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Properties of Functions Formed Using the Sakaguchi and Gao-Zhou Concept

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Abstract: This paper introduces a new class related to close-to-convex functions denoted by $\mathcal{K}_s^{k,N}$. This class is based on combining the concepts of starlike functions with respect to N-ply symmetry points of the order α, introduced by Chand and Singh; and $\mathcal{K}_s^{(k)}$, introduced by Wang, Gao, and Yuan, which are generalizations of the classes of functions introduced by Sakaguchi and Gao and Zhou, respectively. We investigate the class for several properties including coefficient estimates, distortion and growth theorems, and the radius of convexity.

Keywords: close-to-convex; starlike with respect to N-ply symmetric points; univalent

MSC: 30C45

1. Introduction

The study of geometric functions is the study of the geometric properties of analytic functions. They have been studied extensively by many authors throughout the decades, pioneered by mathematicians Cauchy and Riemann. Example results of their work are the Cauchy Integral Formula ([1], p. 2), which states that $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{\zeta}{(\zeta-z)^{n+1}}d\zeta$, where C is a rectifiable Jordan curve, f is analytic inside and on C, and z is inside C; and the Riemann Mapping Theorem ([2], p. 10), which allows the mapping of any simply connected domain in $\mathbb{C}$ conformally onto the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{A}$ be the class of analytic functions in the unit disc U of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of univalent analytic functions in $\mathcal{A}$. Furthermore, we define the classes of starlike, convex, and close-to-convex functions, denoted by $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{K}$, respectively, with the following definitions.

Definition 1. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{S}^*$ is called starlike if, and only if,*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ (z \in U).$$

Definition 2. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{C}$ is called convex if, and only if,*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ (z \in U).$$

Definition 3. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}$ is called close-to-convex if there exists $g \in \mathcal{S}^*$ such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \ (z \in U).$$

Remark 1. *It is well known that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. The class $\mathcal{S}^*$ is generalized by the class $\mathcal{S}^*(\alpha)$, which consists of starlike functions of the order α $(0 \leq \alpha < 1)$, satisfying the inequality*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in U).$$

The class $\mathcal{P}$ consists of functions with a positive real part of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, which satisfies the conditions $\Re\{p(z)\} > 0$ and $p(0) = 1$. Duren [1] and Goodman [2] provide a more in depth study on the class $\mathcal{S}$ and its subclasses, as well as class $\mathcal{P}$.

Sakaguchi [3] introduced a class of functions that are starlike with respect to symmetric points, denoted by $\mathcal{S}_s^*$, with the following definition,

Definition 4. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{S}_s^*$ if, and only if,*

$$\Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \ (z \in U).$$

Chand and Singh [4] generalized $\mathcal{S}_s^*$ by introducing the class of starlike functions with respect to N-ply symmetry points of order α $(0 \leq \alpha < 1)$ for $N \in \mathbb{N}$, which they denoted using $\mathcal{S}_s^{*,N}(\alpha)$, with the following definition.

Definition 5. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{S}_s^{*,N}(\alpha)$ for $0 \leq \alpha < 1$ if, and only if,*

$$\Re\left\{\frac{zf'(z)}{f_N(z)}\right\} > \alpha, \ (z \in U),$$

where for $\gamma = e^{i\frac{2\pi}{N}}$

$$f_N(z) = \frac{1}{N}\sum_{p=0}^{N-1} \gamma^{-p} f(\gamma^p z).$$

As can be seen, $\mathcal{S}_s^{*,2}(0) = \mathcal{S}_s^{*,2} = \mathcal{S}_s^*$. They also introduced the class of close-to-convex functions with respect to N-ply symmetry points of order α $(0 \leq \alpha < 1)$ for $N \in \mathbb{N}$, denoted by $\mathcal{K}_s^N(\alpha)$, with the following definition.

Definition 6. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}_s^N(\alpha)$ for $0 \leq \alpha < 1$ if there exists $g \in \mathcal{S}_s^{*,N}(0)$, such that*

$$\Re\left\{\frac{zf'(z)}{g_N(z)}\right\} > \alpha, \ (z \in U),$$

where for $\gamma = e^{i\frac{2\pi}{N}}$

$$g_N(z) = \frac{1}{N}\sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p z).$$

In another direction, Gao and Zhou [5] introduced a subclass of close-to-convex functions, denoted by $\mathcal{K}_s$, where the authors considered the products of functions in the denominators as opposed to the sums used by Chand and Singh [4]. The definition of $\mathcal{K}_s$ is shown as follows.

Definition 7. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}_s$ if there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$, such that*

$$\Re\left\{\frac{-z^2 f'(z)}{g(z)g(-z)}\right\} > 0, \ (z \in U).$$

Wang, Gao, and Yuan [6] introduced another subclass of close-to-convex functions, denoted by $\mathcal{K}_s^{(k)}(\alpha, \beta)$ where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$ and $k \in \mathbb{N}$, with the following definiton.

Definition 8. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}_s^{(k)}(\alpha, \beta)$ if there exists a function $g \in \mathcal{S}^*(\frac{k-1}{k})$, such that*

$$\left|\frac{z^k f'(z)}{g_k(z)} - 1\right| < \beta\left|\frac{\alpha z^k f'(z)}{g_k(z)} + 1\right|, \ (z \in U), \tag{2}$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$ and for $\varepsilon = e^{i\frac{2\pi}{k}}$

$$g_k(z) = \prod_{j=0}^{k-1} \varepsilon^{-j} g(\varepsilon^j z).$$

Setting $\alpha = \beta = 1$, for convenience, we write $\mathcal{K}_s^{(k)}(1,1) = \mathcal{K}_s^{(k)}$ and condition (2) becomes equivalent to the following

$$\Re\left\{\frac{z^k f'(z)}{g_k(z)}\right\} > 0, \ (z \in U).$$

Also, we observe that $\mathcal{K}_s^{(2)}(1,1) \equiv \mathcal{K}_s^{(2)} = \mathcal{K}_s$, thus $\mathcal{K}_s^{(k)}(\alpha, \beta)$ generalizes $\mathcal{K}_s$. Apart from these classes, several other authors have worked on classes that relates to the class $\mathcal{S}_s^*$ introduced by Sakaguchi such as Chung et al. [7], Darwish et al. [8], Goyal and Singh [9], Kant [10], Kowalyczk and Leś-Bomba [11], Seker [12], and Wang et al. [13].

Motivated by these papers, the authors considered combining these two approaches to generalize the classes further and developed a unified class of functions. The following definition introduces the class $\mathcal{K}_s^{k,N}$, which unites $\mathcal{K}_s^{(k)}$ and $\mathcal{K}_s^N(0)$.

Definition 9. *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}_s^{k,N}$ if there exists $g \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ where $k, N \in \mathbb{N}$, such that*

$$\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0, \ (z \in U), \tag{3}$$

where for $\varepsilon = e^{i\frac{2\pi}{k}}$

$$G_{k,N}(z) = \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} g_N(\varepsilon^j z), \tag{4}$$

and for $\gamma = e^{i\frac{2\pi}{N}}$

$$g_N(z) = \frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p z). \tag{5}$$

As it can be observed, $\mathcal{K}_s^{k,1} = \mathcal{K}_s^{(k)}$ and $\mathcal{K}_s^{1,N} = \mathcal{K}_s^N(0)$. This paper looks into properties for the class $\mathcal{K}_s^{k,N}$. Part of this study is to observe if these properties are in conjunction with the previously obtained results.

2. Preliminary Results

In establishing the properties, some preliminary lemmas are needed to prove these results are stated and established as follows. Lemmas 1 and 2 are known results for functions in $\mathcal{P}$ [2].

Lemma 1. *Let $p \in \mathcal{P}$. Then $|c_n| \leq 2$ for each $n \in \mathbb{N}$. The result is sharp for $p(z) = \frac{1+z}{1-z}$.*

Lemma 2. *Let $p \in \mathcal{P}$ and $z = re^{i\theta}$. Then*

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r} \tag{6}$$

and

$$|p'(z)| \leq \frac{2}{(1-r)^2}.$$

Lemma 3 is cited from a work by Silverman in [14].

Lemma 3. *Suppose that $p \in \mathcal{P}$, $\gamma \in [0,1)$, $\delta = \frac{\gamma}{1-\gamma}$. Moreover, suppose that $|z| = r$ and $a = \frac{1+r^2}{1-r^2}$. Then*

$$\Re\left\{\frac{zp'(z)}{p(z)+\delta}\right\} \geq \begin{cases} \frac{-2r}{(1+r)(1+\delta-(1-\delta)r)} & \text{for } 0 \leq r \leq r_\gamma, \\ 2\sqrt{\delta^2+a\delta} - a - 2\delta & \text{for } r_\gamma < r < 1, \end{cases} \tag{7}$$

where r_γ is the unique root of the equation

$$(1-2\gamma)r^3 - 3(1-2\gamma)r^2 + 3r - 1 = 0 \tag{8}$$

in the interval $(0,1]$. This result is sharp with equality in (7) attained at the point $z = -r$ for

$$\tilde{p}(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } 0 \leq r \leq r_\gamma, \\ \frac{1-z^2}{1-2z\ cos\ \theta_0+z^2} & \text{for } r_\gamma < r < 1, \end{cases} \tag{9}$$

and $cos\ \theta_0$ is defined by the equation

$$cos\ \theta_0 = \frac{1-r_0^2-(1+r_0^2)(\sqrt{\delta^2+a\delta})}{-2r_0(\sqrt{\delta^2+a\delta})},$$

with

$$r_0 = 2\sqrt{\delta^2+a\delta} - (a+2\delta).$$

In Lemma 4, we modify Lemma 1 obtained in [6] by Wang et al.

Lemma 4. *Let $\phi_i \in \mathcal{S}_s^{*,N}(\alpha_i)$ where $0 \leq \alpha_i < 1$ $(i = 0,1,2,\ldots,k-1;\ N \in \mathbb{N})$. Then for $k-1 \leq \sum_{i=0}^{k-1} \alpha_i < k$,*

$$\frac{1}{z^{k-1}} \prod_{i=0}^{k-1} \phi_{i,N}(z) \in \mathcal{S}^*\left(\sum_{i=0}^{k-1} \alpha_i - (k-1)\right),$$

where for $\gamma = e^{i\frac{2\pi}{N}}$

$$\phi_{i,N}(z) = \frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \phi_i(\gamma^p z).$$

Proof. If $\phi_i \in \mathcal{S}_s^{*,N}(\alpha_i)$ $(i = 0,1,2,\ldots,k-1;\ N \in \mathbb{N})$, then $\phi_{i,N} \in \mathcal{S}^*(\alpha_i)$. This makes it simple to deduce that

$$\Re\left\{\frac{z\phi_{0,N}'(z)}{\phi_{0,N}(z)}\right\} > \alpha_0,\ \Re\left\{\frac{z\phi_{1,N}'(z)}{\phi_{1,N}(z)}\right\} > \alpha_1, \ldots,\ \Re\left\{\frac{z\phi_{k-1,N}'(z)}{\phi_{k-1,N}(z)}\right\} > \alpha_{k-1}.$$

Next, let

$$F(z) = \frac{\phi_{0,N}(z)\phi_{1,N}(z)\dots\phi_{k-1,N}(z)}{z^{k-1}} = \frac{\prod_{i=0}^{k-1}\phi_{i,N}(z)}{z^{k-1}}. \tag{10}$$

Differentiating (10) logarithmically gives

$$\frac{zF'(z)}{F(z)} = \frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)} + \frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)} + \dots + \frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)} - (k-1).$$

Thus,

$$\begin{aligned}\Re\left\{\frac{zF'(z)}{F(z)}\right\} &= \Re\left\{\frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)} + \frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)} + \dots + \frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)}\right\} - (k-1)\\ &= \Re\left\{\frac{z\phi'_{0,N}(z)}{\phi_{0,N}(z)}\right\} + \Re\left\{\frac{z\phi'_{1,N}(z)}{\phi_{1,N}(z)}\right\} + \dots + \Re\left\{\frac{z\phi'_{k-1,N}(z)}{\phi_{k-1,N}(z)}\right\} - (k-1)\\ &> \sum_{i=0}^{k-1}\alpha_i - (k-1).\end{aligned}$$

Thus, if $0 \le \sum_{i=0}^{k-1}\alpha_i - (k-1) < 1$, then

$$F(z) = \frac{\prod_{i=0}^{k-1}\phi_{i,N}(z)}{z^{k-1}} \in \mathcal{S}^*\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right).$$

This completes the proof. □

3. Results

This section presents the properties found for the class $\mathcal{K}_s^{k,N}$. In the following, we present the distortion and growth theorems, coefficient estimates, and the radius of convexity. Before proceeding to the properties, the following theorem proves that $G_{k,N}$ given by (4) is starlike.

Theorem 1. *Let* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ *for* $k, N \in \mathbb{N}$. *Then the function* $G_{k,N}$, *given by (4), is starlike and can be written as*

$$G_{k,N}(z) = z + \sum_{n=2}^{\infty} B_{N(n-1)+1} z^{N(n-1)+1}, \tag{11}$$

where

$$B_{N(n-1)+1} = \begin{cases} b_{N(n-1)+1} & , k = 1,\\ \varphi_{k-1,n} & , k \ge 2, \end{cases}$$

where

$$\varphi_{k-1,n} = \sum_{p_{k-1}=1}^{n} \Psi\left[\sum_{p_{k-2}=1}^{p_{k-1}} \psi_{k-2}\left(\sum_{p_{k-3}=1}^{p_{k-2}} \psi_{k-3}\left(\dots\left(\sum_{p_2=1}^{p_3} \psi_2\left(\sum_{p_1=1}^{p_2} \psi_1 b_{N(p_1-1)+1}\right)\right)\right)\right)\right],$$

with $\Psi = \varepsilon^{N(k-1)(n-p_{k-1})} b_{N(n-p_{k-1})+1}$ *and* $\psi_j = \varepsilon^{Nj(p_{j+1}-p_j)} b_{N(p_{j+1}-p_j)+1}$ *for* $1 \le j \le k-2$.

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ where $k, N \in \mathbb{N}$. From (4) and (5),

$$\begin{aligned}
G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} g_N(\varepsilon^j z) \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p \varepsilon^j z) \right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \left\{ \gamma^p \varepsilon^j z + \sum_{n=2}^{\infty} b_n \gamma^{np} \varepsilon^{nj} z^n \right\} \right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \left\{ \varepsilon^j z + \sum_{n=2}^{\infty} b_n \gamma^{p(n-1)} \varepsilon^{nj} z^n \right\} \right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[\frac{1}{N} \sum_{p=0}^{N-1} \left\{ z + \sum_{n=2}^{\infty} b_n \gamma^{p(n-1)} \varepsilon^{j(n-1)} z^n \right\} \right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[z + \sum_{n=2}^{\infty} \phi_n \varepsilon^{j(n-1)} b_n z^n \right],
\end{aligned}$$

where $\phi_n = \frac{1}{N}(1 + \gamma^{n-1} + \gamma^{2(n-1)} + \ldots + \gamma^{(N-1)(n-1)})$. If $\frac{n-1}{N} = t$ is a positive integer, then $\gamma^{n-1} = \gamma^{Nt} = 1$ as $\gamma^N = 1$, which then implies that $\phi_{Nt+1} = 1$. Otherwise, $\gamma^{n-1} \neq 1$ and

$$\phi_n = \frac{1}{N}\left(\frac{1 - \gamma^{N(n-1)}}{1 - \gamma^{n-1}}\right) = \frac{1}{N}\left(\frac{1 - 1^{n-1}}{1 - \gamma^{n-1}}\right) = 0.$$

This simplification allows $\sum_{n=2}^{\infty} \phi_n \varepsilon^{j(n-1)} b_n z^n$ to be written as $\sum_{t=1}^{\infty} \varepsilon^{Njt} b_{Nt+1} z^{Nt+1}$ and with re-indexing gives

$$G_{k,N}(z) = \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1} \right]. \tag{12}$$

Let $P_j(z) = z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1}$ for $0 \leq j \leq k-1$ and denote $D_j(z)$ as follows

$$\begin{aligned}
&D_0(z) = P_0(z), \\
&D_1(z) = \frac{1}{z} P_1(z) P_0(z), \\
&D_2(z) = \frac{1}{z^2} P_2(z) P_1(z) P_0(z), \\
&\ldots \\
&D_j(z) = \frac{1}{z^j} P_j(z) P_{j-1}(z) \ldots P_1(z) P_0(z).
\end{aligned}$$

As $D_0(z) = P_0(z)$, then, by induction

$$D_j(z) = \frac{1}{z} P_j(z) D_{j-1}(z),$$

which, upon expansion, gives

$$\begin{aligned} D_j(z) &= \frac{1}{z}P_j(z)D_{j-1}(z) \\ &= z + \sum_{n=2}^{\infty} \varphi_j z^{N(n-1)+1}, \end{aligned}$$

where

$$\varphi_{j,n} = \sum_{p_j=1}^{n} \Psi \Bigg[\sum_{p_{j-1}=1}^{p_j} \psi_{j-1} \Bigg(\sum_{p_{j-2}=1}^{p_{j-1}} \psi_{j-2} \Bigg(\cdots \Bigg(\sum_{p_2=1}^{p_3} \psi_2 \Bigg(\sum_{p_1=1}^{p_2} \psi_1 b_{N(p_1-1)+1} \Bigg) \Bigg) \Bigg) \Bigg) \Bigg],$$

with $\Psi = \varepsilon^{Nj(n-p_j)} b_{N(n-p_j)+1}$ and $\psi_J = \varepsilon^{NJ(p_{J+1}-p_J)} b_{N(p_{J+1}-p_J)+1}$ for $1 \leq J \leq j-1$. Thus,

$$\begin{aligned} G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \Bigg[z + \sum_{n=2}^{\infty} \varepsilon^{Nj(n-1)} b_{N(n-1)+1} z^{N(n-1)+1} \Bigg] \\ &= \frac{1}{z^{k-1}} P_{k-1}(z) P_{k-2}(z) (\dots P_1(z) P_0(z) \\ &= D_{k-1}(z) \\ &= z + \sum_{n=2}^{\infty} B_{N(n-1)+1} z^{N(n-1)+1}, \end{aligned}$$

where

$$B_{N(n-1)+1} = \begin{cases} b_{N(n-1)+1} & , \, k = 1, \\ \varphi_{k-1,n} & , \, k \geq 2. \end{cases}$$

As $g \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$, then by Lemma 4 in the case $\phi_i = g$ and $\alpha_i = \frac{k-1}{k}$ for all $i = 0, 1, (\dots, k-1$, $G_{k,N} \in \mathcal{S}^*$. This completes the proof. □

Remark 2. *As $G_{k,N} \in \mathcal{S}^*$, the above theorem shows that the class $\mathcal{K}_s^{k,N}$ is a subclass of the close-to-convex functions $\mathcal{K}$, $\mathcal{K}_s^{k,N} \subset \mathcal{K}$.*

Theorem 2. *The distortion and growth bounds for $f \in \mathcal{K}_s^{k,N}$ are given as follows*

$$\frac{1-r}{(1+r)(1+r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq |f'(z)| \leq \frac{1+r}{(1-r)(1-r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}$$

and

$$\int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho \leq |f(z)| \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho,$$

where $|z| = r < 1$ and m is the highest common factor of N and k. Equality is attained at the right-hand side for the function

$$f_1(z) = \int_0^z \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho,$$

with respect to $g_1(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}} \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$.

Proof. Suppose $f \in \mathcal{K}_s^{k,N}$ for $k, N \in \mathbb{N}$, then there exists a function $g \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$, such that (3) holds. Now suppose $g(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}$ where m is the highest common factor of N and k. Then,

$$g'(z) = \frac{1 + (\frac{2-k}{k})z^{\frac{Nk}{m}}}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}+1}} \text{ and } g_N(z) = \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}.$$

Thus,

$$\Re\left\{\frac{zg'(z)}{g_N(z)}\right\} = \Re\left\{z \cdot \frac{1 + (\frac{2-k}{k})z^{\frac{Nk}{m}}}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}+1}} \cdot \frac{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}{z}\right\} = \Re\left\{\frac{1 + (\frac{2-k}{k})z^{\frac{Nk}{m}}}{1 - z^{\frac{Nk}{m}}}\right\} > \frac{k-1}{k},$$

which implies that $g(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}} \in \mathcal{S}_s^{*,N}(\frac{k-1}{k})$. Substituting $g(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}$ into (4) gives

$$\begin{aligned}
G_{k,N}(z) &= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} g(\gamma^p \varepsilon^j z)\right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \varepsilon^{-j} \left[\frac{1}{N} \sum_{p=0}^{N-1} \gamma^{-p} \frac{\gamma^p \varepsilon^j z}{\left(1 - (\gamma^p \varepsilon^j z)^{\frac{Nk}{m}}\right)^{\frac{2m}{Nk^2}}}\right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \left[\frac{1}{N} \sum_{p=0}^{N-1} \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}\right] \\
&= \frac{1}{z^{k-1}} \prod_{j=0}^{k-1} \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}} \\
&= \frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}},
\end{aligned}$$

and for $|z| < r = 1$,

$$|G_{k,N}(z)| = \left|\frac{z}{(1 - z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}\right| \leq \frac{r}{(1 - r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{13}$$

Similarly with $g(z) = \frac{z}{(1+z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}$, we find

$$\left|G_{k,N}(z)\right| \geq \frac{r}{(1 + r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{14}$$

Combining (13) and (14) results in

$$\frac{r}{(1 + r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq \left|G_{k,N}(z)\right| \leq \frac{r}{(1 - r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}. \tag{15}$$

As there exists a $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z), \tag{16}$$

the following is obtained

$$\frac{1 - r}{(1 + r)(1 + r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)(1 - r^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}, \tag{17}$$

using (6), (15), and (16). From (17), the upper bound for $|f(z)|$ is

$$|f(z)| = \Big| \int_0^z f'(\rho)d\rho \Big| \leq \int_0^r |f'(\rho)|d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho.$$

To prove the lower bound, $z_0 \in U$ with $|z_0| = r$ $(0 < r < 1)$ such that $|f(z_0)| = \min\{|f(z) : |z| = r\}$. It is sufficient to prove that that the left-hand side inequality holds for the point z_0. Morever, $|f(z)| \geq |f(z_0)|$ with $|z| = r$ $(0 \leq r < 1)$. As f is univalent in the unit disc U, as f is a close-to-convex function, the original image of the line segment $\big[0, f(z_0)\big]$ is a piece of arc R in $\{|z| \leq r\}$, then, in accordance to (17),

$$|f(z_0)| = \int_{f(R)} |dw| = \int_R |f'(\rho)|d\rho \geq \int_R \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho.$$

Therefore,

$$\int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho \leq |f(z)| \leq \int_0^r \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho.$$

With $g_1(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}}$, we obtain

$$\Re\left\{\frac{zf_1'(z)}{G_{1,k,N}(z)}\right\} = \Re\left\{z \cdot \frac{1+z}{(1-z)(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} \cdot \frac{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk}}}{z}\right\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0$$

and

$$\Re\left\{\frac{zg_1'(z)}{g_{1,N}(z)}\right\} = \Re\left\{\frac{zg_1'(z)}{g_1(z)}\right\} > \frac{k-1}{k},$$

because $g_{1,N}(z) = \frac{1}{N}\sum_{p=0}^{N-1} \gamma^{-p} g_1(\gamma^p z) = \frac{1}{N}\sum_{p=0}^{N-1} g_1(z) = g_1(z)$ as $\gamma^N = 1$. This proves that $f_1 \in \mathcal{K}_s^{k,N}$ with respect to $g_1 \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$ and completes the proof. □

Remark 3. *Setting $N = 1$ and $k = 2$, we have the result obtained in [5], Theorem 3.*

Theorem 3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{k,N}$ for $k, N \in \mathbb{N}$. Then*

$$|a_n| \leq \begin{cases} \frac{1}{n}\left(2 + 2\sum_{d=1}^{d_1-1} |B_{Nd+1}| + |B_n|\right) & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \frac{2}{n}\left(1 + \sum_{d=1}^{d_2} |B_{Nd+1}|\right) & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}$$

The inequalities are sharp with the extremal function

$$f_1(z) = \int_0^z \frac{1+\rho}{(1-\rho)(1-\rho^{\frac{Nk}{m}})^{\frac{2m}{Nk}}} d\rho,$$

with respect to $g_1(z) = \frac{z}{(1-z^{\frac{Nk}{m}})^{\frac{2m}{Nk^2}}} \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$.

Proof. Let $f \in \mathcal{K}_s^{k,N}$. Then there exists $g \in \mathcal{S}_s^{*,N}\left(\frac{k-1}{k}\right)$, such that

$$\Re\left\{\frac{zf'(z)}{G_{k,N}(z)}\right\} > 0, \ (z \in U),$$

where $k, N \in \mathbb{N}$. $G_{k,N}$ of the form (12) is a starlike function by Theorem 2 and there exists $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z). \tag{18}$$

Using the series of the representations for p, f and $G_{k,N}$ in (18) gives

$$\begin{aligned}
&z + 2a_2z^2 + 3a_3z^3 + \ldots + Na_Nz^N + (N+1)a_{N+1}z^{N+1} + (N+2)a_{N+2}z^{N+2} + \ldots + na_nz^n + \ldots \\
&= z(1 + c_1z + c_2z^2 + \ldots + c_Nz^N + c_{N+1}z^{N+1} + \ldots + c_nz^n + \ldots) + \\
&B_{N+1}z^{N+1}(1 + c_1z + c_2z^2 + \ldots + c_Nz^N + c_{N+1}z^{N+1} + \ldots + c_nz^n + \ldots) + \\
&B_{2N+1}z^{2N+1}(1 + c_1z + c_2z^2 + \ldots + c_Nz^N + c_{N+1}z^{N+1} + \ldots + c_nz^n + \ldots) + \ldots + \\
&B_{N(n-1)+1}z^{N(n-1)+1}(1 + c_1z + c_2z^2 + \ldots + c_Nz^N + c_{N+1}z^{N+1} + \ldots + c_nz^n + \ldots) + \ldots .
\end{aligned}$$

Comparing the coefficients on both sides yieds

$$na_n = \begin{cases} c_{n-1} + \sum\limits_{d=1}^{d_1-1} c_{n-(Nd+1)}B_{Nd+1} + B_n & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ c_{n-1} + \sum\limits_{d=1}^{d_2} c_{n-(Nd+1)}B_{Nd+1} & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}$$

Using Lemma 1 and the triangle inequality, the following inequalities are obtained.

$$|a_n| \leq \begin{cases} \frac{1}{n}\left(2 + 2\sum\limits_{d=1}^{d_1-1} |B_{Nd+1}| + |B_n|\right) & , \ n = Nd_1 + 1 \text{ for } d_1 \geq 1, \\ \frac{2}{n}\left(1 + \sum\limits_{d=1}^{d_2} |B_{Nd+1}|\right) & , \ Nd_2 + 1 < n < N(d_2+1) + 1 \text{ for } d_2 \geq 0. \end{cases}$$

This completes the proof. □

Theorem 4. *The radius of convexity R_c for the class $\mathcal{K}_s^{k,N}$ is the positive real root of the equation*

$$1 - 2r - r^2 - r^{\frac{Nk}{m}} - 2r^{\frac{Nk}{m}+1} + r^{\frac{Nk}{m}+2} = 0.$$

Proof. Suppose $f \in \mathcal{K}_s^{k,N}$ where $k, N \in \mathbb{N}$. Then there exists a $p \in \mathcal{P}$, such that

$$\frac{zf'(z)}{G_{k,N}(z)} = p(z),$$

which upon logarithmic differentation gives

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + \frac{zG'_{k,N}(z)}{G_{k,N}(z)}. \tag{19}$$

As

$$\frac{zG'_{k,N}(z)}{G_{k,N}(z)} = 1 + C_{\frac{Nk}{m}}z^{\frac{Nk}{m}} + C_{2\frac{Nk}{m}}z^{2\frac{Nk}{m}} + \ldots$$

and $G_{k,N} \in \mathcal{S}^*$ for $|z| < 1$, then for $|z| = r$

$$\frac{1 - r^{\frac{Nk}{m}}}{1 + r^{\frac{Nk}{m}}} \leq \Re\left\{\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right\} \leq \left|\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right| \leq \frac{1 + r^{\frac{Nk}{m}}}{1 - r^{\frac{Nk}{m}}}. \tag{20}$$

Using (19), (20) and Lemma 3 when $\delta = 0$,

$$\begin{aligned}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= \Re\left\{\frac{zp'(z)}{p(z)}\right\} + \Re\left\{\frac{zG'_{k,N}(z)}{G_{k,N}(z)}\right\} \\ &\geq \frac{-2r}{1-r^2} + \frac{1-r^{\frac{Nk}{m}}}{1+r^{\frac{Nk}{m}}} \\ &= \frac{-2r(1+r^{\frac{Nk}{m}}) + (1-r^2)(1-r^{\frac{Nk}{m}})}{(1-r^2)(1+r^{\frac{Nk}{m}})} \\ &= \frac{1-2r-r^2-r^{\frac{Nk}{m}}-2r^{\frac{Nk}{m}+1}+r^{\frac{Nk}{m}+2}}{(1-r^2)(1+r^{\frac{Nk}{m}})}.\end{aligned}$$

The next step is to find r such that $\Re\{1+\frac{zf''(z)}{f'(z)}\} > 0$. Note that $(1-r^2)(1+r^{\frac{Nk}{m}}) > 0$ for any $N, k \in \mathbb{N}$ as $1-r^2 > 0$ and $1+r^{\frac{Nk}{m}} > 0$ in $0 \leq r < 1$. Let $F(r) = 1-2r-r^2-r^{\frac{Nk}{m}}-2r^{\frac{Nk}{m}+1}+r^{\frac{Nk}{m}+2}$, $0 \leq r < 1$. Differentiating F gives

$$F'(r) = -2-2r-\frac{Nk}{m}r^{\frac{Nk}{m}-1}-2\left(\frac{Nk}{m}+1\right)r^{\frac{Nk}{m}}+\left(\frac{Nk}{m}+2\right)r^{\frac{Nk}{m}+1},$$

which can be rearranged to

$$F'(r) = -2(1+r)-r^{\frac{Nk}{m}-1}\left[2r\left(\frac{Nk}{m}-r\right)+2r+\frac{Nk}{m}(1-r^2)\right] < 0,$$

for all $0 \leq r < 1$. This shows that F is a monotonically decreasing function within $r \in [0,1)$. As $F(0) = 1$ and $F(1) = -4$, this implies that there exists a root R_c within $(0,1)$ such that $F(R_c) = 0$. Therefore, $\Re\{1+\frac{zf''(z)}{f'(z)}\} > 0$ for $0 \leq |z| = r \leq R_c$, i.e., f is convex. This completes the proof. □

Remark 4. *Setting $N = 1$ and $k = 2$, we obtain the result obtained by [5], Theorem 4.*

4. Conclusions

All results obtained are consistent with prior results. In particular, when $N = k = 1$, the results obtained for the growth and distortion theorems (Theorem 2), coefficient estimates (Theorem 3), and radius of convexity (Theorem 4) are equal to those obtained by Libera [15] in 1964 for the class of close-to-convex functions. This suggests that our obtained results are accurate and sharp. Similarly, for the case $N = k \in \mathbb{N}$, the results concur with those obtained by Chand and Singh [4] in 1979, which again illustrate that the results are accurate and that equality occurs for certain extremal functions. However, new results for Theorems 1 and 3 can be found by considering a different function

$$G_{k,N}(z) = z + \sum_{n=2}^{\infty} B_{\frac{Nk}{m}(n-1)+1} z^{\frac{Nk}{m}(n-1)+1}.$$

From a different perspective, the idea used in introducing the new class demonstrates that it is feasible to merge multiple approahces, generalizing the classes and results. Thus, it is foreseeable to utilise subordination principles in merging multiple approaches to expand other classes (such as class $\mathcal{K}_p^s(\phi)$ by Kant [10] and class $\mathcal{K}_s^{(k)}(\gamma,\mu,\varphi)$ by Chung et al. [7]).

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References

1. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.
2. Goodman, A.W. *Univalent Functions*; Mariner Publication Company: Tampa, FL, USA, 1983.
3. Sakaguchi, K. On a Certain Univalent Mapping. *J. Math. Soc. Jpn.* **1958**, *11*, 72–75. [CrossRef]
4. Chand, R.; Singh, P. On Certain Schlicht Mappings. *Indian J. Pure Appl. Math.* **1979**, *10*, 142–149.
5. Gao, C.Y.; Zhou, S.Q. On a Class of Analytic Functions Related to the Starlike Functions. *Kyungpook Math. J.* **2005**, *45*, 123–130.
6. Wang, Z.G.; Gao, C.Y.; Yuan, S.M. On Certain New Subclass of Close-to-Convex Functions. *Mat. Vesn.* **2006**, *58*, 119–124.
7. Chung, Y.L.; Mohd, M.H.; Lee, S.K. On a Subclass of Close-to-Convex Functions. *Bull. Iran. Math. Soc.* **2018**, *44*, 611–621. [CrossRef]
8. Darwish, H.E.; Lashin, A.Y.; Sowileh, S.M. Some Properties for α-Starlike Functions with Respect to k-Symmetric Points of Complex Order. *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **2017**, *71*, 1–9. [CrossRef]
9. Goyal, S.P.; Singh, O. Certain Subclasses of Close-to-Convex Functions. *Vietnam J. Math.* **2014**, *42*, 53–62. [CrossRef]
10. Kant, S. Sharp Fekete-Szegö Coefficients Functional, Distortion and Growth Inequalities for Certain p-valent Close-to-Convex Functions. *J. Class. Anal.* **2018**, *12*, 99–107. [CrossRef]
11. Kowalczyk, J.; Leś-Bomba, E. On a Subclass of Close-to-Convex Functions. *Appl. Math. Lett.* **2010**, *23*, 1147–1151. [CrossRef]
12. Seker, B. On Certain New Subclass of Close-to-Convex Functions. *Appl. Math. Comput.* **2011**, *218*, 1041–1045.
13. Wang, Z.G.; Gao, C.Y.; Orhan, H.; Akbulut, S. Some Subclasses of Close-to-Convex and Quasi-Convex Functions with Respect to k-Symmetric Points. *Gen. Math.* **2007**, *15*, 107–119.
14. Silverman, H. Convexity Theorems for Subclasses of Univalent Functions. *Pac. J. Math.* **1976**, *1*, 253–263. [CrossRef]
15. Libera, R.J. Some Radius of Convexity Problems. *Duke Math. J.* **1964**, *31*, 143–158. [CrossRef]



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