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New Oscillation Results for Third-Order Half-Linear Neutral Differential Equations

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Abstract: The main purpose of this paper is to obtain criteria for the oscillation of all solutions of a third-order half-linear neutral differential equation. The main result in this paper is an oscillation theorem obtained by comparing the equation under investigation to two first order linear delay differential equations. An additional result is obtained by using a Riccati transformation technique. Examples are provided to show the importance of the main results.

Keywords: third-order; neutral; oscillation; delay differential equation

MSC: 34C10; 34K11

1. Introduction

In this paper, we study the oscillation of all solutions of the third-order neutral differential equation

$$(g(t)((h(t)z'(t))')^{\alpha})' + f(t)y^{\alpha}(t) = 0, \; t \geq t_0, \tag{1}$$

where $z(t) = y(t) + p(t)y(\sigma(t))$, subject to the following assumptions:

(H_1) $\sigma \in C^1([t_0,\infty),\mathbb{R})$ with $\sigma(t) \leq t$, and $\lim_{t\to\infty}\sigma(t) = \infty$;
(H_2) $p, f \in C([t_0,\infty),[0,\infty))$, $0 < p(t) \leq p < 1$, and f does not vanish identically;
(H_3) α is a ratio of odd positive integers;
(H_4) $g, h \in C([t_0,\infty),(0,\infty))$ and satisfy

$$\int_{t_0}^{\infty} \frac{1}{g^{\frac{1}{\alpha}}(t)}dt = \int_{t_0}^{\infty} \frac{1}{h(t)}dt = \infty.$$

The function $y(t)$ is said to be a solution of Equation (1) if the corresponding function $z(t) \in C^1([T_y,\infty))$, $T_y \geq t_0$, $(h(t)(z'(t)^{\alpha})' \in C([T_y,\infty))$, $g(t)(h(t)(z'(t))^{\alpha})' \in C([T_y,\infty))$, and $y(t)$ satisfies Equation (1) on $[T_y,\infty)$. We assume Equation (1) possesses solutions satisfying $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$ i.e., Equation (1) has continuable solutions. Such a solution of Equation (1) is called oscillatory if it has infinitely many zeros on $[T_y,\infty)$, and *nonoscillatory* otherwise. We say that an equation is oscillatory if all of its solutions are oscillatory.

In recent years there has been great interest in investigating the oscillatory and asymptotic behavior of third-order functional differential equations. A significant difference between the results here and those in [1–11] is that in all these papers the results obtained are of the form that a solution is either oscillatory or it converges to zero. This is often the "expected" result if direct proofs are attempted. In the present paper, we are able to obtain the oscillation of all solutions due to the technique of proof, namely, by comparing the equation under consideration to an inequality whose oscillatory behavior is known.

In [12], the authors used a relation of the form

$$y(t) \geq (1 - p(t))z(t) \tag{2}$$

where y is positive and z is positive and increasing. In addition, based on a result given in ([13], page 28), if y and z are positive and z is decreasing, they then assume that y is also nonincreasing. This leads to the following relation between y and z:

$$y(t) \geq \left(\frac{1}{1 + p(t)}\right) z(t). \tag{3}$$

However, in a very nice paper [14], the authors present a counter example to show that if z is decreasing, then y does not need to be decreasing, and therefore the relation Equation (3) is not correct. Motivated by this observation, in this paper we first obtain a valid relation between y and z if both are positive and z is decreasing (see Lemma 3 below). Then using this relation, we present some new oscillation criteria for Equation (1). Thus, the results established in this paper are new and complement results already reported in the literature.

2. Main Results

We begin with the following result that gives the basic properties of positive nonoscillatory solutions of Equation (1). An analogous result holds for eventually negative solutions.

Lemma 1. *Assume that $y(t)$ is a positive solution of Equation (1). Then the corresponding function $z(t)$ satisfies one of two cases for all sufficiently large t:*

(I) $z(t) > 0$, $z'(t) < 0$, $(h(t)z'(t))' > 0$, *and* $(g(t)((h(t)z'(t))')^\alpha)' \leq 0$;
(II) $z(t) > 0$, $z'(t) > 0$, $(h(t)z'(t))' > 0$, $(g(t)((h(t)z'(t))')^\alpha)' \leq 0$.

Proof. The structure of positive nonoscillatory solutions of Equation (1) under condition (H_4) follows from well known results of Kiguradze and Chanturia [15]. □

In the following two lemmas we obtain useful relationships between the functions y and z. These will be used in place of the incorrect inequality (3) described in the previous section of this paper.

Lemma 2. *Let $y(t)$ be a positive solution of Equation (1) and let $z(t)$ satisfy Case (II) of Lemma 1. Then*

$$y(t) \geq (1 - p(t))z(\sigma(t)) \tag{4}$$

for all sufficiently large t.

Proof. From the definition of $z(t)$, we have $z(t) \geq y(t)$ and

$$y(t) \geq z(t) - p(t)z(\sigma(t)) \geq (1 - p(t))z(\sigma(t))$$

since z is increasing. □

To simplify our notation, for any $T \geq t_0$, we set

$$\begin{aligned} G_T(t) &= \int_T^t \frac{1}{g^{\frac{1}{\alpha}}(s)} ds, \\ H_T(t) &= \int_T^t \frac{G_T(s)}{h(s)} ds, \\ Q_T(t) &= \frac{1}{h(t)} \int_t^\infty \left(\frac{1}{g(s)} \int_s^\infty f(u)du \right)^{\frac{1}{\alpha}} ds, \\ \phi_T(t) &= \exp\left(\int_T^t Q_T(s) ds \right) \end{aligned}$$

for all $t \geq T$.

Lemma 3. *Let $y(t)$ be a positive solution of Equation (1) with $z(t)$ satisfying Case (I) of Lemma 1 and assume that $\psi(t) = \left(\frac{\phi_T(\sigma(t))}{\phi_T(t)} - p(t) \right) > 0$ for $t \geq T$. Then,*

$$z(t)\phi_T(t) \text{ is increasing} \tag{5}$$

and

$$y(t) \geq \psi(t) z(\sigma(t)) \tag{6}$$

for $t \geq T$.

Proof. Assume that $y(t)$ is a positive solution of Equation (1) with the corresponding function $z(t)$ satisfying Case (I) of Lemma 1 for all $t \geq T$, for some $T \geq t_0$. Then, it is easy to verify that $\lim_{t\to\infty} h(t)z'(t) = 0$ and $\lim_{t\to\infty} g(t)((h(t)z'(t))')^\alpha = 0$. An integration of Equation (1) then yields

$$g(t)((h(t)z'(t))')^\alpha = \int_t^\infty f(s)y^\alpha(s)ds \leq \int_t^\infty f(s)z^\alpha(s)ds \leq z^\alpha(t) \int_t^\infty f(s)ds.$$

Integrating again, we obtain

$$h(t)z'(t) \geq -z(t) \int_t^\infty \left(\frac{1}{g(s)} \int_s^\infty f(u)du \right)^{\frac{1}{\alpha}} ds,$$

or

$$z'(t) \geq -z(t)Q_T(t).$$

Hence,

$$(z(t)\phi_T(t))' = z'(t)\phi_T(t) + z(t)\phi_T'(t) \geq z(t)(\phi_T'(t) - Q_T(t)\phi_T(t)) = 0,$$

so $z(t)\phi_T(t)$ is increasing.

From the definition of z, we have

$$\begin{aligned} y(t) &\geq z(t) - p(t)z(\sigma(t)) = \frac{z(t)\phi_T(t)}{\phi_T(t)} - p(t)z(\sigma(t)) \\ &\geq \left(\frac{\phi_T(\sigma(t))}{\phi_T(t)} - p(t) \right) z(\sigma(t)) \end{aligned}$$

since $z(t)\phi_T(t)$ is increasing. This proves the lemma. □

Our final lemma provides some inequalities involving z and z' and the functions G_T and H_T defined above. They are used in the proofs of Theorems 1 and 2 below.

Lemma 4. *Assume that $y(t)$ is a positive solution of Equation (1) and $z(t)$ satisfies Case (II) of Lemma 1 for all $t \geq T$. Then*

$$z'(t) \geq \frac{g^{\frac{1}{\alpha}}(t)}{h(t)}(h(t)z'(t))'G_T(t), \tag{7}$$

$$z(t) \geq g^{\frac{1}{\alpha}}(t)(h(t)z'(t))'H_T(t), \tag{8}$$

and

$$z(\sigma(t)) \geq H_T(\sigma(t))\frac{h(t)z'(t)}{G_T(t)} \tag{9}$$

for all $t \geq T$.

Proof. The relation Equation (7) easily follows from Lemma 5 in [11], and Equation (8) follows by integrating Equation (7) from T to t and simplifying. From Equation (7), it is easy to see that

$$\left(\frac{h(t)z'(t)}{G_T(t)}\right)' \leq 0,$$

and therefore $\frac{h(t)z'(t)}{G_T(t)}$ is decreasing for $t \geq T$. Furthermore,

$$z(t) = z(T) + \int_T^t \frac{h(s)z'(s)}{G_T(s)}\frac{G_T(s)}{h(s)}ds \geq \frac{h(t)z'(t)}{G_T(t)}H_T(t).$$

Hence,

$$z(\sigma(t)) \geq H_T(\sigma(t))\frac{h(\sigma(t))z'(\sigma(t))}{G_T(\sigma(t))} \geq H_T(\sigma(t))\frac{h(t)z'(t)}{G_T(t)}$$

since $\frac{h(t)z'(t)}{G_T(t)}$ is decreasing. This completes the proof of the lemma. □

We are now ready to state and prove our main results.

Theorem 1. *Let $\sigma'(t) > 0$ and assume that there is a function $\xi(t) \in C^1([t_0, \infty))$ such that*

$$\xi'(t) \geq 0,\ \xi(t) > t \text{ and } \eta(t) = \sigma(\xi(\xi(t))) < t. \tag{10}$$

If both of the first order delay differential equations

$$w'(t) + \left[\frac{1}{h(t)}\int_t^{\xi(t)}\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)} f(s_1)\psi^\alpha(s_1)ds_1\right)^{\frac{1}{\alpha}} ds_2\right] z(\eta(t)) = 0 \tag{11}$$

and

$$w'(t) + f(t)(1-p(t))^\alpha H_T^\alpha(\sigma(t))w(\sigma(t)) = 0 \tag{12}$$

are oscillatory, then Equation (1) is oscillatory.

Proof. Let $y(t)$ be a positive solution of Equation (1). Then there is a $T \geq t_0$ such that $y(t) > 0$ and $y(\sigma(t)) > 0$ for all $t \geq T$. From the definition of $z(t)$, we have $z(t) > 0$ for all $t \geq T$, where T is also chosen so that Lemmas 1–4 hold for all $t \geq T$.

Case (I). Using Equation (6) in Equation (1), we obtain

$$(g(t)((h(t)z'(t))')^\alpha)' + f(t)\psi^\alpha(t)z^\alpha(\sigma(t)) \leq 0, \tag{13}$$

and integrating from t to $\xi(t)$ yields

$$\begin{aligned} g(t)((h(t)z'(t))')^{\alpha} &\geq \int_t^{\xi(t)} f(s_1)\psi^{\alpha}(s_1)z^{\alpha}(\sigma(s_1))ds_1 \\ &\geq z^{\alpha}(\sigma(\xi(t)))\int_t^{\xi(t)} f(s_1)\psi^{\alpha}(s_1)ds_1. \end{aligned}$$

Thus,

$$(h(t)z'(t))' \geq z(\sigma(\xi(t)))\left(\frac{1}{g(t)}\int_t^{\xi(t)} f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}}.$$

Integrating again from t to $\xi(t)$, we obtain

$$-h(t)z'(t) \geq \int_t^{\xi(t)} z(\sigma(\xi(s_2)))\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)} f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}} ds_2,$$

or

$$-z'(t) \geq z(\eta(t))\left(\frac{1}{h(t)}\int_t^{\xi(t)}\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)} f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}} ds_2\right).$$

Finally, integrating from t to ∞ gives

$$z(t) \geq \int_t^{\infty}\frac{z(\eta(s_3))}{h(s_3)}\int_{s_3}^{\xi(s_3)}\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)} f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}} ds_2 ds_3. \tag{14}$$

Let us denote the right side of Equation (14) by $w(t)$. Then $w(t) > 0$ is decreasing, $w(t) \leq z(t)$, and it is easy to see that $w(t)$ is a positive solution of the differential inequality

$$w'(t) + \left[\frac{1}{h(t)}\int_t^{\xi(t)}\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)} f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}} ds_2\right] w(\eta(t)) \leq 0.$$

Then Theorem 1 in [16] shows that the corresponding differential Equation (11) also has a positive solution, which is a contradiction.

Case (II). Using Equation (4) in Equation (1), we have

$$(g(t)((h(t)z'(t))')^{\alpha})' + f(t)(1-p(t))^{\alpha}z^{\alpha}(\sigma(t)) \leq 0,\ t \geq T. \tag{15}$$

From Equation (8),

$$z^{\alpha}(\sigma(t)) \geq g(\sigma(t))[(h(\sigma(t))z'(\sigma(t)))']^{\alpha}H_T^{\alpha}(\sigma(t)),\ t \geq T, \tag{16}$$

so combining Equations (16) and (15) yields

$$(g(t)((h(t)z'(t))')^{\alpha})' + f(t)(1-p(t))^{\alpha}H_T^{\alpha}(\sigma(t))g(\sigma(t))[(h(\sigma(t))z'(\sigma(t)))']^{\alpha} \leq 0,\ t \geq T.$$

Let $w(t) = g(t)((h(t)z'(t))')^{\alpha} > 0$; then we see that $w(t)$ is a positive solution of the inequality

$$w'(t) + f(t)(1-p(t))^{\alpha}H_T^{\alpha}(\sigma(t))w(\sigma(t)) \leq 0.$$

Again by Theorem 1 in [16], this implies that the corresponding differential Equation (12) also has a positive solution, which is a contradiction. This completes the proof of the theorem. □

As an example of how to use our above theorem to obtain explicit oscillation criteria, we have the following corollary.

Corollary 1. *Let conditions of Theorem 1 hold. If*

$$\liminf_{t\to\infty}\int_{\eta(t)}^{t}\frac{1}{h(s_3)}\int_{s_3}^{\xi(s_3)}\left(\frac{1}{g(s_2)}\int_{s_2}^{\xi(s_2)}f(s_1)\psi^{\alpha}(s_1)ds_1\right)^{\frac{1}{\alpha}}ds_2ds_3>\frac{1}{e} \tag{17}$$

and

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}f(s)(1-p(s))^{\alpha}H_T^{\alpha}(\sigma(s))ds>\frac{1}{e} \tag{18}$$

are satisfied, then Equation (1) is oscillatory.

Proof. By ([17], Theorem 1), conditions Equations (17) and (18) imply that Equations (11) and (12) are oscillatory. □

We conclude this section with the following theorem. The proof uses a Riccati transformation technique.

Theorem 2. *Let $\sigma'(t)>0$ and let $\xi(t)\in C^1([t_0,\infty))$ satisfy Equations (10) and (17). If there exists a real valued nondecreasing differentiable function $\rho(t)$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)f(s)(1-p(s))^{\alpha}\frac{H_T^{\alpha}(\sigma(s))}{G_T^{\alpha}(s)}-\frac{g(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right]ds=\infty, \tag{19}$$

then Equation (1) is oscillatory.

Proof. Let $y(t)$ be a positive solution of Equation (1). Proceeding as in the proof of Theorem 1, we see that $z(t)$ satisfies one of the cases in Lemma 1. Case (I) can be eliminated by using condition Equation (17) as in the proof of Theorem 1. Now consider Case (II). Define

$$F(t)=\rho(t)\frac{g(t)((h(t)z'(t))')^{\alpha}}{(h(t)z'(t))^{\alpha}}\quad\text{for } t\geq T; \tag{20}$$

then $F(t)>0$ for all $t\geq T$. Differentiating Equation (20) using Equation (15),and simplifying, we obtain

$$F'(t)\leq\frac{\rho'(t)}{\rho(t)}F(t)-\rho(t)f(t)(1-p(t))^{\alpha}\frac{z^{\alpha}(\sigma(t))}{(h(t)z'(t))^{\alpha}}-\frac{\alpha F^{\frac{\alpha+1}{\alpha}}(t)}{\rho^{\frac{1}{\alpha}}(t)g^{\frac{1}{\alpha}}(t)},\quad t\geq T. \tag{21}$$

Applying the inequality $Au-Bu^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}}$ with $A=\frac{\rho'(t)}{\rho(t)}$, $B=\frac{\alpha}{\rho^{\frac{1}{\alpha}}(t)g^{\frac{1}{\alpha}}(t)}$, and $u=F(t)$ in Equation (21), we obtain

$$F'(t)\leq-\rho(t)f(t)(1-p(t))^{\alpha}\frac{z^{\alpha}(\sigma(t))}{(h(t)z'(t))^{\alpha}}+\frac{g(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}. \tag{22}$$

Finally, using Equation (9) in Equation (22) and then integrating the resulting inequality from T to t, gives

$$\int_{T}^{t}\left[\rho(s)f(s)(1-p(s))^{\alpha}\frac{H_T^{\alpha}(\sigma(s))}{G_T^{\alpha}(s)}-\frac{g(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right]ds\leq F(T)<\infty.$$

This contradicts Equation (19) and completes the proof of the theorem. □

3. Examples

In this section, we provide some examples to illustrate the importance of the main results.

Example 1. Consider the third-order neutral delay differential equation

$$(\sqrt{t}(\sqrt{t}(y(t)+py(\lambda t))')')' + \frac{1}{t^2}y(t) = 0,\ t \geq 1. \tag{23}$$

Here we have $g(t) = h(t) = \sqrt{t}$, $f(t) = \frac{1}{t^2}$, $\alpha = 1$, $\sigma(t) = \lambda t$ with $0 < \lambda < 1$, and we take $p < \lambda^2$. Simple calculations show that $G_1(t) = 2(\sqrt{t}-1)$, $H_1(t) = 2(\sqrt{t}-1)^2$, $Q_1(t) = 2/t$, $\phi_1(t) = t^2$, and $\psi(t) = (\lambda^2 - p) > 0$. Choose $\xi(t) = \beta t$ with $\beta > 1$ and $\lambda\beta^2 < 1$; then $\eta(t) = \lambda\beta^2 t < t$ and condition Equation (17) becomes

$$\begin{aligned}\liminf_{t\to\infty} \int_{\lambda\beta^2 t}^{t} \frac{1}{\sqrt{s_3}} \int_{s_3}^{\beta s_3} \frac{1}{\sqrt{s_2}} \int_{s_2}^{\beta s_2} \frac{1}{s_1^2}(\lambda^2 - p)ds_1 ds_2 ds_3 \\ = 2(\lambda^2-p)(1-\frac{1}{\sqrt{\beta}})(1-\frac{1}{\beta})\ln\frac{1}{\lambda\beta^2}.\end{aligned}$$

Condition Equation (18) becomes

$$\liminf_{t\to\infty}\int_{\lambda t}^{t} \frac{2}{s^2}(1-p)(\sqrt{\lambda s}-1)^2 ds = 2(1-p)\lambda\ln\frac{1}{\lambda}.$$

Therefore, by Corollary 1, Equation (23) is oscillatory if $2(1-p)\lambda\ln\frac{1}{\lambda} > \frac{1}{e}$ and $2(\lambda^2-p)(1-\frac{1}{\sqrt{\beta}})(1-\frac{1}{\beta})\ln\frac{1}{\lambda\beta^2} > \frac{1}{e}$.

Example 2. Consider the equation

$$(t^{\frac{2}{9}}((t^{\frac{1}{3}}(y(t)+py(\lambda t))')')^{\frac{1}{3}})' + \frac{1}{t^{\frac{4}{3}}}y^{\frac{1}{3}}(t) = 0,\ t \geq 1. \tag{24}$$

We have $g(t) = t^{\frac{2}{9}}$, $h(t) = t^{\frac{1}{3}}$, $f(t) = \frac{1}{t^{\frac{4}{3}}}$, $\sigma(t) = \lambda t$, $0 < \lambda < 1$, $\alpha = \frac{1}{3}$, and we take $p < \lambda^{\frac{81}{2}}$. Calculations show that $G_1(t) = 3(t^{\frac{1}{3}} - 1)$, $H_1(t) = \frac{3}{2}(2t - 3t^{\frac{2}{3}} + 1)$, $Q_1(t) = \frac{81}{2t}$, $\phi_1(t) = t^{\frac{81}{2}}$, $\psi(t) = (\lambda^{\frac{81}{2}} - p) > 0$. Condition Equations (17) and (18) are satisfied if

$$\frac{81}{2}(\lambda^{\frac{81}{2}}-p)(1-\frac{1}{\beta^{\frac{1}{3}}})(1-\frac{1}{\beta^{\frac{2}{3}}})\ln\frac{1}{\lambda\beta^2} > \frac{1}{e},$$

and

$$3^{\frac{1}{3}}\lambda^{\frac{1}{3}}(1-p)^{\frac{1}{3}}\ln\frac{1}{\lambda} > \frac{1}{e}.$$

Then by Corollary 1, Equation (24) is oscillatory.

Example 3. Consider the third-order equation

$$(y(t)+py(\lambda t))''' + \frac{2\gamma}{t^3}y(t) = 0,\ t \geq 1. \tag{25}$$

In this case we have $g(t) = h(t) = 1$, $\sigma(t) = \lambda t$ with $0 < \lambda < 1$, $\alpha = 1$, and $f(t) = \frac{2\gamma}{t^3}$ where $\gamma > 0$ is a constant. We take $p < \lambda$, and it is not hard to see that $G_1(t) = (t-1)$, $H_1(t) = \frac{(t-1)^2}{2}$, $Q_1(t) = \frac{\gamma}{t}$, $\phi_1(t) = t^{\gamma}$, and $\psi(t) = (\lambda - p) > 0$.

Taking $\xi(t) = \beta t$ with $\beta > 1$ and $\lambda\beta^2 < 1$, we have $\eta(t) = \lambda\beta^2 t$ and condition Equation (17) takes the form

$$\liminf_{t\to\infty}\int_{\lambda\beta^2 t}^{t}\int_{s_3}^{\beta s_3}\int_{s_2}^{\beta s_2}\frac{2\gamma}{s_1^3}(\lambda - p)ds_1 ds_2 ds_3 = \gamma(\lambda-p)(1-\frac{1}{\beta^2})(1-\frac{1}{\beta})\ln\frac{1}{\lambda\beta^2}.$$

By choosing $\rho(t) = t$, we see that condition Equation (19) becomes

$$\limsup_{t\to\infty} \int_1^t \left[\frac{2\gamma}{s^2}(1-p)\frac{(\lambda s-1)^2}{2(s-1)} - \frac{1}{4s} \right] ds$$
$$> \limsup_{t\to\infty} \int_1^t \left\{ \left[\gamma(1-p)\lambda^2 - \frac{1}{4} \right] \frac{1}{s} - \frac{2\gamma\lambda(1-p)}{s(s-1)} + \frac{\gamma(1-p)}{s^2(1-s)} \right\} \to \infty$$

as $t \to \infty$ provided $\gamma(1-p)\lambda^2 > 1/4$. Hence, by Theorem 2, Equation (25) is oscillatory if

$$\frac{\gamma(\lambda - p)(\beta-1)^2(\beta+1))}{\beta^3} \ln \frac{1}{\lambda\beta^2} > \frac{1}{e} \quad \text{and} \quad 4\gamma(1-p)\lambda^2 > 1.$$

4. Conclusions

In this paper, we have obtained some new oscillation criteria for Equation (1) in the cases where $0 < p(t) \le p < 1$ and $\alpha \in (0, \infty)$. These results ensure that all solutions of the equation studied are oscillatory. Our results are new in the sense that the results in the papers [1–11] will not ensure that all solutions of Equation (1) are oscillatory.

It would also be of interest to extend the results here to the cases where $-1 < p(t) \le 0$ or $p(t)$ is oscillatory. The extension of the results here to higher order equations such as

$$(g(t)((h(t)z'(t))')^\alpha)^{(n)} + f(t)y^\alpha(t) = 0,\ t \ge t_0,$$

where $z(t) = y(t) + p(t)y(\sigma(t))$ and $n \ge 3$ is an odd integer would also be of interest.

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