; Vizer, M. The variety of domination games. *Aequ. Math.* **2019**, *93*, 1085–1109. [CrossRef]
17. Phillips, J.B.; Slater, P.J. An introduction to graph competition independence and enclaveless parameters. *Graph. Theory Notes N.Y.* **2001**, *41*, 37–41.
18. Phillips, J.B.; Slater, P.J. Graph competition independence and enclaveless parameters. *Congr. Numer.* **2002**, *154*, 79–100.
19. Goddard, W.; Henning, M.A. The competition-independence game in trees. *J. Combin. Math. Combin. Comput.* **2018**, *104*, 161–170.