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# Regarding New Wave Patterns of the Newly Extended Nonlinear (2+1)-Dimensional Boussinesq Equation with Fourth Order

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**Abstract:** This paper applies the sine-Gordon expansion method to the extended nonlinear (2+1)-dimensional Boussinesq equation. Many new dark, complex and mixed dark-bright soliton solutions of the governing model are derived. Moreover, for better understanding of the results, 2D, 3D and contour graphs under the strain conditions and the suitable values of parameters are also plotted.

**Keywords:** the extended nonlinear (2+1)-dimensional Boussinesq equation; the sine-Gordon expansion method; complex; mixed dark-bright soliton solutions; contour surfaces

## 1. Introduction

This century, soliton theory has been one of the most important theories of nonlinear sciences. Mathematically, one of such fields where this theory has been considered has the aim of explaining the propagation of water waves. Such wave propagation has been observed in quantum mechanics, electricity, optical soliton, optical fibers, viscoelasticity, mathematics, physics, chemistry and in many other areas. A lot of researchers from all over the world have developed various methods to investigate new properties of this propagation: the variational iteration method [1], the modified simply equation method [2], the simplified Hirota's method [3], the extended tanh method [4,5], the tanh method [6], the inverse scattering transform method [7], Hirota's bilinear method [8], the sine-Gordon expansion method [9], the $(\frac{G'}{G})$ expansion method [10], the tanh and extended tanh methods [11], the new Jacobi elliptic function expansion method [12], the extended and generalized tanh-function method [13–15], the hyperbolic ansatz method [16], the rational sinh-cosh method and so on [17–29]. One of these water wave propagations belongs to the Boussinesq who proposed a model in 1871 given as

$$u_{tt} - u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0,$$

where $\beta$ and $\gamma$ are real constants with non-zero figures. The Boussinesq equation can be further divided into two cases: the good Boussinesq (BSQ) equation with $\gamma > 0$, and the bad Boussinesq equation with $\gamma < 0$. Hirota was among the first one who solved the bad Boussinesq equation ($\gamma < 0$) for the multi-soliton solution [30]. After then, several other types of solutions of this equation were obtained in [31] by using the Wronskian formulation. As for the good Boussinesq equation with $\gamma > 0$, the multi-soliton solution was derived in [32] by using the Hirota method. In that work, the bilinear form of the good BSQ equation was obtained and the multi-soliton solution was

subsequently derived from that transformed bilinear equation. A rigourous proof was also provided in [32]. The Wronskian formulation of the good Boussinesq equation can be derived and found in [33]. Several kinds of solutions, including solitons, negations, positions and complexitons, were found by using this Wronskian formulation. The above-mentioned works employed the Hirota bilinear form of the Boussinesq equation to derive the analytical solutions. A systematic method for deriving such a bilinear form is the homogeneous balance method. In fact, a modified version of this method was introduced in [34] to extend the Hirota bilinear form of the Boussinesq equation which had been derived in [30].

This model describes the propagation of shallow water waves within the small amplitudes as they propagate at a uniform speed in a water canal of constant depth. Moreover, it arises in several other fluid dynamical theories. Investigating deeper into properties of this model, some powerful methods [35,36] have been applied successfully. One model based on this equation, namely, the extended nonlinear (2+1)-dimensional Boussinesq equation [37,38] defined by

$$u_{tt} + \alpha u_{yt} - \alpha u_{yy} + \alpha_1 \epsilon u_{xy} + \alpha_2 \epsilon (u^2)_{xx} + \alpha_3 \epsilon^2 u_{xxxx} = 0, \epsilon^2 = \pm 1, \tag{1}$$

was newly presented to the literature. In Equation (1), $\alpha, \alpha_1, \alpha_2$ and $\alpha_3$ are real constants and non-zero. Equation (1) has been derived from the Boussinesq model which is used to explain water waves arising in fluid dynamics as special cases of Whitham theory [39,40].

This paper is distributed in different sections. In Section 2, we present the general properties of sine-Gordon expansion method (SGEM) in a detailed manner. In Section 3, we apply SGEM to the Equation (1) to extract many new dark, complex and mixed dark-bright soliton solutions. Moreover, strain conditions are also derived for validity of results. In the last section, we underline the novelty of these results via a conclusion section.

## 2. General Facts of SGEM

In this section we shall consider the following sine-Gordon equation [41]:

$$u_{xx}(x,t) - u_{tt}(x,t) = m^2 \sin(u(x,t)), \tag{2}$$

in which $m$ is a real non-zero constant. When we apply the traveling wave transformation as $u(x,t) = U(\xi)$, $\xi = \mu(x - ct)$ into Equation (2), it may be rewritten as the following nonlinear ordinary differential equation (NODE):

$$U'' = \frac{m^2}{\mu^2(1-c^2)} \sin(U), \tag{3}$$

where $U = U(\xi)$, $U'' = \frac{d^2U}{d\xi^2}$ and c, $\mu$, are also non-zero real constants. Integrating Equation (3) by multiplying $U'$ both sides for calculating, we obtain

$$\left( (\frac{U}{2})' \right)^2 = \frac{m^2}{\mu^2(1-c^2)} \sin^2(\frac{U}{2}) + k, \tag{4}$$

where $k$ is an integral constant. For simplicity, we consider $k = 0$, $w = \frac{U}{2}$, and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$, Equation (4) reads as

$$w' = a \sin(w), \tag{5}$$

where $w = w(\xi)$. By taking $a = 1$ in Equation (5), we obtain the following two interesting and important relationships

$$\sin(w) = \sin[w(\xi)] = \frac{2pe^{\xi}}{p^2e^{2\xi} + 1} \downarrow p_{=1} = \operatorname{sech}(\xi), \tag{6}$$

$$\cos(w) = \cos[w(\xi)] = \frac{2pe^{\xi}}{p^2e^{2\xi}+1} \downarrow p_{=1} = \tanh(\xi). \tag{7}$$

We shall consider in a general case the following nonlinear partial differential equation defined by

$$P(u, u_x, u_{xt}, u^2, \cdots) = 0. \tag{8}$$

In Equation (8), by applying the traveling wave transform as $u = u(x,t) = U(\xi)$, $\xi = \mu(x - ct)$, we obtain the following NODE

$$N(U, U', U'', U^2, \cdots) = 0. \tag{9}$$

where $U = U(\xi)$, $U^{'} = \frac{dU}{d\xi}$. In this NODE, supposing the trial solution function may be considered as

$$U(\xi) = \sum_{i=1}^{n} \tanh^{i-1}(\xi)[B_i \text{sech}(\xi) + A_1 \tanh(\xi)] + A_0. \tag{10}$$

Equation (10) may be rewritten with the help of Equations (6) and (7) as following

$$U(w) = \sum_{i=1}^{n} \cos^{i-1}(w)[B_i \sin(w) + A_1 \cos(w)] + A_0. \tag{11}$$

in which the value of $n$ will be determined later via balance principle. After putting the necessary derivations of Equation (11) into Equation (9), we obtain an equation of $\sin^i(w)\cos^j(w)$. Taking all these terms to zero yields a system of equations. Solving this system by using some computational programs, gives the values of $A_i, B_i, \mu$ and $c$. Via these values of parameters $A_i, B_i, \mu$ and $c$ in Equation (10), we obtain the new traveling wave solutions to Equation (8).

## 3. Application of SGEM

In this section, we apply SGEM to the Equation (1) for obtaining new traveling wave solutions. Applying the traveling wave transformation defined by

$$u(x, y, t) = U(\xi),\ \xi = kx + wy - ct, \tag{12}$$

into Equation (1) results in the following NODE:

$$(c^2 - \alpha wc - \alpha w^2 + \alpha_1 \epsilon kw)U^{''} + k^2\alpha_2\epsilon(U^2)^{''} + k^4\epsilon^2 U^{(4)} = 0.$$

Integrating twice and setting the integral constants to zero yields the following NODE

$$\alpha_3 k^4\epsilon^2 U^{''} + (c^2 - \alpha wc - \alpha w^2 + \alpha_1\epsilon kw)U + k^2\alpha_2\epsilon U^2 = 0. \tag{13}$$

Balancing in Equation (13) yields $n = 2$. Putting $n = 2$ in Equation (11), we get the following

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + B_2 \cos(w)\sin(w) + A_2 \cos^2(w) + A_0, \tag{14}$$

and

$$\begin{aligned} U^{''}(w) = {} & B_1 \cos^2(w)\sin(w) - B_1 \sin^3(w) - 2A_1 \sin^2(w)\cos(w) + \\ & B_2 \cos^3 \sin(w) - 5B_2 \sin^3(w)\cos(w) - 4A_2 \cos^2(w)\sin^2(w) + 2A_2 \sin^4(w). \end{aligned} \tag{15}$$

Substituting Equations (14) and (15) into Equation (13), we obtain an equation of $\sin^i(w)\cos^i(w)$. Getting all coefficients of these terms to zero, we gain a system of equations. Solving this system via some powerful package programs, we find the values of parameters $A_i, B_i, \mu$ and $c$, which produce many entirely new traveling wave solutions to the Equation (1).

**Case-1** When we select these values of parameters as $A_0 = \frac{3k^2\varepsilon\alpha_3}{\alpha_2}$, $A_1 = 0$, $A_2 = -\frac{3k^2\varepsilon\alpha_3}{\alpha_2}$, $B_1 = 0$, $B_2 = \frac{3ik^2\varepsilon\alpha_3}{\alpha_2}$, $c = \frac{1}{2}(\alpha w - \sqrt{\alpha(4+\alpha)w^2 - 4k\varepsilon(w\alpha_1 + k^3\varepsilon\alpha_3)})$ for placement into Equation (10) with $n = 2$, it yields the following mixed dark-bright soliton solution to the governing model.

$$\begin{aligned} u_1(x,y,t) = {} & \frac{3k^2\varepsilon\alpha_3}{\alpha_2} + \frac{3ik^2\varepsilon\alpha_3}{\alpha_2}\text{sech}(kx+wy-ct)\tanh(kx+wy-ct) \\ & - \frac{3k^2\varepsilon\alpha_3}{\alpha_2}\tanh^2(kx+wy-ct), \end{aligned} \tag{16}$$

where $c$ is defined as $\frac{1}{2}(\alpha w - \sqrt{\alpha(4+\alpha)w^2 - 4k\varepsilon(w\alpha_1 + k^3\varepsilon\alpha_3)})$ with the strain condition $\alpha(4+\alpha)w^2 - 4k\varepsilon(w\alpha_1 + k^3\varepsilon\alpha_3) \geq 0$.

**Case-2** Choosing as $A_0 = \frac{2k^2\varepsilon\alpha_3}{\alpha_2}$, $A_1 = 0$, $A_2 = -\frac{3k^2\varepsilon\alpha_3}{\alpha_2}$, $B_1 = 0$, $B_2 = -\frac{3ik^2\varepsilon\alpha_3}{\alpha_2}$, $c = \frac{1}{2}(\alpha w + \sqrt{\alpha(4+\alpha)w^2 - 4k\varepsilon w\alpha_1 + 4k^4\varepsilon^2\alpha_3})$ for Equation (10) with $n = 2$, it produces another new mixed dark-bright soliton solution as follows.

$$\begin{aligned} u_2(x,y,t) = {} & \frac{2k^2\varepsilon\alpha_3}{\alpha_2} - \frac{\alpha_3}{\alpha_2}3ik^2\varepsilon\text{sech}(kx+wy-ct)\tanh(kx+wy-ct) \\ & - \frac{\alpha_3}{\alpha_2}3k^2\varepsilon\tanh^2(kx+wy-ct), \end{aligned} \tag{17}$$

where the strain condition is $\alpha w + \sqrt{\alpha(4+\alpha)w^2 - 4k\varepsilon w\alpha_1 + 4k^4\varepsilon^2\alpha_3} \geq 0$.

**Case-3** Once we set $A_0 = -A_2$, $A_1 = B_1 = 0$, $B_2 = -iA_2$, $\alpha_1 = \frac{-3c^2+3\alpha cw+3\alpha w^2+k^2\varepsilon A_2\alpha_2}{3kw\varepsilon}$, $\alpha_3 = -\frac{A_2\alpha_2}{3k^2\varepsilon}$, and inserting these values into Equation (10) along with $n = 2$, we obtain another new complex dark-bright soliton as follows.

$$\begin{aligned} u_3(x,y,t) = {} & -A_2 + iA_2\text{sech}(-kx-wy+ct)\tanh(-kx-wy+ct) \\ & + A_2\tanh^2(-kx-wy+ct); \end{aligned} \tag{18}$$

in here, $A_2, k, w, c$ are real constants and non-zero.

**Case-4** If it is chosen that $A_0 = \frac{6k^2\varepsilon\alpha_3}{\alpha_2}$, $A_1 = 0$, $A_2 = -\frac{6k^2\varepsilon\alpha_3}{\alpha_2}$, $B_1 = 0$, $B_2 = 0, \alpha_1 = \frac{-c^2+\alpha cw+\alpha w^2-4k^4\varepsilon^2\alpha_3}{kw\varepsilon}$, the dark soliton solution to the Equation (1), when considering Equation (10) to have $n = 2$, is obtained as follows.

$$u_4(x,y,t) = \frac{6k^2\varepsilon\alpha_3}{\alpha_2}(1 - \tanh^2(-kx-wy+ct)), \tag{19}$$

in which $k, w, c, \alpha_3$ are real constants and non-zero.

**Case-5** By considering another coefficients defined by $A_0 = \frac{-A_2}{3}$, $A_1 = B_1 = B_2 = 0$, $\alpha = \frac{-2A_2^2\alpha_2^2+3i\alpha_1\sqrt{6}w\sqrt{\varepsilon A_2\alpha_2\alpha_3}+18c^2\alpha_3}{18w(c+w)\alpha_3}$, $k = \frac{i\sqrt{A_2\alpha_2}}{\sqrt{6\varepsilon\alpha_3}}$, we find another complex dark soliton solution given by

$$u_5(x,y,t) = \frac{-A_2}{3} + A_2\tanh^2(\frac{-i\sqrt{A_2\alpha_2}}{\sqrt{6\varepsilon\alpha_3}}x - wy + ct), \tag{20}$$

in which $A_2, \varepsilon, \alpha_2, \alpha_3, w, c$ are real constants and non-zero.

**Case-6** Taking these coefficients as $A_0 = -A_2$, $A_1 = B_1 = B_2 = 0$, $\alpha = \frac{A_2^2\alpha_2^2-3i\alpha_1 w\sqrt{1.5\varepsilon A_2\alpha_2\alpha_3}+9c^2\alpha_3}{9w(c+w)\alpha_3}$, $k = \frac{-i\sqrt{A_2\alpha_2}}{\sqrt{6\varepsilon\alpha_3}}$, we find another complex dark soliton solution given by

$$u_6(x,y,t) = -A_2 + A_2\tanh^2(\frac{i\sqrt{A_2\alpha_2}}{\sqrt{6\varepsilon\alpha_3}}x - wy + ct), \tag{21}$$

in which $A_2, \varepsilon, \alpha_2, \alpha_3, w, c$ are real constants and non-zero. Choosing the suitable values of parameters along with strain conditions, we plot some various simulations as being Figures 1–8 with the help of computational programs.

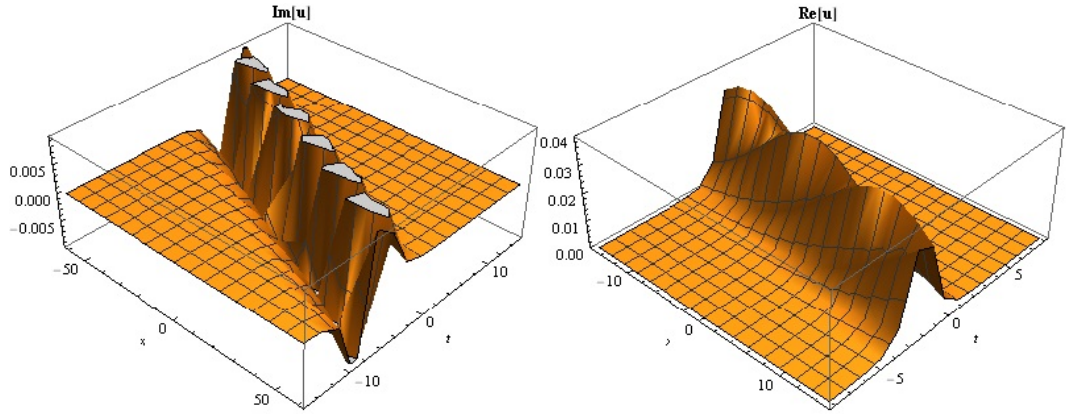



**Figure 1.** The 3D figures of $u_1$ for k = 0.1, $w$ = 2, $\varepsilon$ = 1, $\alpha$ = 0.4,, $\alpha_1$ = 0.3, $\alpha_2$ = 4, $\alpha_3$ = 0.5, y = 0.05.

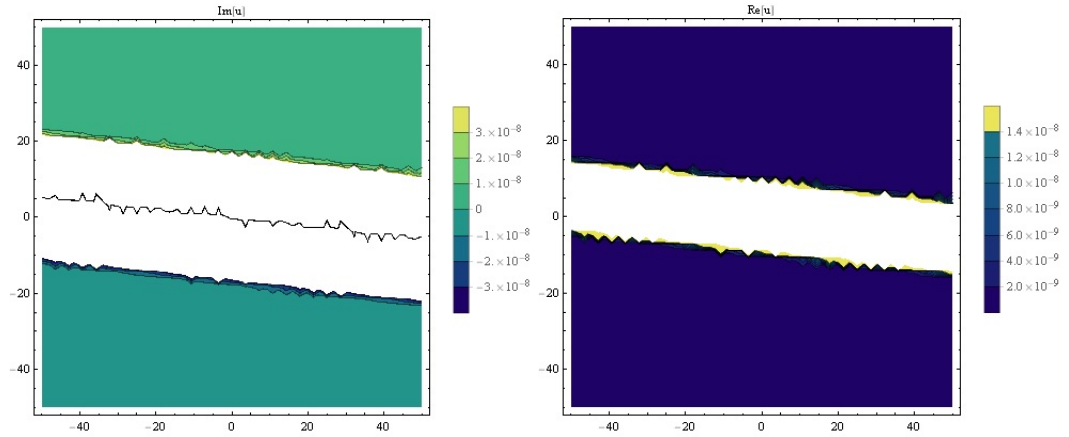



**Figure 2.** The contour simulations of $u_1$ for k = 0.1, $w$ = 2, $\varepsilon$ = 1, $\alpha$ = 0.4, $\alpha_1$ = 0.3, $\alpha_2$ = 4, $\alpha_3$ = 0.5, y = 0.05.

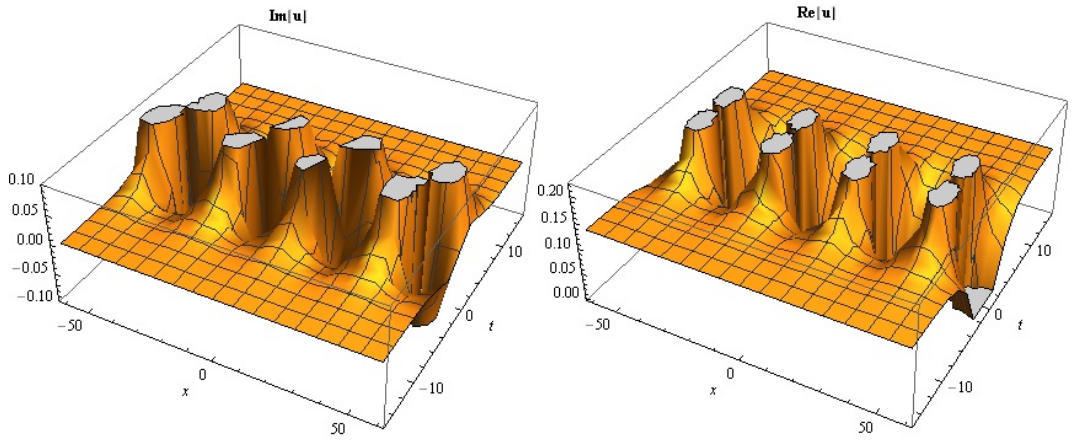



**Figure 3.** The 3D figures of $u_5$ for $\alpha_2$ = 0.1, $\alpha_3$ = 0.32, $\varepsilon$ = 1,$w$ = 0.012, $A_2$ = 0.2, c = 0.3, y = 0.05.

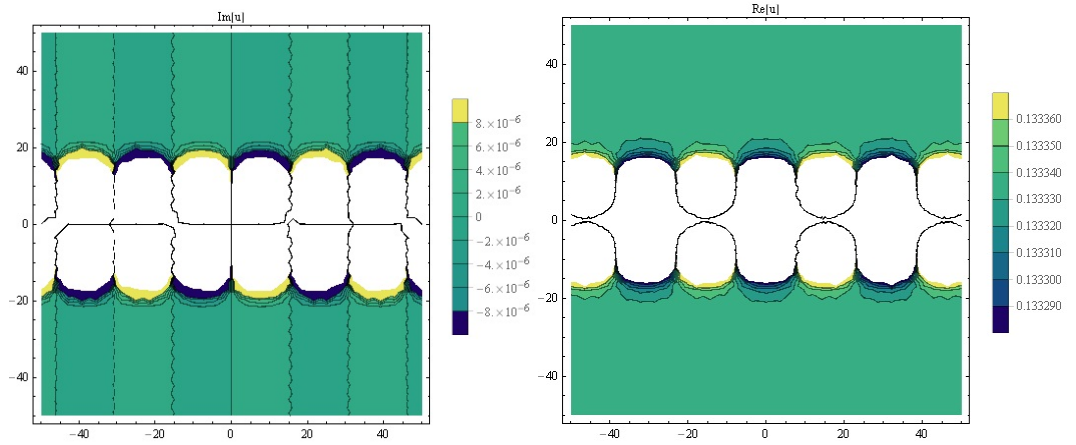



**Figure 4.** The contour simulations of $u_5$ for $\alpha_2 = 0.1$, $\alpha_3 = 0.32$, $\varepsilon = 1$, $w = 0.012$, $A_2 = 0.2$, c = 0.3, y = 0.05.

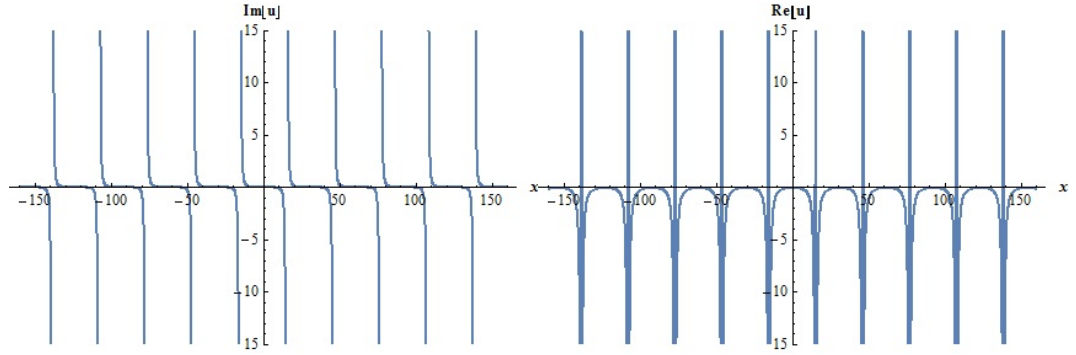



**Figure 5.** The 2D graphs of $u_5$ for $\alpha_2 = 0.1$, $\alpha_3 = 0.32$, $\varepsilon = 1$, $w = 0.012$, $A_2 = 0.2$, c = 0.3, $t = 0.13$, y = 0.05.

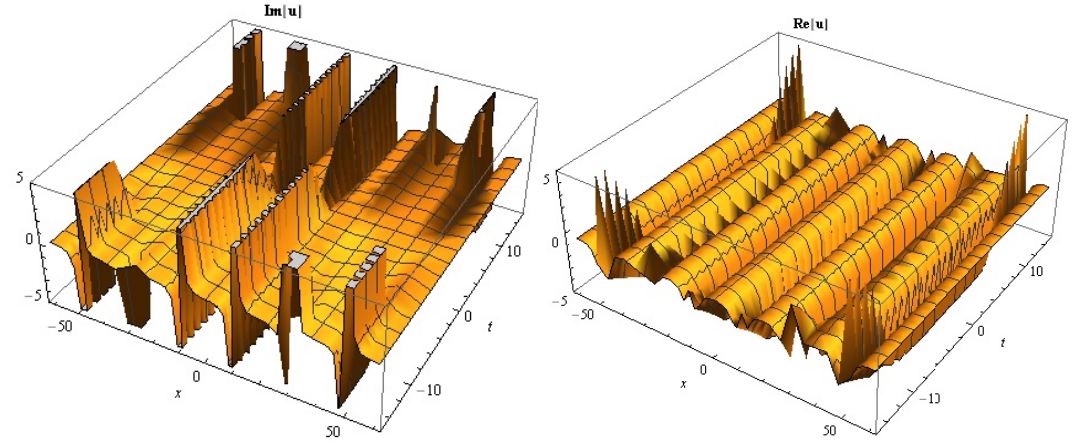



**Figure 6.** The 3D figures of $u_6$ for $\alpha_2 = 0.1$, $\alpha_3 = 0.32$, $\varepsilon = 1$, $w = 0.012$, $A_2 = 0.5$, c = 0.01, y = 0.05.

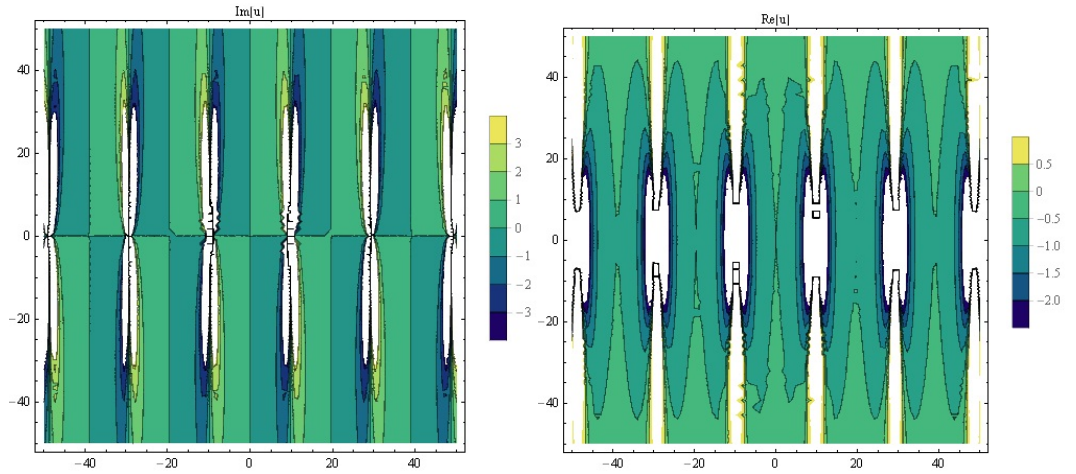



**Figure 7.** The contour simulations of $u_6$ for $\alpha_2 = 0.1$, $\alpha_3 = 0.32$, $\varepsilon = 1$, $w = 0.012$, $A_2 = 0.5$, c = 0.01, y = 0.05.

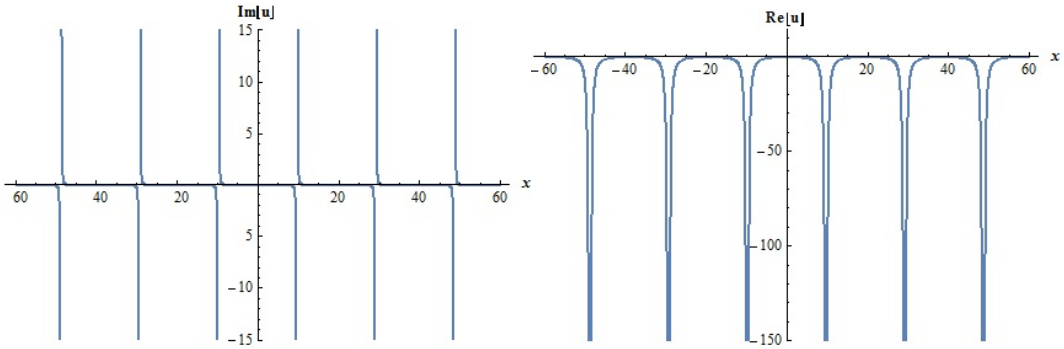



**Figure 8.** The 2D graphs of $u_6$ for $\alpha_2 = 0.1$, $\alpha_3 = 0.32$, $\varepsilon = 1$,$w = 0.012$, $A_2 = 0.5$, c = 0.01, $t = 0.01$, y = 0.05.

## 4. Conclusions

In this paper, we have successfully applied SGEM into a governing model. Many entirely new soliton solutions, such as dark, mixed dark-bright and complex, have been extracted. Strain conditions have been also given for valid the solutions. By putting suitable values of coefficients in to Equation (10) along with $n = 2$, we have plotted the 2D, 3D and contour surfaces of some results. It has been observed that wave patterns from Figures 1–8 show their estimated behaviors physically. Moreover, it has been also observed that these results satisfy the considered model. Furthermore, it is also estimated that these dark solitons are related to the gravitational potential [42].

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