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# Nonadditivity Index Based Quasi-Random Generation of Capacities and Its Application in Comprehensive Decision Aiding

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Received: 5 February 2020; Accepted: 18 February 2020; Published: 24 February 2020



**Abstract:** The capacity is a powerful tool with exponential coefficients to represent the interaction phenomenon among decision criteria, but its random generation becomes a tough issue for dealing with the monotonicity with all inclusion subsets as well as the complex constraints of decision preference. In this paper, we adopt a kind of explicit interaction index, the nonadditivity index, to construct two types of quasi-random generation methods of capacity under a given decision interaction preference. Compared to the existing random generation algorithms, the methods have relatively satisfactory performance on the statistics characteristic of generated capacities but need rather less calculation effort on the generation process. We also show the effectiveness of proposed quasi-random generation methods by an illustrative decision example.

**Keywords:** capacity; nonadditivity index; random generation; decision dominance relationship

## 1. Introduction

The capacity [1], also called fuzzy measure [2,3], or nonadditive measure [4,5], extends the probability measure's additivity with respect to disjoint subsets into its nonadditivity (subadditivity, superadditivity, or additivity), which means the capacity only needs to satisfy the monotonicity with respect to set inclusion [6–8].

In the capacity based decision making context, the decision preference information can be represented through the values of subsets' capacity or its equivalent representations such as the simultaneous interaction index [9,10], nonadditivity index [5,11] and nonmodularity index [12]. Nonadditivity is an explicit interaction index specially for describing the interaction phenomenon among criteria associated with nonadditivity and meanwhile has some good mathematic properties and intuitive explanations in accordance with decision common sense [5]. This is also the reason we choose nonadditivity to mainly describe the interaction among criteria in the following.

Besides using the comparison or interval form of the equivalent transformations' values of criteria subsets, the decision preference can also be represented by the overall evaluations of typical or historical decision alternatives [13], or by some special types of capacities, like  $\lambda$ -capacity [2],  $k$ -additive capacity [14],  $p$ -symmetric capacity [15],  $k$ -maxitive and minitive capacity [16],  $k$ -interactive capacity [17], and  $k$ -order upper and lower representative capacity [18]. Almost all of these three types of decision preference information can be commonly turned into a collection of linear constraints [3].

Generally speaking, there are two approaches to transform all the decision preferences into the final decision result. One is to adopt a principle to obtain the optimal capacity, and the corresponding nonlinear integral values, like Choquet integral or Sugeno integral values, of decision alternatives are taken as references to make the final decision. These principles include least-squares principle [13,19,20],

the least absolute deviation criterion [21], the maximum split approaches [20,22], the maximum entropy principle [23–26], the maximum log-likelihood principle [27–29], the compromise principle [30] and the MCCPI (multicriteria correlation preference information) based minimum divergence principle [31,32]. However, one can figure out that different principles may lead to different results and every single principle can neither reveal all dominance relationships among alternatives nor the frequencies, probabilities or credibilities of these dominances.

The other approach is to use the random generation of capacity within the feasible region defined by decision preference to simulate the whole dominance relationship among decision alternatives together with the estimated probabilities, and finally get the most confidential decision result. Combarro et al. [33–35] proposed the random node and minimum plus based capacity random generation methods, but these methods only consider the boundary and monotonicity conditions (see definition of capacity), without imposing any further preference information constraints. In order to simultaneously consider the complex decision preference constraints, Wu et al. [36] provided a Choquet integral and linear programming based random generation method as well as the comprehensive decision method. However, this method involves a great many times of linear programming construction and solving. The computation cost is a huge burden for a personal computer when decision criteria number  $n \geq 7$  or the generated capacity number is large; for instance, 5000 capacity with five criteria. So, its applicability is limited in some situations.

In this paper, we propose two types of quasi-random generation methods based on the nonadditivity index which involve no linear programmings and then work with rather efficient execution performance. The method can also flexibly show the relatively complex interaction preference upon decision criteria. We found that, through some empirical tests, the quasi-random generation methods have relatively good performances on statistics characteristic of generated capacities compared with the result by real random generation methods. The decision example further shows that the proposed methods are relatively easier to establish and apply for saving the effort of constructing the optimal model and the objective function and can be taken as competent substitutes for the real random generation based decision making scheme in some situations. In summary, compared to the existing capacity random generation methods, the proposed methods can not only represent additional decision preference about complex interaction phenomena among decision criteria but also have some relative advantages in dealing with large scale decision making problems.

This paper is organized as follows. After the introduction, we present some information about the capacity, nonadditivity index and the Choquet integral in Section 2. In Section 3, we briefly summarize the presentation form of the decision preference information and show the Choquet integral based random generation algorithm as well as its comprehensive decision aid algorithm. Section 4 is devoted to the rationale and process of the nonadditivity index based quasi-random generation algorithms. An empirical analysis of the proposed methods is given in Section 5. Finally, we conclude the paper in Section 6.

## 2. Preliminaries

Let  $N = \{1, 2, \dots, n\}$ ,  $n \geq 2$ , be the set of decision criteria,  $\mathcal{P}(N)$  be the power set of  $N$ , and  $|S|$  be the cardinality of subset  $S \subseteq N$ .

**Definition 1** ([1,2]). A capacity on  $N$  is a set function  $\mu : \mathcal{P}(N) \rightarrow [0, 1]$  such that (Boundary condition)  $\mu(\emptyset) = 0$ ,  $\mu(N) = 1$ ; (Monotonicity condition)  $\forall A, B \subseteq N$ ,  $A \subseteq B$  implies  $\mu(A) \leq \mu(B)$ .

The quantity of  $\mu(A)$  is considered as the importance of  $A$  to the decision problem [37,38]. The nonadditivity is one of the explicit properties produced from monotonicity, which can be described through the following definition.

**Definition 2** ([5,11]). A capacity  $\mu$  on  $N$  is said to be  $\star$ -additive, if

$$\mu(A \cup B) \stackrel{\star}{=} \mu(A) + \mu(B), \forall A, B \subseteq N, A, B \neq \emptyset, A \cap B = \emptyset. \tag{1}$$

Furthermore,  $\mu$  is said to be  $\star$ -additive within  $S \subseteq N$ , if

$$\mu(A \cup B) \stackrel{\star}{=} \mu(A) + \mu(B), \forall A, B \subseteq S, A, B \neq \emptyset, A \cap B = \emptyset. \tag{2}$$

where “ $\star$ -additive” stands for “additive (resp. superadditive, subadditive, strict superadditive, and strict subadditive)”, “ $\stackrel{\star}{=}$ ” stands for “= (resp.  $\geq$ ,  $\leq$ ,  $>$ , and  $<$ )”.

It is a common interpretation that an additive capacity means that the decision criteria are independent; a strict superadditive (respectively strict subadditive, superadditive, and subadditive) capacity indicates that all criteria are mutually complementary (substitutive, nonsubstitutive, and noncomplementary).

**Definition 3** ([5]). The nonadditivity index of  $\mu$  is defined as

$$n_\mu(A) = \mu(A) - \frac{1}{2^{|A|-1} - 1} \sum_{CCA} \mu(C), A \subseteq N. \tag{3}$$

**Theorem 1** ([5]). If a capacity  $\mu$  on  $N$  is  $\star$ -additive, then  $n_\mu(A) \stackrel{\star}{=} 0, \forall A \subseteq N, |A| \geq 2$ . If a capacity  $\mu$  on  $N$  is  $\star$ -additive within  $S \subseteq N$ , then  $n_\mu(A) \stackrel{\star}{=} 0, \forall A \subseteq S, |A| \geq 2$ .

This means that the nonadditivity index reflects the above common interpretation [5,12,39].

**Theorem 2** ([5]).  $-1 \leq n_\mu(A) \leq 1, \forall A \subseteq N$ .

This means the nonadditivity index has uniform range  $[-1, 1]$ , then the interaction of subsets with different cardinalities can be directly compared.

**Theorem 3** ([5]).  $n_\mu(A) = 1 \Leftrightarrow \mu(S) = \varepsilon_A^+(S), S \subseteq A$ , and  $n_\mu(A) = -1 \Leftrightarrow \mu(S) = \varepsilon_A^-(S), S \cap A \neq \emptyset$ , where  $\varepsilon_A^+$  is a capacity on  $N$  such that  $\varepsilon_A^+(B) = 1$  if  $B \supseteq A, B \subseteq N$ , and  $\varepsilon_A^+(B) = 0$  otherwise;  $\varepsilon_A^-$  is a capacity on  $N$  such that  $\varepsilon_A^-(B) = 1$  if  $B \cap A \neq \emptyset$ , and  $\varepsilon_A^-(B) = 0$  otherwise.

One can figure out that  $\varepsilon_A^+$  and  $\varepsilon_A^-$  describe two opposite extreme interaction situations between criteria within  $A$ . The former, also known as an unanimity game, corresponds to a fully complementary case, while the latter corresponds to a fully redundant case, and the upper and lower boundaries of the nonadditivity index are reached by these two extreme capacities.

The Choquet integral is a widely accepted type of fuzzy integral for flexibly aggregating the partial evaluations of the decision alternatives on multiple interdependent criteria, and hence can be accordingly used to present the decision maker’s preferences on the alternatives which are regarded as the implicit form of preferences on decision criteria.

**Definition 4.** Let  $x$  be a real-valued vector function on  $N, x := (x_1, \dots, x_n) \in [0, 1]^n$ , the Choquet integral of  $x$  with respect to a capacity  $\mu$  on  $N$  is defined as

$$C_\mu(x) = \sum_{i=1}^n [x_{(i)} - x_{(i-1)}] \mu(N_{(i)}) = \sum_{i=1}^n [\mu(N_{(i)}) - \mu(N_{(i+1)})] x_{(i)} \tag{4}$$

where the parentheses used for indices represent a permutation on  $N$  such that  $x_{(1)} \leq \dots \leq x_{(n)}, x_{(0)} = 0, N_{(i)} = \{(i), \dots, (n)\}$ , and  $N_{(n+1)} = \emptyset$ .

### 3. Choquet Integral Based Random Generation and Comprehensive Decision Method

Almost all of the decision preference can be represented as a collection of linear constraints [3]. Taking the nonadditivity index as an example, we can represent the decision maker’s preference by comparison and interval forms:

- the sign of interaction of criteria subset  $A$ ,  $|A| \geq 2$ , is positive or negative:

$$n_{\mu}(A) \geq (\leq)0; \tag{5}$$

- the interaction of criteria subset  $A$  is larger than that of  $B$ ,  $|A|, |B| \geq 2$ :

$$I_{\mu}(A) \geq I_{\mu}(B) + \delta; \tag{6}$$

- the interaction of criteria subset  $A$ ,  $|A| \geq 2$ , belongs to interval  $[0.25, 0.5]$ :

$$0.25 \leq n_{\mu}(A) \leq 0.5; \tag{7}$$

where  $\delta$  is small threshold, e.g., 0.05.

Combing the boundary and monotonicity conditions of capacity with the decision preference constraints, we can get the domain of feasible capacities if those conditions and constraints are all compatible. Otherwise, we can adopt some inconsistency methods and adjustment strategies, see, e.g., [39–41].

With the compatible decision preference constraints, the Choquet integral based random generation method is proposed in reference [36], as shown in Algorithm 1.

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**Algorithm 1:** Capacity random generation with mixed constraints

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**Input:** Decision criteria set  $N$ ; decision alternatives set  $A$  and their partial evaluations on  $N$ ; decision maker’s preference constraint set  $D$ ; the chosen alternative’s Chouquet integral,  $r$ , and its reachable range  $[r_{\min}, r_{\max}]$ .

**Output:** A set of randomly generated capacities, denoted as matrix  $\mathbf{U}$ .

**Initialize:** The number of generated capacities, denoted as  $\mathcal{K}$ .

**for**  $k$  in  $1 : \mathcal{K}$  **do**

Get a random value in the range of  $[r_{\min}, r_{\max}]$ , denoted as  $\bar{r}$ .

Calculate the optimal capacity, denoted as  $\mu_k$ , with which the Choquet integral value of chosen alternative,  $r$ , is the closest to the value  $\bar{r}$  obtained through an optimization model (the objective is to minimize the distance between  $r$  and  $\bar{r}$ , and the constraints include  $D$  and the boundary and monotonicity conditions).

Append  $\mu_k$  to  $\mathbf{U}$ .

**end**

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As mentioned above, different capacities may lead to different overall evaluations and ranking orders of decision alternatives. Sketching the whole view of the dominance of the decision alternatives becomes a meaningful tool in practice. A random simulation based comprehensive decision aid algorithm is also provided in [36], see Algorithm 2.

From Algorithm 1, one can find out that, for each capacity, at least one turn of construction and execution of linear programming is needed, which really leads into a large computation difficulty, especially when the criteria number  $n$  or the capacity amount  $\mathcal{K}$  is not small. In the following, we give an approach to release this burden by using the nonadditivity index to show the decision maker’s relatively complex preference information.

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**Algorithm 2:** Random simulation based comprehensive decision aid algorithm.

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**Input:** Decision criteria set  $N$ ; decision alternatives set  $A$  and their partial evaluations on  $N$ ; the randomly generated matrix of capacities  $\mathbf{U}_{\mathcal{K} \times 2^{|N|}}$ .

**Output:** The comprehensive dominance relationships among all decision alternatives, the overall evaluations matrix  $\mathbf{E}_{\mathcal{K} \times |A|}$  and the ranking orders matrix  $\mathbf{O}_{\mathcal{K} \times |A|}$ .

**for**  $k$  in  $1 : \mathcal{K}$  **do**

    Calculate the Choquet integrals and ranking orders of all alternatives w.r.t.  $\mu_k$ , stored as the  $k$ th row of  $\mathbf{E}$  and  $\mathbf{O}$ , respectively.

**end**

**for** each pair of decision alternatives  $\{a, b\} \subseteq A$  **do**

    Calculate the frequencies of dominance of  $a$  over  $b$ ,  $b$  over  $a$ ,  $a$  equal to  $b$ , denoted as  $d^>(a, b)$ ,  $d^<(a, b)$  and  $d^{\sim}(a, b)$ .

**end**

By choosing the largest frequency dominance relationship, the most creditable dominance relationship of all alternatives is generally found (for more detailed rules for dealing with some special cases, please see [36])

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#### 4. Nonadditivity Index Driven Capacity Random Generation

Now we focus on the preference information only represented through the nonadditivity index. The nonadditivity index based quasi-random generation processes are established on the following facts and rules:

- (i) The capacity value  $\mu(A)$  is the importance of subset  $A$  and the monotonicity means  $\mu(A) \geq \max_{B \subset A} \mu(B)$ .
- (ii) a convex combination of capacities is also a capacity, where the convex combination of  $\mu_1, \dots, \mu_n$  is defined as  $\sum_{i=1}^n \lambda_i \mu_i$ , where  $\lambda_i \geq 0$ ,  $\sum_{i=1}^n \lambda_i = 1$ .
- (iii) superadditivity commonly corresponds to positive interaction and positive nonadditivity index value; subadditivity negative.
- (iv)  $\varepsilon_A^+$  and  $\varepsilon_A^-$  represent the extremely positive and negative interaction of criteria within  $A$ , respectively.

Here, we give two algorithms for the nonadditivity index driven capacity quasi-random generation.

##### 4.1. Quasi-Random Generation Algorithm (I)

The first algorithm is presented as Algorithm 3. In this algorithm,  $\text{rnorm}(\mu, \sigma)$  denotes the function that generates normally distributed random numbers with mean  $\mu$  and standard deviation  $\sigma$  [42]. The function  $\text{mean}()$  is the arithmetic mean of its list of arguments.

**Remark 1.** In Algorithm 3, the preference information matrix  $\mathbf{D}$  does not involve the decision alternatives and only includes the the decision maker's preference on the importance of each criterion and the interaction of decision criteria in terms of nonadditivity index (for the singleton criterion, its nonadditivity index is just equivalent to its capacity). Furthermore, these preferences are specifically presented through the degree of interaction coefficients,  $\alpha$ ,  $\beta$  and  $\gamma$ . The capacity matrix  $\mathbf{U}$  consists of  $\mathcal{K}$  random capacities, so its size is  $\mathcal{K} \times 2^n$ .

**Remark 2.** In Algorithm 3, the capacities are arranged in the cardinality based order [3], i.e., if  $|A| < |B|$  then the index of  $A$  should be smaller than that of  $B$ .

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**Algorithm 3:** Nonadditivity driven capacity quasi-random generation algorithm (I)

---

**Input:** Decision criteria set  $\{1, \dots, N\}$ ; decision maker’s explicit preferences on importance and interaction of criteria, denoted as  $D$ .

**Output:** A set of randomly generated capacities, stored in matrix  $\mathbf{U}$ .

**Initialize:** the number of generated capacities, denoted as  $\mathcal{K}$ .

```

for  $k$  in  $1 : \mathcal{K}$  do
  for  $A$  in  $\{\emptyset, \{1\}, \{2\}, \dots, N\}$  do
    if  $A == \emptyset$  then
      |  $\mu_k(A) = 0$ 
    end
    if  $|A| == 1$  then
      |  $\mu_k(A) = \min[\max(\text{rnorm}(\frac{\alpha}{n}, \frac{1}{3n}), 0), 1]$ ,
      | where  $\alpha \in [0, n]$  are the importance coefficients of the criteria.
    end
    if  $2 \leq |A| \leq n$  then
      | if the interaction in  $A$  is negative then  $\eta = -1$ ;
      | else if the interaction in  $A$  is zero then  $\eta = 0$ ;
      | else if the interaction in  $A$  is positive then  $\eta = 1$ ;
      | else /* the interaction in  $A$  is not given */
      | |  $\eta = \text{random value in } \{-1, 0, 1\}$ .
      | end
      | if  $\eta == -1$  then
      | |  $\mu_k(A) = \max_{B \subset A} \mu_k(B) +$ 
      | |  $\beta(\max[\max_{B \subset A} \mu_k(B), \min(2\text{mean}_{B \subset A} \mu_k(B), 1)] - \max_{B \subset A} \mu_k(B))$ ,
      | | where  $\beta \in [0, 1]$  is the negative interaction coefficient.
      | else if  $\eta == 0$  then
      | |  $\mu_k(A) = \max[\max_{B \subset A} \mu_k(B), \min(2\text{mean}_{B \subset A} \mu_k(B), 1)]$ .
      | else /*  $\eta == 1$  */
      | |  $\mu_k(A) = \max[\max_{B \subset A} \mu_k(B), \min(2\text{mean}_{B \subset A} \mu_k(B), 1)] +$ 
      | |  $\gamma(1 - \max[\max_{B \subset A} \mu_k(B), \min(2\text{mean}_{B \subset A} \mu_k(B), 1)])$ ,
      | | where  $\gamma \in [0, 1]$  is the positive interaction coefficient.
      | end
      | end
    end
    if  $|A| == n$  then
      |  $\mu_k(A) = 1$ 
    end
  end
  end
  Set the  $k$ th row of  $\mathbf{U}$  as  $\mu_k$ .
end

```

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**Remark 3.** In the random generation function  $\text{rnorm}()$ , the standard deviation parameter  $\frac{1}{3n}$  is an empirically selected value, the aim of which is to force most of the random values to be in the range  $[\frac{\alpha-1}{n}, \frac{\alpha+1}{n}]$ ; as for the importance coefficient  $\alpha \in [0, n]$ , the more important the criterion is, the larger  $\alpha$  should be. The value  $\alpha = 1$  can be regarded as a default case that the criterion has the standard importance  $1/n$ .

**Remark 4.** In Algorithm 3, the interaction degree coefficients  $\beta$  and  $\gamma$  represent the negative and positive degrees. The smaller the value of  $\beta$ , the stronger the degree of negative interaction; the larger the value of  $\gamma$ , the stronger the degree of positive interaction. Tactically, we can divide the  $[0, 1]$  into four intervals  $[0, 0.25]$ ,  $[0.25, 0.5]$ ,  $[0.5, 0.75]$ ,  $[0.75, 1]$ . For  $\beta$ , we can use random values from these four intervals to show the extremely

negative, strongly negative, negative and slightly negative interaction, respectively. For  $\gamma$ , we can use random values from these four intervals to show the slightly positive, positive, strongly positive and extremely positive interaction, respectively. If the interaction degree is not given, we just assign to  $\beta$  or  $\gamma$  a random value in the interval  $[0, 1]$ .

**Remark 5.** Setting zero interaction is rooted in the definition of the nonadditivity index; see the case when the following equation equals 0.

$$n_{\mu}(A) = \mu(A) - \frac{1}{2^{|A|-1} - 1} \sum_{B \subset A} \mu(B) = \mu(A) - 2 \text{mean}_{B \subset A} \mu(B). \tag{8}$$

#### 4.2. Quasi-Random Generation Algorithm (II)

The second quasi-random generation method mainly uses the convex combination of capacities of  $\varepsilon^+$  and  $\varepsilon^-$ , as shown in Algorithm 4.

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#### Algorithm 4: Nonadditivity driven capacity quasi-random generation algorithm (II)

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**Input:** Decision criteria set  $\{1, \dots, N\}$ ; decision maker’s explicit preferences on importance and interaction of the criteria, denoted as  $D$ .

**Output:** A set of randomly generated capacities, stored in the matrix  $\mathbf{U}$ .

**Initialize:** The number of generated capacities, denoted as  $\mathcal{K}$ .

**for**  $k$  in  $1 : \mathcal{K}$  **do**

$\mu_k(\emptyset) = \mu_k(\{1\}) = \mu_k(\{2\}) = \dots = \mu(N) = 0$ .

**for**  $A$  in  $\{\emptyset, \{1\}, \{2\}, \dots, N\}$  **do**

**if**  $|A| == 1$  **then**

$\mu_k = \mu_k + \min[\max(\text{rnorm}(\frac{\alpha}{n}, \frac{1}{3n}), 0), 1] \times \varepsilon^+(A)$

**end**

**if**  $2 \leq |A| < n$  **then**

**if** the interaction of  $A$  is negative **then**  $\eta = -1$ ;

**else if** the interaction of  $A$  is zero **then**  $\eta = 0$ ;

**else if** the interaction of  $A$  is positive **then**  $\eta = 1$ ;

**else**

/\* the interaction of  $A$  is not given \*/

$\eta =$  random value in  $\{-1, 0, 1\}$ .

**end**

**if**  $\eta == -1$  **then**

$\mu_k = \mu_k + \beta \times \varepsilon^-(A)$ ,

where  $\beta \in [0, 1]$  is the negative interaction coefficient.

**else if**  $\eta == 0$  **then**

$\mu_k = \mu_k + \frac{1}{2} \times (\varepsilon^-(A) + \varepsilon^+(A))$ .

**else**

/\*  $\eta == 1$  \*/

$\mu_k = \mu_k + \gamma \times \varepsilon^+(A)$ ,

where  $\gamma \in [0, 1]$  is the positive interaction coefficient.

**end**

**end**

**if**  $|A| = n$  **then**

$\mu_k = \mu_k / \mu(N)$

**end**

**end**

Set the  $k$ th row of  $\mathbf{U}$  as  $\mu_k$ .

**end**

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**Remark 6.** In Algorithm 4, the operations on capacities are given as:

$$\lambda \times \mu := \lambda\mu(A), \forall A \subseteq N, \tag{9}$$

$$v + \mu := v(A) + \mu(A), \forall A \subseteq N, \tag{10}$$

where  $v$  and  $\mu$  are two capacities and  $\lambda \in [0, \infty]$  is a nonnegative real number. Since the convex combination of capacities is still a capacity, the result of operation  $\mu_k = \mu_k / \mu(N)$  is also a capacity.

**Remark 7.** The coefficients  $\beta$  and  $\gamma$  play the same role as in Algorithm 3. The only different thing of  $\beta$  in Algorithm 4 is: The larger the value of  $\beta$ , the stronger degree of the negative interaction is.

**Remark 8.** It should be pointed out that the case of zero interaction in Algorithm 4,  $\frac{1}{2} \times (\varepsilon^-(A) + \varepsilon^+(A))$  may exaggerate the interaction situation of subsets of  $A$  in some situations. To avoid this situation, in practice, we can use a very lightly negative or positive value to show the almost zero interaction, e.g., set  $\beta$  and  $\gamma$  a value in  $[0.0001, 0.0005]$ .

One can figure out that Algorithms 3 and 4 do not utilize any optimization model to calculate the range of the subset’s capacity value and only adopt plus, min or max operations on simple real numbers or 0–1 extreme capacities. Hence, it is not a difficult thing for an ordinary personal computer to execute these two algorithms even when the criteria is set with more than ten elements or the number of generated capacities  $\mathcal{K}$  with a ten thousands scale. As mentioned before, the execution efficiency is one of the main courses for us to develop these algorithms.

Unlike the real random generation methods proposed in [33,34], these algorithms do not adopt the random nodes (subsets) or random minimum plus, but both use the cardinality based order to select criteria subsets in sequence, and hence they are not strictly random generation and cannot ensure that the generated capacities follow the uniform or normal distribution. However, these quasi-random methods still hold some random operations or ingredients, like  $\text{rnorm}(\mu, \sigma)$  in both algorithms, and the generated capacities have good statistic characteristics and empirical results, see Table 1 and the illustrative example in the next section. Table 1 shows the Choquet integral of alternative  $(1/n, 2/n, \dots, 1)$  of 10,000 generated capacities by two quasi-random generation algorithms with  $\alpha$  as a random value in  $[0.5, 2]$ ,  $\beta$  and  $\gamma$  as random values in  $[0, 1]$ . From the definition of the Choquet integral, we know that the theoretical maximum and minimum of  $(1/n, 2/n, \dots, 1)$  are just  $1/n$  and 1, respectively. The Table 1 means that the quasi-random algorithms can cover the whole feasible capacity domain to a rather good extent.

**Table 1.** The Choquet integral of alternative  $(1/n, 2/n, \dots, 1)$ .

		$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Max	Algorithm 3.	0.9847188	0.9995541	0.9821527	0.9829890	0.9948813	0.9836805
	Algorithm 4.	0.9804507	0.9945898	0.9847443	0.9928669	0.9974471	0.9904509
Min	Algorithm 3.	0.2006147	0.1670042	0.1434162	0.1257456	0.1113552	0.1009193
	Algorithm 4.	0.2005896	0.1674521	0.1428751	0.1256290	0.1112416	0.1001236
Median	Algorithm 3.	0.5118252	0.5091925	0.5025637	0.5134141	0.5077784	0.5091297
	Algorithm 4.	0.4913858	0.5062988	0.4941002	0.5067579	0.4989028	0.4972489

### 5. An Illustrative Example

Consider an example adapted from [39]. Suppose a buyer adopts five criteria to evaluate a vehicle: The selling price, the space and comfort, the safety and convenience, the engine and transmission system and the maintenance cost for five years, denoted as criteria 1–5, respectively. The normalized partial evaluations of the 10 candidate cars are given in Table 2.

**Table 2.** The partial evaluations of candidate cars.

Car	Criterion 1	Criterion 2	Criterion 3	Criterion 4	Criterion 5
a	0.573	0.877	0.257	0.643	0.653
b	0.886	0.692	0.877	0.820	0.360
c	0.706	0.387	0.551	0.832	0.346
d	0.228	0.586	0.973	0.113	0.395
e	0.710	0.510	0.301	0.940	0.458
f	0.925	0.817	0.468	0.325	0.647
g	0.259	0.725	0.409	0.624	0.722
h	0.147	0.400	0.664	0.326	0.387
i	0.842	0.701	0.708	0.688	0.807
j	0.2	0.4	0.6	0.8	1.0

Using the nonadditivity index to describe the explicit preference on criteria, the decision maker gives the following preference information:

- (a) From the perspective of partial importance shown by capacity value or equivalently nonadditivity index value, criterion 1 is at least as important as criterion 4, criterion 3 is at least as important as criterion 5, criterion 2 is more important than criteria 1 and 3 with threshold 0.05.
- (b) the nonadditivity interactions of {1, 2}, {1, 3}, {1, 4}, {1, 5} are positive (larger than or equal to 0.05); and the nonadditivity interactions of {2, 3}, {2, 4}, {2, 5}, {3, 4} and {4, 5} are negative (less than or equal to −0.05).
- (c) the nonadditivity interaction of {1, 3} is greater than those of {1, 4} and {1, 5} with threshold 0.05, the nonadditivity interaction of {1, 5} is greater than that of {3, 5} with threshold 0.05; the nonadditivity interactions of {2, 3}, {2, 5} and {3, 4} are greater than that of {2, 4} with threshold 0.1, the nonadditivity interaction of {4, 5} is at least as great as that of {3, 4}.
- (d) the nonadditivity interaction index of {1, 3, 5} belongs to [0.5, 0.75], the nonadditivity indices of {1, 2, 3} and {1, 3, 4} belong to [0.25, 0.5], the nonadditivity indices of {1, 2, 4}, {1, 4, 5}, {1, 2, 3, 4}, {1, 2, 3, 5} and {1, 3, 4, 5} belong to [0, 0.25], the nonadditivity indices of {2, 3, 4}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5}, {1, 2, 4, 5} and {1, 2, 3, 4, 5} belong to [−0.25, 0], and the nonadditivity index of {2, 3, 4, 5} belongs to [−0.5, −0.25].

First, we use the nonadditivity index to represent the above preference as follows:

$$\mathbf{D}^N \Leftarrow \begin{cases} n_\mu(\{1\}) - n_\mu(\{4\}) \geq 0, \\ \dots \\ n_\mu(\{1, 2\}) \geq 0.05, \\ \dots \\ n_\mu(\{1, 3\}) - n_\mu(\{1, 4\}) \geq 0.05, \\ \dots \\ n_\mu(\{1, 3, 5\}) \geq 0.5, \\ n_\mu(\{1, 3, 5\}) \leq 0.75, \\ \dots \\ \mu(\emptyset) = 0, \quad \mu(N) = 1, \\ \mu(B \cup \{i\}) - \mu(B) \geq 0, \forall i \in N, B \subseteq N \setminus \{i\}. \end{cases} \tag{11}$$

Based on  $\mathbf{D}^N$ , we can execute the capacity random generation algorithm with mixed constraints (Algorithm 1), in which we choose the Choquet integral value of  $\mathbf{j}$  (range with [0.431875, 0.5053731]) as the driven index to generate capacities with  $\mathcal{K} = 1,000$ , and denote the output random capacities matrix as  $\mathbf{U}^1$ , the corresponding nonadditivity indices matrix  $\mathbf{N}^1$ .

For Algorithms 3 and 4, we can use the following parameters to represent the above decision preference information:

- The importance coefficients  $\alpha$  of five criteria are given as (0.75, 1.5, 0.5, 1, 0.5);
- the interaction degrees  $\beta(A)$  of subsets  $A$  with positive nonadditivity indices in Algorithm 3 are given as follows:

$$\beta(\{1,2\}) \in [0.05, 0.125], \beta(\{1,3\}) \in [0.25, 0.5], \beta(\{1,4\}) \in [0.05, 0.125], \beta(\{1,5\}) \in [0.05, 0.125],$$

$$\beta(\{1,2,3\}) \in [0.25, 0.5], \beta(\{1,2,4\}) \in [0, 0.25], \beta(\{1,3,4\}) \in [0.25, 0.5], \beta(\{1,3,5\}) \in [0.5, 0.75],$$

$$\beta(\{1,4,5\}) \in [0, 0.25], \beta(\{1,2,3,4\}) \in [0, 0.25], \beta(\{1,2,3,5\}) \in [0, 0.25], \beta(\{1,3,4,5\}) \in [0, 0.25];$$

- the interaction degrees  $\beta(A)$  of subsets  $A$  with positive nonadditivity indices in Algorithm 4 are given as follows:

$$\beta(\{1,2\}) \in [0.875, 0.95], \beta(\{1,3\}) \in [0.5, 0.75], \beta(\{1,4\}) \in [0.875, 0.95], \beta(\{1,5\}) \in [0.875, 0.95],$$

$$\beta(\{1,2,3\}) \in [0.5, 0.75], \beta(\{1,2,4\}) \in [0.75, 1], \beta(\{1,3,4\}) \in [0.5, 0.75], \beta(\{1,3,5\}) \in [0.25, 0.5],$$

$$\beta(\{1,4,5\}) \in [0.75, 1], \beta(\{1,2,3,4\}) \in [0.75, 1], \beta(\{1,2,3,5\}) \in [0.75, 1], \beta(\{1,3,4,5\}) \in [0.75, 1];$$

- the interaction degrees  $\gamma(A)$  of subsets  $A$  with negative nonadditivity indices in Algorithms 3 and 4 are given as follows:

$$\gamma(\{2,3\}) \in [-0.125, -0.05], \gamma(\{2,4\}) \in [-0.5, -0.25], \gamma(\{2,5\}) \in [-0.125, -0.05], \gamma(\{3,4\}) \in [-0.125, -0.05],$$

$$\gamma(\{3,5\}) \in [-0.125, -0.05], \gamma(\{4,5\}) \in [-0.125, -0.05], \gamma(\{2,3,4\}) \in [-0.25, 0],$$

$$\gamma(\{2,3,5\}) \in [-0.25, 0], \gamma(\{2,4,5\}) \in [-0.25, 0], \gamma(\{3,4,5\}) \in [-0.25, 0], \gamma(\{1,2,4,5\}) \in [-0.25, 0],$$

$$\gamma(\{2,3,4,5\}) \in [-0.25, 0], \gamma(\{1,2,3,4,5\}) \in [-0.25, 0].$$

By executing the Algorithms 3 and 4 with  $\mathcal{K} = 10,000$ , we can have the output random capacities matrices  $\mathbf{U}^2$  and  $\mathbf{U}^3$ , and their corresponding nonadditivity indices matrices  $\mathbf{N}^2$  and  $\mathbf{N}^3$ , respectively.

The matrices  $\mathbf{N}^1, \mathbf{N}^2$  and  $\mathbf{N}^3$  are shown in Figures 1–3, and the column means of three matrices are also given in Figure 1 for the purpose of comparison, where for the horizontal axis “ $c(1, 2)$ ” means the coalition of decision criteria 1 and 2, i.e.,  $\{1, 2\}$ . Taking nonadditivity index  $\mathbf{N}^1$  as a reference (the solid lines), we can find out that the nonadditivity indices  $\mathbf{N}^2$  and  $\mathbf{N}^3$  have the same inflection trends, the positivity or negativity of almost all criteria subsets are the same, and especially on some subsets, e.g.,  $\{2, 3\}, \{2, 4\}, \{3, 5\}, \{2, 3, 5\}$  and  $\{1, 2, 4, 5\}$ , three nonadditivity indices have very close values. Compared with  $\mathbf{N}^3, \mathbf{N}^2$  has a trend relatively more similar to  $\mathbf{N}^1$ . From Figures 2 and 3, we can get that nonadditivity index  $\mathbf{N}^2$  has a relatively dispersive distribution, whereas  $\mathbf{N}^3$  is more concentrative.

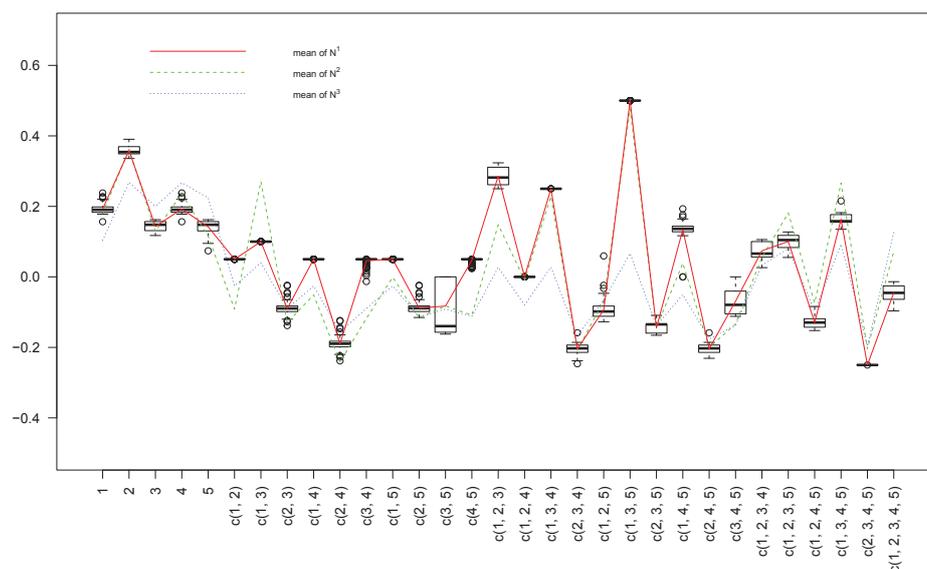


Figure 1. The nonadditivity index matrix  $\mathbf{N}^1$  by Algorithm 1.

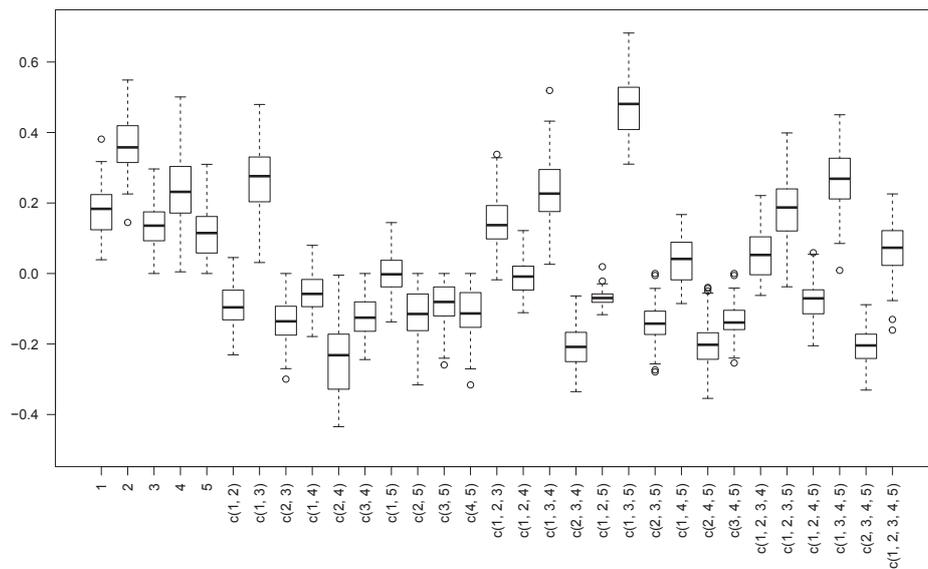


Figure 2. The nonadditivity index matrix  $N^2$  by Algorithm 3.

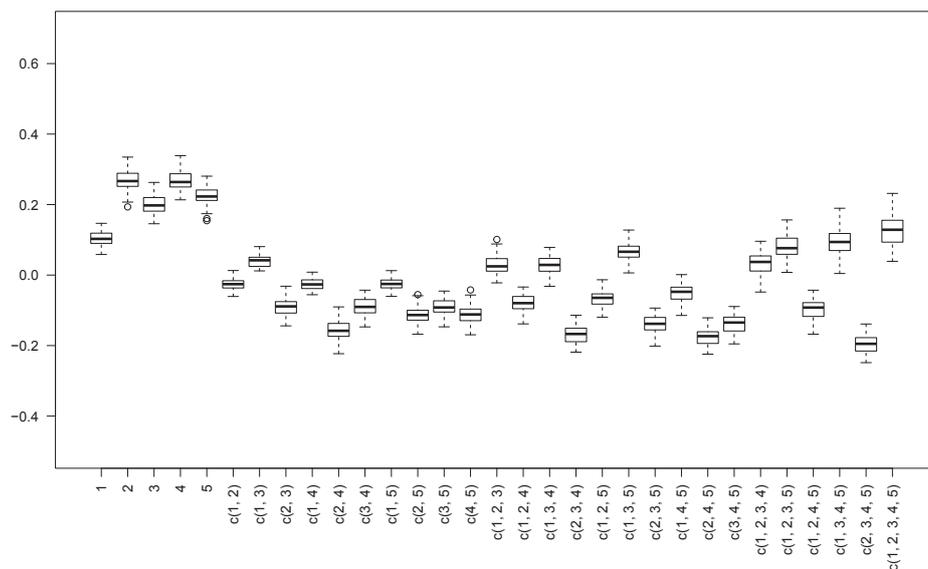


Figure 3. The nonadditivity index matrix  $N^3$  by Algorithm 4.

By executing the random simulation based comprehensive decision aid algorithm, Algorithm 2, with the three capacity matrices  $U^1$ ,  $U^2$  and  $U^3$ , we can have the corresponding dominance relationships of all alternatives and three ranking orders as follows:

$$U^1 : b \succ i \succ f \succ c \succ a \succ e \succ j \succ g \succ d \succ h;$$

$$U^2 : b \succ i \succ f \succ c \succ a \succ e \succ g \succ d \succ j \succ h;$$

$$U^3 : i \succ b \succ f \succ c \succ e \succ a \succ j \succ g \succ d \succ h.$$

One can see that the ranking order from  $U^2$ , compared with that from  $U^3$ , relatively caters the ranking order from  $U^1$ . Hence, if we take the random generation algorithm with mix constraints of nonadditivity index as a benchmark, Algorithm 3 can be a better choice than Algorithm 4 on the basis of comparison analysis of this example.

Finally, we should emphasise the major advantage of Algorithms 3 and 4: They do not need the sequential and taxing operations of the traditional random generation methods, such as construct preference constraints, establishment of the monotonicity conditions of all the set inclusion, checking and adjusting the inconsistency among the preference, choosing the selection principle of most satisfactory capacity and solving the optimization models of capacity identification. Hence, they are more efficient in dealing with a large scale of decision criteria and can accomplish relatively comprehensive decision making in practice.

## 6. Conclusions

In this paper, we proposed two quasi-random generation algorithms of capacity under the case of using the nonadditivity index to show the decision maker's preference information. The nonadditivity index has some good mathematical properties and rather intuitive explanations on the interaction phenomenon of decision criteria, and it is also a competent and flexible tool to represent the complex interaction cases among criteria. These two types of nonadditivity index oriented quasi-random generation algorithms are not unblemished in rigorously fitting the decision preference but have an incomparable execution efficiency in practice on account of not involving the preference inconsistency checking and adjusting as well as the optimization model establishing and solving, which will be really helpful for efficiently dealing with a large scale decision making problem with deeply correlative criteria. Future research will focus on some effective measures to improve the fitting accuracy of nonadditivity oriented algorithms to the decision maker's complicated preference as well as the random generation algorithm based decision aid packages.

**Author Contributions:** Methodology, L.H. and J.-Z.W.; Software, R.-J.X.; Writing—original draft, L.H. All authors have read and agreed to the published version of the manuscript.

**Funding:** The work was supported by the National Natural Science Foundation of China (No. 71671096) and the K.C.Wong Magna Fund in Ningbo University.

**Acknowledgments:** The authors would like to thank Gleb Beliakov for his suggestions on improving the random generation algorithms. The authors are grateful to the four anonymous reviewers for their valuable comments and suggestions which have led to significant improvement of this paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

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