

Ideals on the Quantum Plane's Jet Space

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Received: 26 December 2019; Accepted: 27 February 2020; Published: 6 March 2020



Abstract: The goal of this paper is to introduce some rings that play the role of the jet spaces of the quantum plane and unlike the quantum plane itself possess interesting nontrivial prime ideals. We will prove some results (Theorems 1–4) about the prime spectrum of these rings.

Keywords: quantum plane; noncommutative geometry; quantum curves

1. Introduction

According to the classical perception of plane geometry the affine plane corresponds to the algebra freely generated by two variables x and y subject to the trivial commutation relation $yx = xy$. When the commutation relation $yx = xy$ is replaced by $yx = qxy$ the resulting associative k -algebra is called the quantum plane [1,2]. The different models of noncommutative configuration space were developed by physicists, for example, by Hartland Snyder [3].

Objects like “planes” are expected to possess some analog of “curves”. However, the quantum plane possesses very few prime ideals. The idea of the paper, originally motivated by p -derivation [4], is to look at certain rings that play the role of jet spaces of the quantum planes. This is done by introducing a procedure of prolongation [5] of given variables to form a jet space in the style of Kolchin’s differential algebra [6] and by considering commutation relations among these variables which are compatible with the action of the natural derivations on these rings. These are the multiplicative relations unlike the ones of Weyl type considered in particular in [7]. It turns out these new rings possess plenty of prime ideals which are related to the (commutative) geometry of $\mathbb{P}^n \times \mathbb{P}^n$, $n \geq 1$. The Representation Theory of Quantum Group is well established subject [8–14] and summarized in Kassel’s book [15] and not be discussed at this time.

2. Background and Motivation

Quantum Symmetry (Basic Example)

A quantum group is the q -deformed universal enveloping algebra introduced by Drinfeld [16] and Jimbo [17] in their study of the integral system. The word “quantum” in quantum plane denotes a plane-like object on which the quantum group is applied similarly to rotations on a regular plane. A quantum group is a Hopf algebra endowed with comultiplication Δ , counit ϵ , and the antipode S [2]. Its theory has been developed in different directions [15,18]. In the quantum space approach [1], the initial object is a quadratic algebra which is considered being as the polynomial algebra on a quantum linear space. Quantum group appears like a group of automorphisms of the quantum linear space.

The basic example is a Quantum Group $GL_q(2)$ [2] (see also Definition IV.3.2, Theorems IV.3.1, IV.3.3, Proposition I.4.1 in Kassel’s book [15]). Let k be a ground field, $q \in k^*$. By definition, the ring of polynomial

functions $F = F[GL_q(2)]$ is a Hopf algebra which can be described in the following way. As a k -algebra, it is generated by a, b, c, d and a formal inverse of the central element

$$D = DET_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc,$$

where a, b, c, d satisfy the following commutation relations:

$$\begin{aligned} ab &= q^{-1}ba \\ ac &= q^{-1}ca \\ cd &= q^{-1}dc \\ bd &= q^{-1}db \\ bc &= cb \\ ad - da &= (q^{-1} - q)bc \end{aligned}$$

The comultiplication $\Delta : F \rightarrow F \otimes F$ is defined by [2,15]

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the $\otimes$ product denotes the usual product of matrices in which products like ab are replaced by $a \otimes b$. The counit is given by

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

The antipode map $S : F \rightarrow F$ is

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^{-1} \begin{pmatrix} d & -qb \\ -c/q & a \end{pmatrix}.$$

It can be checked directly that all these structures are well defined and satisfy the Hopf algebra axioms.

3. Quantum Plane: Gauss Polynomials and the q -Binomial Formula

Definition 1. Let k be a field. Let $q \neq 1$ be an invertible element of the ground field k and let I_q be the two-sided ideal of the free algebra $k\langle x, y \rangle$ of noncommutative polynomials in x and y generated by the element $f = yx - qxy$. The quantum plane is defined as the quotient algebra $\frac{k\langle x, y \rangle}{I_q}$. For future developments, we need to compute the powers of $x + y$ in the quantum plane. To this end, we have to consider Gauss polynomials.

Gauss polynomials are polynomials in one variable q whose values at $q = 1$ are equal to the classical binomial coefficients. For any integer $n > 0$, set

$$(n)_q = 1 + q + q^2 + \dots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

Define the q -factorial of n by $(0)!_q = 1$ and

$$(n)!_q = (1)_q(2)_q \dots (n)_q = \frac{(q-1)(q^2-1)\dots(q^n-1)}{(q-1)^n},$$

when $n > 0$. The q -factorial is a polynomial in q with integral coefficients and with value at $q = 1$ equal to the usual factorial $n!$. We define the Gauss polynomials for $0 \leq k \leq n$ by

$$\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q (n-k)!_q},$$

with following properties:

1. $\binom{n}{k}_q$ is a polynomial in q with integral coefficients and with value at $q = 1$ equal to the binomial coefficient $\binom{n}{k}$ (see Proposition IV.2.1 in [15]).
2. The q -Pascal identity holds:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q$$

(see Proposition IV.2.1 in [15]).

3. There is a q -analog of the Chu-Vandermonde formula. For $m \geq p \geq n$ we have

$$\binom{m+n}{p}_q = \sum_{0 \leq k \leq p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q$$

(see Proposition IV.2.3 in [15]).

4. For all $n > 0$,

$$(x+y)^n = \sum_{0 \leq k \leq n} \binom{n}{k}_q x^k y^{n-k}$$

(see Proposition IV.2.2 in [15]).

If q is a root of unity of order $p > 0$, then [2]

$$(x+y)^p = x^p + y^p$$

5. The formal series in z can be defined as following :

$$e_q(z) = \sum_{n \geq 0} \frac{z^n}{(n)!_q}$$

with the property of q -exponential function: $e_q(x+y) = e_q(x)e_q(y)$

(see Proposition IV.2.4 in [15])

In this paper, q will eventually be assumed not a root of unity. However, some of the results can be extended to the case when q is a root of unity in which case the q -binomial formulae become relevant.

4. Quantum Plane and Quantum Group

A more conceptual approach to $GL_q(2)$ [2] consists in introducing quantum plane $\frac{k\langle x, y \rangle}{I_q}$ and obtaining the commutation relations of $GL_q(2)$ from the following matrix relations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

such that x', y' and x'', y'' are on Quantum Plane and

$$\begin{aligned} y'x' &= qx'y' \\ y''x'' &= qx''y''. \end{aligned}$$

In this way, $GL_q(2)$ emerges merely as a quantum automorphism group of noncommutative linear space.

5. The Problem and the Main Results

The family of prime ideals of the quantum plane has a simple structure as we shall presently review. Recall that an ideal P is prime if $P \neq (1)$ and if for any two elements a and b of the quantum plane from $ab \in P$ it follows that $a \in P$ or $b \in P$.

We denote by $\text{Spec } B$ the set of prime ideals in any ring B . $\text{Spec} \left(\frac{k\langle x, y \rangle}{I_q} \right)$ consists of the following prime ideals: $\{\langle 0 \rangle, \langle x, y \rangle, \langle x - \alpha, y \rangle, \langle x, y - \beta \rangle\}$, where $\alpha, \beta \in k^*$.

$\langle S \rangle$ denotes the two-sided ideal generated by set S .

Due to the commutation relation $yx = qxy$ the above set of ideals can be rewritten as $\{\langle 0 \rangle, \langle x, y \rangle, \langle x - \alpha \rangle, \langle y - \beta \rangle\}$ since, for example,

$$y(x - \alpha) - q(x - \alpha)y = (q - 1)\alpha y,$$

so $y \in \langle x - \alpha \rangle$ and $\langle x - \alpha, y \rangle = \langle x - \alpha \rangle$.

The fact that the ring structure of the quantum plane is so trivial prevents us from considering "curves". That is a motivation to attempt to introduce new rings that play the role of the jet spaces [5] of the quantum plane and possess interesting nontrivial prime ideals.

Let us consider the noncommutative ring $B^{(n)} = k\langle x, x', x'', \dots, x^{(n)}, y, y', y'', \dots, y^{(n)} \rangle$ where $x', x'', \dots, x^{(n)}, y', y'', \dots, y^{(n)}$ are new indeterminates.

Consider the unique k -derivation $\delta : B^{(n-1)} \rightarrow B^{(n)}$ (a k -linear map satisfying the usual Leibniz rule: $\delta(FG) = \delta(F)G + F\delta(G)$ such that $\delta x = x', \delta x' = x'', \dots$ and $\delta y = y', \delta y' = y'', \dots$ Assuming $f = yx - qxy$ let us define the following elements of $B^{(1)}$:

$$g_1 = y'x - qxy', g_2 = yx' - qx'y, g_3 = y'x' - qx'y', h = xx' - x'x, \bar{h} = yy' - y'y.$$

By the Leibnitz Rule,

$$\delta f = g_1 + g_2$$

In similar fashion let us define the following elements of $B^{(n)}$:

$$g_{ij} = y^{(i)}x^{(j)} - qx^{(j)}y^{(i)}$$

$$h_{ij} = x^{(i)}x^{(j)} - x^{(j)}x^{(i)}$$

$$\bar{h}_{ij} = y^{(i)}y^{(j)} - y^{(j)}y^{(i)}$$

for integers $i, j \leq n$.

We can consider a noncommutative ring $A^{(n)} = \frac{B^{(n)}}{\langle f, h_{ij}, \bar{h}_{ij}, g_{ij} \rangle}$ as well as a ring of the usual (commutative) polynomials $A_c^{(n)} = k[x, x', x'', \dots, x^{(n)}, y, y', y'', \dots, y^{(n)}]$.

Definition 2. A polynomial F is called bi-homogeneous of bi-degree (p, q) , if F is homogeneous of degree p (resp. q) when considered as a polynomial in $X_0, \dots, X_n$ (resp. in the $Y_0, \dots, Y_n$). Any monomial $x^{i_0} (x')^{i_1} (x'')^{i_2} \dots (x^{(n)})^{i_n}$

$y^{j_0}(y')^{j_1}(y'')^{j_2} \dots (y^{(n)})^{j_n}$ has the bi-degree (i, j) where the total degree in $x, x', \dots, x^{(n)}$ is $i = i_0 + i_1 + \dots + i_n$ and the total degree in $y, y', y'', \dots, y^{(n)}$ is $j = j_0 + j_1 + \dots + j_n$.

Let us consider a k -linear bijective map $A^{(n)} \cong A_c^{(n)}$ sending the class of any monomial into the same monomial viewed as an element of $A_c^{(n)}$. Via this bijection we have a multiplication law $\bullet_c$ on $A_c^{(n)}$ such that for any two bi-homogeneous polynomials of bi-degrees (i, j) and (k, l) respectively,

$$a \bullet_c b = ab \cdot q^{-jk}.$$

The bijection is not an isomorphism of rings. From now on we shall identify $A_c^{(n)}$ and $A^{(n)}$ as sets via above bijection. Note that $A_c^{(n)}$ is bi-graded in the usual way. In the following let q be not a root of unity. Our main results about $\text{Spec}(A^{(n)})$ can be presented as the following Theorems 1–4.

Theorem 1. If $0 \neq P \subset A^{(n)}$ is a prime ideal then P contains a non-zero bi-homogeneous polynomial which is an irreducible element of $A_c^{(n)}$.

Theorem 2. If $f \in A^{(n)}$ bi-homogeneous such that its image $f \in A_c^{(n)}$ is irreducible then $\langle f \rangle \subset A^{(n)}$ is prime.

Theorem 3. Any prime ideal $P \subset A^{(n)}$ not containing any of the ideals $\langle x, x', \dots, x^{(n)} \rangle$ or $\langle y, y', \dots, y^{(n)} \rangle$ is of the form $P = \langle T \rangle$, where T is the family of all bi-homogeneous polynomials in P .

Theorem 4a. Any prime ideal $P \subset A^{(n)}$ such that $\langle x, x', \dots, x^{(n)} \rangle \subset P$ is of the form $\langle x, x', \dots, x^{(n)}, \varphi_1(y, y', \dots, y^{(n)}), \dots, \varphi_k(y, y', \dots, y^{(n)}) \rangle$ where $\varphi_i(y, y', \dots, y^{(n)}) \in k[y, y', \dots, y^{(n)}]$ for $i = 1, \dots, k$ generates a prime ideal of $k[y, y', \dots, y^{(n)}]$.

Theorem 4b. Any prime ideal $P \subset A^{(n)}$ such that $\langle y, y', \dots, y^{(n)} \rangle \subset P$ is of the form $\langle \psi_1(x, x', \dots, x^{(n)}), \dots, \psi_k(x, x', \dots, x^{(n)}), y, y', \dots, y^{(n)} \rangle$ where $\psi_i(x, x', \dots, x^{(n)}) \in k[x, x', \dots, x^{(n)}]$ for $i = 1, \dots, k$ generates a prime ideal of $k[x, x', \dots, x^{(n)}]$.

6. δ -Prime Ideals

Let us recall the previously defined derivation $\delta : B^{(n-1)} \rightarrow B^{(n)}$. Let $A = \varinjlim A^{(n)}$. Then δ induces a k -derivation $\delta : A \rightarrow A$. For each n we have $x^{(n)} = \delta x^{(n-1)}$ and $y^{(n)} = \delta y^{(n-1)}$.

Define a δ -prime ideal to be a prime ideal P such that $\delta P \subset P$. As in Theorem 3 let $T = \{f \in P \mid f \text{ is bi-homogeneous}\}$ so $P = \langle T \rangle$. We can prove the following proposition:

$$\delta P \subset P \iff \delta T \subset T.$$

Proof. Implication $\implies$:

$$\delta T \subset \delta P \cap \{\text{bi-homogenous elements of } A\} \subset P \cap \{\text{bi-homogenous elements of } A\} = T$$

The $\impliedby$ part follows because if $f \in P$ then

$$f = \sum_i \alpha_i f_i \beta_i, f_i \in T$$

$$\delta f = \sum_i \delta \alpha_i f_i \beta_i + \sum_i \alpha_i \delta f_i \beta_i + \sum_i \alpha_i f_i \delta \beta_i$$

Since $\delta T \subset T$, then $\delta f \in \langle T \rangle = P$. $\square$

7. Proofs of the Main Results

For the proofs of Theorems 1–4 we need the following definition of the lexicographical ordering in $\mathbb{N}^2$.

$$(a, b) \leq (c, d) \iff \left\{ \begin{array}{l} \text{either } a < c \\ \text{or } a = c, b \leq d. \end{array} \right\}$$

Let us consider a polynomial $g \in A^{(n)}$. Write $g = \sum_{ij} g_{ij}$, such that g_{ij} is bi-homogeneous of bi-degree (i, j) . Let's consider the set $\Gamma_g = \{(i, j), g_{ij} \neq 0\} \subset \mathbb{N}^2$. The size of a polynomial g in $A^{(n)}$ will be defined as $\#\Gamma_g$, a number of points in Γ_g . If g_{ij} has a bi-degree (i, j) then $g_{ij}x$ has a bi-degree $(i+1, j)$ and yg_{ij} has a bi-degree $(i, j+1)$. The size of gx and yg will stay the same as the size of g .

Lemma 1. *If $h = yg - q^v gy$ and $(v, \mu) \in \Gamma_g$, then the size of Γ_h will be strictly less than the size of Γ_g .*

Indeed,

$$h = yg - q^v gy = y \sum_{ij} g_{ij} - q^v \sum_{ij} g_{ij} y = \sum_{ij} g_{ij} y (q^i - q^v).$$

It follows that all points of Γ_g with the first coordinate equal to v will disappear in Γ_h and the size of Γ_h will be strictly less than the size of Γ_g .

Similarly, if $\bar{h} = gx - q^\mu xg$ and $(v, \mu) \in \Gamma_g$, then the size of $\Gamma_{\bar{h}}$ will be strictly less than the size of Γ_g .

7.1. Proof of Theorem 1

We start by showing the following claim: there exists a nonzero bi-homogeneous polynomial in P . Indeed take $0 \neq g \in P$ of smallest possible size. We claim that the size of g equal to 1 which means g is bi-homogeneous. Assume that the size of g is greater or equal than 2.

Case 1. g is not homogeneous in $x, x', \dots, x^{(n)}$.

Let's consider $\bar{g} = yg - q^v gy \in P$ such that there is at least one term with total degree in $x, x', \dots, x^{(n)}$ equal to v . Since g is not homogeneous in $x, x', \dots, x^{(n)}$, $\bar{g} \neq 0$. On the other hand by the Lemma 1 we have $\#\Gamma_{\bar{g}} < \#\Gamma_g$ which contradicts the minimality of $\text{size}(g)$.

Case 2. g is homogeneous in $x, x', \dots, x^{(n)}$ but not in $y, y', \dots, y^{(n)}$.

Let us consider $\hat{g} = gx - q^\mu xg \in P$ such that there is at least one term in g with the total degree in $y, y', \dots, y^{(n)}$ equal to μ . Since g is not homogeneous in $y, y', \dots, y^{(n)}$, $\hat{g} \neq 0$. On the other hand by the Lemma 1 we have $\#\Gamma_{\hat{g}} < \#\Gamma_g$ which contradicts with minimality of the size of g .

This proves our claim. To conclude the proof of Theorem 1, using our claim one can pick a nonzero bi-homogeneous polynomial $f \in P$ of smallest bi-degree (i^*, j^*) with respect to lexicographical order among the nonzero bi-homogeneous polynomials in P .

We claim that f is irreducible in $A_c^{(n)}$.

If we assume it is not irreducible, then $f = g \bullet_c h$, $g, h \in A_c^{(n)}$, $(g, h \notin k)$.

Write:

$$\begin{aligned} g &= \sum_{ij} g_{ij}, g_{ij} \text{ bi-homogeneous of bidegree } (i, j) \\ h &= \sum_{ij} h_{ij}, h_{ij} \text{ bi-homogeneous of bidegree } (i, j). \end{aligned}$$

Note the following properties of bi-degrees:

1. $\text{bideg}(g_{i_1j_1} \cdot h_{i_2j_2}) = (i_1 + i_2, j_1 + j_2)$.
2. If $(i_1, j_1) \leq (i_0, j_0)$ and $(k_1, l_1) \leq (k_0, l_0)$ then $(i_1 + k_1, j_1 + l_1) \leq (i_0 + k_0, j_0 + l_0)$.

Let (i_0, j_0) be the highest element of Γ_g with respect to lexicographical order, (k_0, l_0) be the highest element of Γ_h with respect to lexicographical order and let (i_1, j_1) be the lowest element of Γ_g , (k_1, l_1) be the lowest element of Γ_h .

Then the highest element of $\Gamma_{g \bullet_c h}$ will be $(i_0 + k_0, j_0 + l_0)$ and the lowest element of $\Gamma_{g \bullet_c h}$ will be $(i_1 + k_1, j_1 + l_1)$. Since $f = g \bullet_c h$ we have $(i_0 + k_0, j_0 + l_0) = (i_1 + k_1, j_1 + l_1) = (i^*, j^*)$

Since $i^* = i_0 + k_0 = i_1 + k_1$ and $i_0 \geq i_1$ it follows that $i_0 = i_1$ because if $i_0 > i_1$ then k_0 has to be less than k_1 which contradicts with the choice of k_0 . It immediately follows that $k_0 = k_1$. Similarly, $j_0 = j_1$ and $l_0 = l_1$, so g and h are both bi-homogeneous of degrees less than (i^*, j^*) .

Since P is a prime ideal, at least one of them belongs to P . This contradicts the choice of f .

7.2. Proof of Theorem 2

Assume f is irreducible in $A_c^{(n)}$ and bi-homogeneous of bi-degree (i, j) .

We prove by induction on the total degree N in $x, x', \dots, x^{(n)}, y, y', \dots, y^{(n)}$ that if f has a total degree N then from $g \cdot h \in \langle f \rangle$ it follows that g or $h \in \langle f \rangle$.

If $N = 0$ the theorem is clear. Assume the theorem is true for total degree less or equal to $N - 1$.

Let N be the total degree of f . We have that from $g \cdot h \in \langle f \rangle$ it follows that $g \cdot h = \sum_i \alpha_i f \beta_i$ where α_i and β_i belong to $A^{(n)}$. We may assume that α_i and β_i are bi-homogeneous.

Since f is bi-homogeneous, $\sum_i \alpha_i f \beta_i = \sum_i \alpha_i \beta_i q^{n_i} f = \sum_i \gamma_i f = \gamma f$, for some n_i, γ_i, γ .

$$\gamma = \sum_{ij} \gamma_{ij}, \gamma_{ij} \text{ bi-homogeneous of bidegree } (i, j).$$

Let (i_0, j_0) be the highest element of Γ_g with respect to lexicographical order, (k_0, l_0) be the highest element of Γ_h and (m_0, n_0) be the highest element of Γ_γ . Then

$$\begin{aligned} g_{i_0j_0} \cdot h_{k_0l_0} &= \gamma_{m_0n_0} \cdot f \\ q^t g_{i_0j_0} \bullet_c h_{k_0l_0} &= q^s \gamma_{m_0n_0} \bullet_c f \end{aligned}$$

for some t and s .

Since f is irreducible in the commutative ring $A_c^{(n)}$, it follows that $g_{i_0j_0} = \eta \bullet_c f = q^{-li} \eta \cdot f$ (f is bi-homogeneous and the bi-degree of η is (k, l)) or $h_{k_0l_0} = \bar{\eta} \bullet_c f = q^{-\bar{l}\bar{i}} \bar{\eta} \cdot f$ (bi-degree of $\bar{\eta}$ is $(\bar{k}, \bar{l})$).

Assume, for example, the former is the case. From $gh = \gamma f$ we get $(g - g_{i_0j_0} + g_{i_0j_0}) \cdot h = (g - g_{i_0j_0}) \cdot h + g_{i_0j_0} \cdot h = g' \cdot h + q^w \eta \cdot f \cdot h$ where $g' = g - g_{i_0j_0}$. Obviously, $g' \cdot h \in \langle f \rangle$.

Since the total degree in $x, x', \dots, x^{(n)}, y, y', \dots, y^{(m)}$ of $g' \cdot h$ is less or equal to $N - 1$, by the induction hypothesis either $g' \in \langle f \rangle$ and $g = g' + q^w \eta \cdot f \in \langle f \rangle$ or $h \in \langle f \rangle$ and the proof is complete.

7.3. Proof of Theorem 3

It is obvious that $\langle T \rangle \subset P$.

To prove $\langle T \rangle \supset P$ assume on the contrary that P does not belong to $\langle T \rangle$. Let $f \in P \setminus \langle T \rangle$ be of minimal size. Since by this assumption f cannot be bi-homogeneous the size f more than 1. There are two cases.

Case 1. f is not homogeneous in $x, x', \dots, x^{(n)}$.

Write

$$f = \sum_{st} f_{st}.$$

For an arbitrary $(k, l) \in \Gamma_f$ there exists a pair $(i, j) \in \Gamma_f$ such that $i \neq k$ otherwise f ought to be homogeneous in $x, x', \dots, x^{(n)}$.

Let $h = yf - q^i f y \in P$.

Then by Lemma 1, the size of h is less than the size of f . It follows that $h \in \langle T \rangle$ so h can be written as

$$h = \gamma_1 B_1 + \gamma_2 B_2 + \dots + \gamma_m B_m$$

where $B_l \in P$ and bi-homogeneous

$$h = \sum_{st} (y f_{st} - q^i f_{st} y) = \sum_{st} \lambda_{si} f_{st} y,$$

where $\lambda_{si} = q^s - q^i$.

Let us pick out the bi-homogeneous components of bi-degree $(k, l + 1)$. Then $\lambda_{ki} \cdot f_{kl} \cdot y = \widetilde{\gamma}_1 B_1 + \widetilde{\gamma}_2 B_2 + \dots + \widetilde{\gamma}_m B_m \in \langle T \rangle$ where $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_m$ are bi-homogeneous. Since $\lambda_{ki} \neq 0$ because $i \neq k$, we have $f_{kl} \cdot y \in \langle T \rangle$. So $f_{kl} \cdot y \in P$.

Similarly let $h^{(s)} = y^{(s)} f - q^i f y^{(s)}$. As above we get $f_{kl} \cdot y^{(s)} \in P$ for all s . Since $\langle y, y', \dots, y^{(n)} \rangle$ is not contained in P it follows that at least one of $y^{(s)} \notin P$. Because P is prime, $f_{kl} \in P$. However, f_{kl} is obviously bi-homogeneous so $f_{kl} \in \langle T \rangle$.

Since the pair (k, l) is arbitrary it follows that

$$f = \sum_{st} f_{st} \in \langle T \rangle,$$

which is a contradiction.

Case 2. f is homogeneous in $x, x', \dots, x^{(n)}$ but not in $y, y', \dots, y^{(n)}$. Write

$$f = \sum_t f_{kt}$$

For an arbitrary $(k, l) \in \Gamma_f$ there exists a pair $(k, j) \in \Gamma_f$ such that $l \neq j$ otherwise f ought to be homogeneous in $y, y', \dots, y^{(n)}$.

Let $\bar{h} = f x - q^j x f \in P$. Then by the Lemma 1 size of $\bar{h}$ is less than size of f . It follows that $\bar{h} \in \langle T \rangle$ so $\bar{h}$ can be written as

$$\bar{h} = \bar{\gamma}_1 \bar{B}_1 + \bar{\gamma}_2 \bar{B}_2 + \dots + \bar{\gamma}_m \bar{B}_m$$

where $\bar{B}_l \in P$ and bi-homogeneous.

Then we also have $\bar{h} = \sum_t (f_{kt} x - q^j x f_{kt}) = \sum_t \lambda_{tj} f_{kt} x$, where $\lambda_{si} = q^t - q^j$. Let us pick out the bi-homogeneous components of bi-degree $(k + 1, l)$. Then $\lambda_{lj} \cdot x \cdot f_{kl} = \widetilde{\gamma}_1 \bar{B}_1 + \widetilde{\gamma}_2 \bar{B}_2 + \dots + \widetilde{\gamma}_m \bar{B}_m \in \langle T \rangle$ where $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_m$ are bi-homogeneous.

Since $\lambda_{lj} \neq 0$ because of $l \neq j$, we have $x \cdot f_{kl} \in \langle T \rangle$, so $x \cdot f_{kl} \in P$.

Similarly let $\overline{h^{(s)}} = f x^{(s)} - q^j x^{(s)} f$. As above we get $x^{(s)} \cdot f_{kl} \in P$ for all s . Since $\langle x, x', \dots, x^{(n)} \rangle$ is not contained in P it follows that at least one of $x^{(s)} \notin P$. Since P is prime, $f_{kl} \in P$, but f_{kl} is obviously bi-homogeneous so $f_{kl} \in \langle T \rangle$. The pair (k, l) is arbitrary so it follows that

$$f = \sum_{st} f_{st} \in \langle T \rangle,$$

which is a contradiction.

7.4. Proof of Theorem 4a

Let us consider the factor ideal $\frac{P}{\langle x, x', \dots, x^{(n)} \rangle}$. Then

$$\frac{P}{\langle x, x', \dots, x^{(n)} \rangle} \subset \frac{k \langle x, x', \dots, x^{(n)}, y, y', \dots, y^{(n)} \rangle}{\langle x, x', \dots, x^{(n)} \rangle} = k[y, y', \dots, y^{(n)}].$$

Due to the structure of prime ideals of $k[y, y', \dots, y^{(n)}]$ we have

$$\frac{P}{\langle x, x', \dots, x^{(n)} \rangle} = \left(\varphi_1(y, y', \dots, y^{(n)}), \dots, \varphi_k(y, y', \dots, y^{(n)}) \right) \cdot \frac{k \langle x, x', \dots, x^{(n)}, y, y', \dots, y^{(n)} \rangle}{\langle x, x', \dots, x^{(n)} \rangle} = \frac{\langle x, x', \dots, x^{(n)}, \varphi_1(y, y', \dots, y^{(n)}), \dots, \varphi_k(y, y', \dots, y^{(n)}) \rangle}{\langle x, x', \dots, x^{(n)} \rangle}$$

It follows that

$$P = \langle x, x', \dots, x^{(n)}, \varphi_1(y, y', \dots, y^{(n)}), \dots, \varphi_k(y, y', \dots, y^{(n)}) \rangle.$$

Theorem 4b can be proved similarly.

8. Concluding Remarks and Open Problems

As an application of the bi-homogeneous ideals introduced above, we would like to approach the Quantum Cubic. The passage from the curve on the Quantum Plane to the plane curve is well-defined. It involves substitution $q = 1$ and semi-classical limit analogous to $\hbar = 1$ for a Weil Algebras, where the differential operator reduces to a multiplication operator vanishing precisely on the plane curve. However, constructing the quantum curve from the plane curve is not canonical. The main issues lie in the ambiguity in ordering the non-commuting x and y . Our approach of prolongating the Quantum Plane to Quantum Jet Space is one more new attempt to remedy this.

Similarly to the classical approach of expressing Weierstrass Cubic $y^2 = x^3 + ax + b$ using homogeneous coordinates $(x : y : z)$ of the Projective Plane $\mathbb{P}^2$

$$y^2 z = x^3 + axz^2 + bz^3.$$

we can propose to consider “bi-homogenization” $x \rightarrow \frac{x'}{x}$ and $y \rightarrow \frac{y'}{y}$ to obtain the following form of a bi-degree (3,2) Cubic curve on $\mathbb{P}^1 \times \mathbb{P}^1$

$$(y')^2 x^3 = y^2 (x')^3 + ay^2 (x') x^2 + by^2 x^3.$$

As it can be done directly for the classical cubic we would like to check directly that the “bi-homogenized” cubic possesses the associative group law. This was attempted by using the computer algebra systems to prove the associativity of the group law, but without immediate success. It remains to be seen whether or not the group law has to be modified. Among many models recently an ad hoc model similar to ours was proposed in [19]. A very interesting nonstandard quantum plane, which we were not aware of, was proposed in [20]. It could be happening that these models proposed independently are

somewhat equivalent. There is an obvious open problem to build a differential geometry on our Jet Space preferably not equivalent to the Wess-Zumino De Rahm Complex of Yu. I. Manin [2] (which includes an anticommutative coordinates). There is an also encouraging substantial interest of physicists in “quantum” curves, such as in [21].

Funding: This research received no external funding

Conflicts of Interest: The author declares no conflict of interest.

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