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Positive Solutions for a Class of p -Laplacian Hadamard Fractional-Order Three-Point Boundary Value Problems

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Abstract: In this paper, using the Avery–Henderson fixed point theorem and the monotone iterative technique, we investigate the existence of positive solutions for a class of p -Laplacian Hadamard fractional-order three-point boundary value problems.

Keywords: Hadamard fractional boundary value problem; Avery–Henderson fixed point theorem; monotone iterative technique; positive solutions

1. Introduction

In this paper, we study the following p -Laplacian Hadamard fractional-order three-point boundary value problem

$$\begin{cases} -D^\alpha(\varphi_p(D^\beta y(t))) = f(t, y(t)), t \in (1, e), \\ y(1) = y(e) = \delta y(1) = \delta y(e) = 0, D^\beta y(1) = 0, D^\beta y(e) = bD^\beta y(\eta), \end{cases} \quad (1)$$

where $\alpha \in (1, 2]$, $\beta \in (3, 4]$ are real numbers, and D^α, D^β are the Hadamard fractional derivatives; δ means that delta derivative, i.e., $\delta y(1) = \left(t \frac{dy}{dt}\right) \Big|_{t=1}$, $\delta y(e) = \left(t \frac{dy}{dt}\right) \Big|_{t=e}$; φ_p is the p -Laplacian, i.e., $\varphi_p(s) = |s|^{p-2}s$ with $s \in \mathbb{R}$, $p > 1$. The constants b, η and the function f satisfy the conditions:

(H1) $b \in [0, +\infty)$ and $\eta \in (1, e)$ with $b^{p-1}(\log \eta)^{\alpha-1} \in [0, 1)$.

(H2) $f \in C([1, e] \times \mathbb{R}^+, \mathbb{R}^+)$.

Fractional calculus arises naturally in describing complex phenomena in many applications. For example, Podlubny [1] introduces a fractional electric circuit model, and gives a closed-loop control system

$$\begin{cases} {}_0D_t^q x(t) = \alpha {}_0D_t^{q-1}(y(t) - x(t)) - \frac{2\alpha}{7}(4x(t) - x^3(t)), \\ \frac{dy(t)}{dt} = x(t) - y(t) + z(t), \\ \frac{dz(t)}{dt} = -\frac{100}{7}y(t), \end{cases}$$

where ${}_0D_t^q, {}_0D_t^{q-1}$ are fractional derivatives. For more details on fractional applications, we refer the reader to [2–18]. In [6], K. Diethelm and N. J. Ford studied numerical solutions for fractional differential equations of the form

$$y^{(\alpha)}(t) = f(t, y(t), y^{(\beta_1)}(t), y^{(\beta_2)}(t), \dots, y^{(\beta_n)}(t)),$$

with $\alpha > \beta_n > \beta_{n-1} > \dots > \beta_1$ and $\alpha - \beta_n \leq 1, \beta_j - \beta_{j-1} \leq 1, 0 < \beta_1 \leq 1$. They not only discussed the analytical question on solutions, but also investigated how the solutions depend on the given data. In [7], Y. Luchko studied the generalized time-fractional diffusion equation

$$D_t^\alpha u(t) = \operatorname{div}(p(x) \operatorname{grad} u) - q(x)u + F(x, t),$$

where $0 < \alpha \leq 1, (x, t) \in G \times (0, T), G \subset \mathbb{R}^n$, and D_t^α is the Caputo–Dzherbashyan fractional derivative. The author used an appropriate maximum principle to obtain a unique solution, and also studied the continuous dependence on the data given in the problem.

Research on Hadamard fractional differential equations is at an early stage; see for example [19–30]. In [19], B. Ahmad and S. K. Ntouyas used fixed point theory to study the existence and uniqueness of solutions for a Hadamard type fractional differential equation involving integral boundary conditions

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), & 1 < t < e, 1 < \alpha \leq 2, \\ x(1) = 0, & x(e) = \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} \frac{x(s)}{s} ds, \beta > 0, \end{cases}$$

where f satisfies the Lipschitz condition.

On the other hand, p -Laplacian equations are extensively used in physics, mechanics, dynamical systems, etc (see [15–18,20–23,31]). For example, Leibenson [31] introduced p -Laplacian differential equations to study a mechanics problem involving turbulent flow in a porous medium. Recently, G. Wang et al. used the tools of Hadamard type fractional-order equations to study turbulent flow models, see [20,21]. In [20], they studied the uniqueness, the existence and nonexistence of solutions for the following Hadamard type fractional differential equation with the p -Laplacian operator

$$\begin{cases} -D^\beta \varphi_p(-D^\alpha \chi(t)) = f(t, \chi(t), \chi(\theta(t))), \\ D^\alpha \chi(1) = D^\alpha \chi(e) = 0, \chi(1) = 0, \\ D^{\alpha-1} \chi(1) = \eta D^{\alpha-1} \chi(e), \end{cases}$$

where $f \in C([1, e] \times \mathbb{R}^2, \mathbb{R})$ and $\theta \in C[1, e]$. In [21], they also studied the unique positive solution for a Caputo–Hadamard-type fractional turbulent flow model

$$\begin{cases} {}^{CH}D^{\chi_2} \varphi_{p(t)}({}^{CH}D^{\chi_1} x(t)) + f(x(t), I_{\eta}^{\gamma, \delta} x(t)) = 0, & t \in [1, e], \\ x'(1) = \lambda x'(e), & x(1) = x''(1) = 0, {}^{CH}D^{\chi_1} x(1) = 0, \end{cases}$$

where ${}^{CH}D$ is Caputo–Hadamard fractional derivative, $I_{\eta}^{\gamma, \delta}$ is the generalized Erdelyi–Kober fractional integral operator: $I_{\eta}^{\gamma, \delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} ds$.

In this paper, we study positive solutions for the p -Laplacian Hadamard fractional-order differential Equation (1). Using the monotone iterative technique we show that (1) has two positive solutions, and we establish iterative formulas for the two solutions. In addition from the Avery–Henderson fixed point theorem, we also obtain that (1) has two positive solutions under some appropriate conditions on the nonlinearity f . It is interesting to note that the methods used in this paper can be applied to very general integral equations (and therefore very general differential equations) if

the kernel has a suitable behavior as described in Section 2. The behavior of the Greens' function of a problem will indicate whether the theory presented in this paper can be used efficiently.

2. Preliminaries

In this section, we provide the definition of the Hadamard fractional derivative; for other related detail materials see the book [5].

Definition 1. The Hadamard derivative of fractional order q for a function $g : [1, \infty) \rightarrow \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s) \frac{ds}{s}, n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

In (1), we let $\varphi_p(D^\beta y(t)) = v(t)$, $t \in [1, e]$. Then we have

$$\begin{aligned} v(1) &= \varphi_p(D^\beta y(t))|_{t=1} = |D^\beta y(1)|^{p-2} D^\beta y(1) = 0, \\ v(e) &= |D^\beta y(e)|^{p-2} D^\beta y(e) = |b D^\beta y(\eta)|^{p-2} b D^\beta y(\eta) = b^{p-1} v(\eta). \end{aligned}$$

Therefore, we obtain

$$\begin{cases} -D^\alpha v(t) = f(t, y(t)), t \in (1, e), \\ v(1) = 0, v(e) = b^{p-1} v(\eta), \end{cases} \quad (2)$$

where $\alpha \in (1, 2]$ and b, η, f satisfy (H1)–(H2). Using a similar argument as in Lemmas 2 and 3 of [24], we obtain the following result.

Lemma 1. The boundary value problem (2) is equivalent to the following Hammerstein-type integral equation

$$v(t) = \int_1^e H(t, s) f(s, y(s)) ds,$$

where

$$\begin{aligned} H(t, s) &= H_0(t, s) + \frac{b^{p-1} (\log t)^{\alpha-1}}{1 - b^{p-1} (\log \eta)^{\alpha-1}} H_0(\eta, s), \text{ for } t, s \in [1, e], \\ H_0(t, s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1}, 1 \leq t \leq s \leq e, \\ (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, 1 \leq s \leq t \leq e. \end{cases} \end{aligned}$$

Proof. From the first equation in (2) we have

$$v(t) = c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s},$$

where $c_i \in \mathbb{R}$, $i = 1, 2$ and $v(1) = 0$ implies that $c_2 = 0$. Then from $v(e) = b^{p-1} v(\eta)$ we have

$$c_1 - \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} = c_1 b^{p-1} (\log \eta)^{\alpha-1} - \frac{b^{p-1}}{\Gamma(\alpha)} \int_1^\eta (\log \eta - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s},$$

and

$$\begin{aligned} c_1 &= \frac{1}{(1 - b^{p-1} (\log \eta)^{\alpha-1}) \Gamma(\alpha)} \int_1^e (1 - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\ &\quad - \frac{b^{p-1}}{(1 - b^{p-1} (\log \eta)^{\alpha-1}) \Gamma(\alpha)} \int_1^\eta (\log \eta - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s}. \end{aligned}$$

Substituting this c_1 , we obtain

$$\begin{aligned}
 v(t) &= \frac{1}{(1-b^{p-1}(\log \eta)^{\alpha-1})\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} (1-\log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} (1-\log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} (1-\log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} - \frac{b^{p-1}(\log t)^{\alpha-1}}{(1-b^{p-1}(\log \eta)^{\alpha-1})\Gamma(\alpha)} \int_1^\eta (\log \eta - \log s)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\
 &= \int_1^e H_0(t, s) f(s, y(s)) \frac{ds}{s} + \frac{b^{p-1}(\log t)^{\alpha-1}}{1-b^{p-1}(\log \eta)^{\alpha-1}} \int_1^e H_0(\eta, s) f(s, y(s)) \frac{ds}{s} \\
 &= \int_1^e H(t, s) f(s, y(s)) \frac{ds}{s}.
 \end{aligned}$$

This completes the proof. $\square$

Note that

$$\varphi_p(D^\beta y(t)) = v(t) = \int_1^e H(t, s) f(s, y(s)) \frac{ds}{s}, t \in [1, e].$$

Then we have

$$D^\beta y(t) = \varphi_p^{-1} \left(\int_1^e H(t, s) f(s, y(s)) \frac{ds}{s} \right), t \in [1, e], \quad (3)$$

and together with the boundary conditions $y(1) = y(e) = \delta y(1) = \delta y(e) = 0$, we have the following result.

Lemma 2. The problem (3) is equivalent to the following Hammerstein-type integral equation

$$y(t) = \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s},$$

where

$$G(t, s) = \begin{cases} \frac{(\log t - \log s)^{\beta-1} + (\log t)^{\beta-2} (1 - \log s)^{\beta-2} [(\log s - \log t) + (\beta - 2) \log s (1 - \log t)]}{\Gamma(\beta)}, & 1 \leq s \leq t \leq e, \\ \frac{(\log t)^{\beta-2} (1 - \log s)^{\beta-2} [(\log s - \log t) + (\beta - 2) \log s (1 - \log t)]}{\Gamma(\beta)}, & 1 \leq t \leq s \leq e. \end{cases}$$

Proof. For convenience, we put $\phi(t) = \varphi_p^{-1} \left(\int_1^e H(t, s) f(s, y(s)) \frac{ds}{s} \right), t \in [1, e]$. Then by a similar argument as in Lemma 1, we have

$$y(t) = c_1 (\log t)^{\beta-1} + c_2 (\log t)^{\beta-2} + c_3 (\log t)^{\beta-3} + c_4 (\log t)^{\beta-4} + \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-1} \phi(s) \frac{ds}{s},$$

where $c_i \in \mathbb{R}, i = 1, 2, 3, 4$ and $y(1) = \delta y(1) = 0$ implies that $c_3 = c_4 = 0$. Consequently, we have

$$\delta y(t) = c_1 (\beta - 1) (\log t)^{\beta-2} + c_2 (\beta - 2) (\log t)^{\beta-3} + \frac{\beta - 1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-2} \phi(s) \frac{ds}{s}.$$

As a result, we have

$$\begin{aligned}
 c_1 + c_2 + \frac{1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-1} \phi(s) \frac{ds}{s} &= 0, \\
 c_1 (\beta - 1) + c_2 (\beta - 2) + \frac{\beta - 1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-2} \phi(s) \frac{ds}{s} &= 0.
 \end{aligned}$$

Solving this system, we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \beta - 2 & -1 \\ 1 - \beta & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-1} \phi(s) \frac{ds}{s} \\ \frac{\beta-1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-2} \phi(s) \frac{ds}{s} \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} y(t) &= \frac{\beta-2}{\Gamma(\beta)} \int_1^e (\log t)^{\beta-1} (1 - \log s)^{\beta-1} \phi(s) \frac{ds}{s} - \frac{\beta-1}{\Gamma(\beta)} \int_1^e (\log t)^{\beta-1} (1 - \log s)^{\beta-2} \phi(s) \frac{ds}{s} \\ &\quad - \frac{\beta-1}{\Gamma(\beta)} \int_1^e (\log t)^{\beta-2} (1 - \log s)^{\beta-1} \phi(s) \frac{ds}{s} + \frac{\beta-1}{\Gamma(\beta)} \int_1^e (\log t)^{\beta-2} (1 - \log s)^{\beta-2} \phi(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-1} \phi(s) \frac{ds}{s} \\ &= \frac{1}{\Gamma(\beta)} \int_1^e (\log t)^{\beta-2} (1 - \log s)^{\beta-2} [(\log s - \log t) + (\beta-2) \log s (1 - \log t)] \phi(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log s)^{\beta-1} \phi(s) \frac{ds}{s} \\ &= \int_1^e G(t, s) \phi(s) \frac{ds}{s}. \end{aligned}$$

Consequently, we find

$$y(t) = \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.$$

This completes the proof. $\square$

Lemma 3. The functions H_0 and G have the following properties:

- (i) $H_0(t, s) \geq 0, G(t, s) \geq 0$ for $(t, s) \in [1, e] \times [1, e]$,
- (ii) $(\beta-2)(\log t)^{\beta-2}(1 - \log t)^2(\log s)^2(1 - \log s)^{\beta-2} \leq \Gamma(\beta)G(t, s) \leq M_0(\log s)^2(1 - \log s)^{\beta-2}$ for $(t, s) \in [1, e] \times [1, e]$,
- (iii) $(\beta-2)(\log s)^2(1 - \log s)^{\beta-2}(\log t)^{\beta-2}(1 - \log t)^2 \leq \Gamma(\beta)G(t, s) \leq M_0(\log t)^{\beta-2}(1 - \log t)^2$, for $(t, s) \in [1, e] \times [1, e]$, where $M_0 = \max \{\beta-1, (\beta-2)^2\}$.

Lemma 3 (ii) and (iii) are direct results from Lemma 3 in [14]. Moreover, by Lemma 3 (i) we have $H(t, s) \geq 0$ for $(t, s) \in [1, e] \times [1, e]$.

Let $E := C[1, e]$, $\|y\| := \max_{t \in [1, e]} |y(t)|$ and $P := \{y \in E : y(t) \geq 0, \forall t \in [1, e]\}$. Then $(E, \|\cdot\|)$ is a real Banach space and P a cone on E . From Lemma 2, we define an operator $A : E \rightarrow E$ as follows:

$$(Ay)(t) = \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \quad y \in E. \quad (4)$$

Note that the continuity of the functions G, H, f , guarantees that the operator A is a completely continuous operator. Moreover if there is a $\bar{y} \in E$ such that $A\bar{y} = \bar{y}$, then from Lemma 2 we have that $\bar{y}$ is a solution for (1). Therefore, in what follows we study the existence of fixed points of the operator A .

Lemma 4. (see [32]). Let E be a partially ordered Banach space, and $x_0, y_0 \in E$ with $x_0 \leq y_0, D = [x_0, y_0]$. Suppose that $A : D \rightarrow E$ satisfies the following conditions:

- (i) A is an increasing operator;
- (ii) $x_0 \leq Ax_0, y_0 \geq Ay_0$, i.e., x_0 and y_0 is a subsolution and a supersolution of A ;

(iii) A is a completely continuous operator.

Then A has the smallest fixed point x^* and the largest fixed point y^* in $[x_0, y_0]$, respectively. Moreover, $x^* = \lim_{n \rightarrow \infty} A^n x_0$ and $y^* = \lim_{n \rightarrow \infty} A^n y_0$.

Lemma 5. (Avery-Henderson fixed point theorem, see [33]). Let P be a cone in a real Banach space and for some positive constants r and γ , there exist increasing, non-negative continuous functionals ξ and ζ on P , and a non-negative continuous functional ϑ on P with $\vartheta(0) = 0$, such that

$$\zeta(y) \leq \vartheta(y) \leq \xi(y), \text{ and } \|y\| \leq \gamma \zeta(y), \forall y \in \overline{P(\zeta, r)} \text{ with } P(\zeta, r) = \{y \in P : \zeta(y) < r\}.$$

Suppose that there exist positive numbers $\mu < \nu < r$ such that

$$\vartheta(\lambda y) \leq \lambda \vartheta(y), \forall \lambda \in [0, 1] \text{ and } y \in \partial P(\vartheta, \nu).$$

If $A : \overline{P(\zeta, r)} \rightarrow P$ is a completely continuous operator satisfying

- (i) $\zeta(Ay) > r$ for all $y \in \partial P(\zeta, r)$,
- (ii) $\vartheta(Ay) < \nu$ for all $y \in \partial P(\vartheta, \nu)$,
- (iii) $P(\zeta, \mu) \neq \emptyset$ and $\xi(Ay) > \mu$ for all $y \in \partial P(\zeta, \mu)$,

Then A has at least two fixed points y_1 and y_2 such that

$$\mu < \zeta(y_1) \text{ with } \vartheta(y_1) < \nu; \nu < \vartheta(y_2) \text{ with } \zeta(y_2) < r.$$

Lemma 6. (see [23], Lemma 6). Let $\theta > 0$ and $\psi \in P$. Then,

$$\left(\int_0^1 \psi(t) dt \right)^\theta \leq \int_0^1 \psi^\theta(t) dt, \text{ if } \theta \geq 1, \text{ and } \left(\int_0^1 \psi(t) dt \right)^\theta \geq \int_0^1 \psi^\theta(t) dt, \text{ if } 0 < \theta \leq 1.$$

3. Main Results

In this section we let

$$p_* = \min\{1, p-1\}, \quad p^* = \max\{1, p-1\},$$

and

$$\omega(t) = (\log t)^2(1 - \log t)^{\beta-2}, \quad \sigma(t) = (\log t)^{\beta-2}(1 - \log t)^2, \quad t \in [1, e].$$

Now, we state our main results and give their proofs.

Theorem 1. Suppose that (H1)–(H2) and the following conditions hold:

$$(H3) \text{ There exist } a_1 \geq \left(\frac{\Gamma(\beta)}{\beta-2} \right)^{p-1} \left[\int_1^e \int_1^e \omega^{p_*}(s) H^{\frac{p_*}{p-1}}(s, \tau) \sigma^{p_*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} \right]^{\frac{1-p}{p_*}}, \quad r_1 > 0 \text{ such that}$$

$$f(t, y) \geq a_1 y^{p-1} \text{ for } y \in [0, r_1], t \in [1, e].$$

$$(H4) \text{ There exist } a_2 \in \left(0, 2 \frac{p-p^*-1}{p^*} \left(\frac{\Gamma(\beta)}{M_0} \right)^{p-1} \left[\int_1^e \int_1^e H^{\frac{p^*}{p-1}}(s, \tau) \sigma^{p^*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} \right]^{\frac{1-p}{p^*}} \right), l_1 > 0$$

such that

$$f(t, y) \leq a_2 y^{p-1} + l_1 \text{ for } y \in \mathbb{R}^+, t \in [1, e].$$

(H5) $f(t, y)$ is an increasing function in the second variable y , i.e., $f(t, y_1) \leq f(t, y_2)$ for $y_1 \leq y_2, \forall t \in [1, e]$.

Then (1) has two positive solutions. Moreover, there exist two iterative sequences uniformly converging to the two solutions.

Proof. Using (H3) we have

$$\begin{aligned} [(Ay)(t)]^{p_*} &= \left[\int_1^e G(t, s) \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{p_*} \\ &= \left[\int_0^1 G(t, e^s) \left(\int_0^1 H(e^s, e^\tau) f(e^\tau, y(e^\tau)) d\tau \right)^{\frac{1}{p-1}} ds \right]^{p_*} \\ &\geq \int_0^1 G^{p_*}(t, e^s) \left(\int_0^1 H(e^s, e^\tau) f(e^\tau, y(e^\tau)) d\tau \right)^{\frac{p_*}{p-1}} ds \\ &\geq \int_0^1 G^{p_*}(t, e^s) \int_0^1 H^{\frac{p_*}{p-1}}(e^s, e^\tau) f^{\frac{p_*}{p-1}}(e^\tau, y(e^\tau)) d\tau ds \\ &= \int_1^e G^{p_*}(t, s) \int_1^e H^{\frac{p_*}{p-1}}(s, \tau) f^{\frac{p_*}{p-1}}(\tau, y(\tau)) \frac{d\tau}{\tau} \frac{ds}{s} \\ &\geq \int_1^e G^{p_*}(t, s) \int_1^e H^{\frac{p_*}{p-1}}(s, \tau) a_1^{\frac{p_*}{p-1}} y^{p_*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s}. \end{aligned}$$

Then, we can choose $\varepsilon_1 > 0$ such that $\varepsilon_1 \sigma \in [0, r_1]$, and from Lemma 3 (iii) we have

$$\begin{aligned} [(A\varepsilon_1 \sigma)(t)]^{p_*} &\geq \int_1^e G^{p_*}(t, s) \int_1^e H^{\frac{p_*}{p-1}}(s, \tau) a_1^{\frac{p_*}{p-1}} \varepsilon_1^{p_*} \sigma^{p_*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} \\ &\geq \int_1^e \left(\frac{\beta-2}{\Gamma(\beta)} \right)^{p_*} \omega^{p_*}(s) \sigma^{p_*}(t) \int_1^e H^{\frac{p_*}{p-1}}(s, \tau) a_1^{\frac{p_*}{p-1}} \varepsilon_1^{p_*} \sigma^{p_*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} \\ &\geq \varepsilon_1^{p_*} \sigma^{p_*}(t). \end{aligned}$$

Thus

$$A\varepsilon_1 \sigma \geq \varepsilon_1 \sigma. \quad (5)$$

On the other hand, from (H4) we have

$$\begin{aligned}
 [(Ay)(t)]^{p^*} &= \left[\int_1^e G(t,s) \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{p^*} \\
 &= \left[\int_0^1 G(t, e^s) \left(\int_0^1 H(e^s, e^\tau) f(e^\tau, y(e^\tau)) d\tau \right)^{\frac{1}{p-1}} ds \right]^{p^*} \\
 &\leq \int_0^1 G^{p^*}(t, e^s) \left(\int_0^1 H(e^s, e^\tau) f(e^\tau, y(e^\tau)) d\tau \right)^{\frac{p^*}{p-1}} ds \\
 &\leq \int_0^1 G^{p^*}(t, e^s) \int_0^1 \frac{p^*}{H^{p-1}}(e^s, e^\tau) f^{\frac{p^*}{p-1}}(e^\tau, y(e^\tau)) d\tau ds \\
 &= \int_1^e G^{p^*}(t, s) \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) f^{\frac{p^*}{p-1}}(\tau, y(\tau)) \frac{d\tau}{\tau} \frac{ds}{s} \\
 &\leq \int_1^e G^{p^*}(t, s) \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \left(a_2 y^{p-1}(\tau) + l_1 \right)^{\frac{p^*}{p-1}} \frac{d\tau}{\tau} \frac{ds}{s}.
 \end{aligned}$$

Then from (H4) we have

$$a_2^{\frac{p^*}{p-1}} 2^{\frac{p^*}{p-1}-1} \left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} \int_1^e \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \sigma^{p^*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} < 1,$$

and hence there is a $M_1 > \varepsilon_1$ and

$$M_1 \geq \sqrt[p^*]{\frac{\left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} 2^{\frac{p^*}{p-1}-1} l_1^{\frac{p^*}{p-1}} \int_1^e \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \frac{d\tau}{\tau} \frac{ds}{s}}{1 - a_2^{\frac{p^*}{p-1}} 2^{\frac{p^*}{p-1}-1} \left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} \int_1^e \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \sigma^{p^*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s}}},$$

such that

$$\begin{aligned}
 [(AM_1\sigma)(t)]^{p^*} &\leq \int_1^e G^{p^*}(t, s) \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \left(a_2 M_1^{p-1} \sigma^{p-1}(\tau) + l_1 \right)^{\frac{p^*}{p-1}} \frac{d\tau}{\tau} \frac{ds}{s} \\
 &\leq \int_1^e G^{p^*}(t, s) \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) 2^{\frac{p^*}{p-1}-1} \left(a_2^{\frac{p^*}{p-1}} M_1^{p^*} \sigma^{p^*}(\tau) + l_1^{\frac{p^*}{p-1}} \right) \frac{d\tau}{\tau} \frac{ds}{s} \\
 &\leq \int_1^e \left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} \sigma^{p^*}(t) \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) 2^{\frac{p^*}{p-1}-1} \left(a_2^{\frac{p^*}{p-1}} M_1^{p^*} \sigma^{p^*}(\tau) + l_1^{\frac{p^*}{p-1}} \right) \frac{d\tau}{\tau} \frac{ds}{s} \\
 &= M_1^{p^*} \sigma^{p^*}(t) \cdot a_2^{\frac{p^*}{p-1}} 2^{\frac{p^*}{p-1}-1} \left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} \int_1^e \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \sigma^{p^*}(\tau) \frac{d\tau}{\tau} \frac{ds}{s} \\
 &\quad + \sigma^{p^*}(t) \cdot \left(\frac{M_0}{\Gamma(\beta)} \right)^{p^*} 2^{\frac{p^*}{p-1}-1} l_1^{\frac{p^*}{p-1}} \int_1^e \int_1^e \frac{p^*}{H^{p-1}}(s, \tau) \frac{d\tau}{\tau} \frac{ds}{s} \\
 &\leq M_1^{p^*} \sigma^{p^*}(t).
 \end{aligned}$$

This implies that

$$AM_1\sigma \leq M_1\sigma. \tag{6}$$

In Lemma 4 we put $D = [\varepsilon_1\sigma, M_1\sigma]$, and from (H5) we have that A is an increasing operator. Thus from Lemma 4, A has the smallest fixed point y_{**} and the largest fixed point y^{**} in D . That is, Equation (1) has two positive solutions y_{**} and y^{**} in D . Moreover, $y_{**} = \lim_{n \rightarrow \infty} A^n \varepsilon_1\sigma = \lim_{n \rightarrow \infty} A^{n-1}(A\varepsilon_1\sigma)$ and $y^{**} = \lim_{n \rightarrow \infty} A^n M_1\sigma = \lim_{n \rightarrow \infty} A^{n-1}(AM_1\sigma)$. This completes the proof. $\square$

Lemma 7. Let $P_0 = \left\{ y \in P : y(t) \geq \frac{(\beta-2)\kappa^*}{M_0} \|y\|, t \in [\kappa_1, \kappa_2] \right\}$. Then $A(P) \subset P_0$, where $\kappa_1, \kappa_2 \in (1, e)$ with $\kappa_1 < \kappa_2$, and $\kappa^* = \min_{t \in [\kappa_1, \kappa_2]} \sigma(t) > 0$.

Proof. For $y \in P$, from Lemm 3 (ii) we have

$$\begin{aligned} (Ay)(t) &= \int_1^e G(t,s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\leq \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} (Ay)(t) &= \int_1^e G(t,s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\geq \frac{\beta-2}{\Gamma(\beta)} \int_1^e \sigma(t) \omega(s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \frac{\beta-2}{M_0} \sigma(t) \cdot \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}. \end{aligned}$$

Therefore, we have

$$(Ay)(t) \geq \frac{\beta-2}{M_0} \sigma(t) \|Ay\|, t \in [1, e].$$

In particular, we have

$$(Ay)(t) \geq \frac{(\beta-2)\kappa^*}{M_0} \|Ay\|, t \in [\kappa_1, \kappa_2].$$

This completes the proof. $\square$

Let

$$I = [\kappa_1, \kappa_2], \zeta(y) = \min_{t \in I} y(t), \vartheta(y) = \max_{t \in I} y(t), \tilde{\zeta}(y) = \max_{t \in [1, e]} y(t), \text{ and } P(\zeta, r) = \{y \in P_0 : \zeta(y) < r\},$$

and

$$C = \frac{(\beta-2)\kappa^*}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_I H(s,\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s}, D = \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s,\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s}.$$

Theorem 2. Suppose that (H1)–(H2) hold, and there exist positive constants $0 < \mu < v < r$ such that the function f satisfies the following conditions:

- (H6) $f(t, y) > \varphi_p \left(\frac{\mu}{C} \right)$ for $t \in I$ and $y \in \left[\frac{(\beta-2)\kappa^*}{M_0} \mu, \mu \right]$,
- (H7) $f(t, y) < \varphi_p \left(\frac{v}{D} \right)$ for $t \in [1, e]$ and $y \in \left[0, \frac{vM_0}{(\beta-2)\kappa^*} \right]$,
- (H8) $f(t, y) > \varphi_p \left(\frac{r}{C} \right)$ for $t \in I$ and $y \in \left[r, \frac{rM_0}{(\beta-2)\kappa^*} \right]$.

Then (1) has at least two positive solutions y_1 and y_2 such that

$$\begin{aligned}\mu &< \max_{t \in [1, e]} y_1(t) \text{ with } \max_{t \in I} y_1(t) < \nu, \\ \nu &< \max_{t \in I} y_2(t) \text{ with } \min_{t \in I} y_2(t) < r.\end{aligned}$$

Proof. Note that $P_0 \subset P$, and from Lemma 7 we have $A(P_0) \subset P_0$. From the definitions of ζ, ξ, ϑ , for each $y \in P_0$ we have

$$\zeta(y) \leq \vartheta(y) \leq \xi(y), \text{ and } \|y\| \leq \frac{M_0}{(\beta-2)\kappa^*} \min_{t \in I} y(t) = \frac{M_0}{(\beta-2)\kappa^*} \zeta(y).$$

For every $y \in P_0, \lambda \in [0, 1]$ we obtain

$$\vartheta(\lambda y) = \max_{t \in I} \lambda y(t) = \lambda \vartheta(y), \text{ and } \vartheta(0) = 0.$$

We first verify condition (iii) in Lemma 5. Since $0 \in P_0$ and $\mu > 0$, $P(\xi, \mu) \neq \emptyset$. Note from $y \in \partial P(\xi, \mu)$, i.e., $\|y\| = \mu$, and thus $\frac{(\beta-2)\kappa^*}{M_0} \mu \leq y(t) \leq \mu$ for $t \in I$. Therefore, using (H6) we have

$$\begin{aligned}\xi(Ay) &= \max_{t \in [1, e]} (Ay)(t) \\ &= \max_{t \in [1, e]} \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\geq \max_{t \in [1, e]} \frac{\beta-2}{\Gamma(\beta)} \int_1^e \sigma(t) \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\geq \max_{t \in I} \frac{\beta-2}{\Gamma(\beta)} \int_1^e \sigma(t) \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\geq \min_{t \in I} \frac{\beta-2}{\Gamma(\beta)} \int_1^e \sigma(t) \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &> \frac{(\beta-2)\kappa^*}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_I H(s, \tau) \varphi_p \left(\frac{\mu}{C} \right) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \frac{\mu}{C} \frac{(\beta-2)\kappa^*}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_I H(s, \tau) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \mu.\end{aligned}$$

We next show condition (ii) in Lemma 5 is true. Since $y \in \partial P(\vartheta, \nu)$, then $0 \leq y(t) \leq \|y\| \leq \frac{\nu M_0}{(\beta-2)\kappa^*}$ for $t \in [1, e]$. Consequently, from (H7) we obtain

$$\begin{aligned}\vartheta(Ay) &= \max_{t \in I} (Ay)(t) \\ &= \max_{t \in I} \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\leq \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &< \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) \varphi_p \left(\frac{\nu}{D} \right) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \frac{\nu}{D} \frac{M_0}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \nu.\end{aligned}$$

Finally, we show condition (i) in that Lemma 5 holds. Since $y \in \partial P(\zeta, r)$, i.e., $\min_{t \in I} y(t) = r$, and thus $r \leq y(t) \leq \|y\| \leq \frac{rM_0}{(\beta-2)\kappa^*}$ for $t \in I$. Hence, (H8) implies that

$$\begin{aligned}\zeta(Ay) &= \min_{t \in I} (Ay)(t) \\ &= \min_{t \in I} \int_1^e G(t, s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &\geq \min_{t \in I} \frac{\beta-2}{\Gamma(\beta)} \int_1^e \sigma(t) \omega(s) \varphi_p^{-1} \left(\int_1^e H(s, \tau) f(\tau, y(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &> \frac{(\beta-2)\kappa^*}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_I H(s, \tau) \varphi_p \left(\frac{r}{C} \right) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= \frac{r}{C} \frac{(\beta-2)\kappa^*}{\Gamma(\beta)} \int_1^e \omega(s) \varphi_p^{-1} \left(\int_I H(s, \tau) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\ &= r.\end{aligned}$$

This completes the proof.

□

4. Conclusions

In this paper, we first used the monotone iterative technique to show that (1) has two positive solutions, and we established iterative formulas for the two solutions when the nonlinearity f grows $(p-1)$ -sublinearly. Next, using the Avery–Henderson fixed point theorem, we showed that (1) has two positive solutions under some appropriate conditions on the nonlinearity f .

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