mathematics

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Evaluation of the One-Dimensional L^p Sobolev Type Inequality

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Received:30 December 2019; Accepted: 17 February 2020 ; Published: 21 February 2020

Abstract: This study applies the extended L^2 Sobolev type inequality to the L^p Sobolev type inequality using Hölder's inequality. The sharp constant and best function of the L^p Sobolev type inequality are found using a Green function for the nth order ordinary differential equation. The sharp constant is shown to be equal to the L^p norm of the Green function and to the pth root of the value of the origin of the best function.

Keywords: sobolev inequality; sharp constant; green function

MSC: 34B27, 41A44, 46E35

1. Introduction

The Sobolev inequality also called the Sobolev embedding theorem, is often the core inequality in partial differential equations and variation calculations. Nonetheless, there have been few studies explicitly seeking the sharp (small) constant and best function calculation of the Sobolev inequality, the so-called "best evaluation of the Sobolev inequality".

In 1976, Aubin [1] and Talenti [2] independently discovered the sharp constants and best functions of Sobolev inequalities calculated the sharp constant, and found the best function by applying symmetric rearrangement to the Sobolev inequality:

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n) = \{u \in L^q(\mathbb{R}^n),\ \nabla u \in L^p(\mathbb{R}^n)\}. \tag{1}$$

On the other hand, in this study, the sharp constant of Sobolev inequality is primarily calculated using the Green function of various boundary value problems when $p \leq 2$ and $q = \infty$. We have found that the sharp constant can be obtained using the Green function, which is the reproducing kernel of a Hilbert space [3]. Specifically, the sharp constant of the corresponding Sobolev inequality is expressed as the maximum diagonal value of the Green function. Kametaka et al. [3] presented an earlier result for this sharp constant, using a unique approach that differs from symmetric rearrangement. The best evaluation of the Sobolev inequality obtained thus far has been applied to beam deflection [4], electrical circuits [5], and a range of other practical problems.

This paper aims to extend the result of the L^2 Sobolev type inequality to the L^p Sobolev type inequality, i.e., to generalize the best evaluation of the Sobolev type inequality. Results from studies [5–7] and [8] are prior works related to the subject of the present paper.

For $n = 1, 2, 3, \cdots$, we consider the following boundary value problem for an nth order ordinary differential operator $P(d/dt)$:

$$P(d/dt)u = f(t) \quad (t \in \mathbb{R}) \tag{2}$$

with the following conditions,

$$u^{(i)}(t) \in L^q(\mathbb{R}) \quad (0 \le i \le n), \tag{3}$$

where exponent q satisfies $1 < q \le 2$. The condition (3) corresponds to boundary condition at $t = \pm\infty$. The characteristic polynomial with real coefficients

$$P(z) = \prod_{j=0}^{n-1}(z + a_j)$$

is assumed to be a Hurwitz polynomial with real characteristic roots $-a_0, -a_1, \cdots, -a_{n-1}$.

For simplicity, we impose the following two assumptions:

Assumption 1. *The real coefficients a_i $(i = 0, 1, \cdots, n-1)$ satisfy the following inequality:*

$$0 < a_0 < a_1 < \cdots < a_{n-1}.$$

Assumption 2. *The two exponents p and q satisfy both $1 < q \le p < \infty$ and the relation*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

To describe the main theorem, we introduce a function $G(t)$, which is defined by

$$G(t) = Y(t)\sum_{j=0}^{n-1} b_j e^{-a_j t} \quad (t \in \mathbb{R}), \quad b_j = \frac{1}{P'(-a_j)} = \frac{1}{\displaystyle\prod_{k=0, k\neq j}^{n-1}(-a_j + a_k)}$$

where $Y(t)$ is a unit step function, i.e., $Y(t) = 1$ $(t \ge 0)$, and 0 $(t < 0)$. It is shown in Section 3 that the function $G(t,s) = G(t-s)$ is the Green function of (2). Note that coefficient b_j is bounded for $0 \le j \le n-1$. Hereafter, for convenience, b_{max} is defined as the maximum of the absolute value for b_j $(0 \le j \le n-1)$, such that

$$b_{max} = \max_{0 \le j \le n-1} |b_j|.$$

This paper is organized as follows. In Section 2, we use the Fourier transform to construct a Green function for the boundary value problem of the nth order ordinary differential equation. In Section 3, we derive a Sobolev type inequality from its solution formula. Section 4 describes the property of the best function of the Sobolev type inequality. Finally, in Section 5, we calculate the sharp constants for two special cases.

2. Derivation of Green Function

In this section, we will obtain a concrete expression of the Green function $G(t,s) = G(t-s)$. Concerning the uniqueness and existence of the solution to (2), we have the following theorem. Strictly speaking, the Green function $G(t,s) = G(t-s)$ is a two-variable function. However, we call the function $G(t)$ the Green function for the sake of convenience.

Theorem 1. *For any function $f \in L^q(\mathbb{R})$, (2) has a unique solution u expressed as*

$$u(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds \quad (t \in \mathbb{R}). \tag{4}$$

By using the functions

$$G_j(t) = Y(t) e^{-a_j t} \quad (t \in \mathbb{R},\ 0 \le j \le n-1),$$

the Green function can be expressed as

$$G(t) = \begin{cases} (G_0 * G_1 * \cdots * G_{n-1}) & (t \ge 0), \\ 0 & (t < 0). \end{cases} \tag{5}$$

The Green function is then rewritten as

$$\begin{aligned} G(t) &= Y(t) \sum_{j=0}^{n-1} \frac{1}{\prod\limits_{k=0, k\neq j}^{n-1} (-a_j + a_k)} G_j(t) \\ &= (-1)^{n-1} Y(t) \left| \begin{array}{c} a_j^i \\ \hline G_j(t) \end{array} \right| \Big/ \left| \; a_j^i \; \right| \quad (t \in \mathbb{R}). \end{aligned} \tag{6}$$

In the above determinants, exponent i and index j are such that $0 \le i \le n-2$ and $0 \le j \le n-1$ in the numerator and $0 \le i, j \le n-1$ in the denominator.

Before proving Theorem 1, the following proposition is prepared. We define the Fourier transform $\mathscr{F} : L^1(\mathbb{R}) \cap L^q(\mathbb{R}) \to L^p(\mathbb{R})$ of the function $u \in L^q(\mathbb{R})$ as

$$\mathscr{F}[u(t)](\omega) \equiv \widehat{u}(\omega) \equiv \int_{-\infty}^{\infty} u(t) e^{-\sqrt{-1}\omega t}\, dt.$$

Proposition 1.

$$G_j(t) = Y(t) e^{-a_j t} \xrightarrow{\mathscr{F}} \widehat{G_j}(\omega) = \frac{1}{\sqrt{-1}\omega + a_j} \quad (t,\ \omega \in \mathbb{R},\ 0 \le j \le n-1).$$

In order to prove Theorem 1, we transform the expansion of $1/P(z)$ to a partial fraction. For the partial fraction expansion

$$\frac{1}{P(z)} = \sum_{j=0}^{n-1} \frac{b_j}{z + a_j}, \quad b_j = \frac{1}{P'(-a_j)} = \frac{1}{\prod\limits_{k=0, k\neq j}^{n-1} (-a_j + a_k)},$$

using well-known facts (see reference [9] p.120 (18))

$$\begin{pmatrix} \\ b_i \\ \\ \end{pmatrix} = \begin{pmatrix} & & \\ & (-a_j)^i & \\ & & \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$ {}^t\mathbf{a}\mathbf{A}^{-1}\mathbf{b} = -\left|\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline {}^t\mathbf{a} & 0 \end{array}\right| \Big/ \left| \ \mathbf{A} \ \right|, $$

where $\mathbf{A}$ is any $n \times n$ regular matrix and $\mathbf{a}$ and $\mathbf{b}$ are $n \times 1$ matrices, we have the following partial fraction expansion:

$$ \frac{1}{P(z)} = \sum_{j=0}^{n-1} \frac{b_j}{z + a_j} = (-1)^{n-1} \left| \frac{a_j^i}{(z+a_j)^{-1}} \right| \Big/ \left| \ a_j^i \ \right|. \tag{7} $$

The above method is a well-known technique in Heaviside calculus (see reference [10] A. §5, 22.2, for example).

Proof of Theorem 1. By applying the Fourier transform to the differential equation of (2), we have

$$ P(d/dt)u = f(t) \xrightarrow{\mathscr{F}} P(\sqrt{-1}\omega)\widehat{u} = \widehat{f}(\omega), $$

and hence

$$ \begin{aligned} &\widehat{u}(\omega) = \widehat{G}(\omega)\widehat{f}(\omega), \\ &\widehat{G}(\omega) = \frac{1}{P(\sqrt{-1}\omega)} = \sum_{j=0}^{n-1} \frac{b_j}{\sqrt{-1}\omega + a_j} = \sum_{j=0}^{n-1} b_j \int_0^\infty e^{-(\sqrt{-1}\omega + a_j)t}\, dt \\ &= \int_0^\infty \left(\sum_{j=0}^{n-1} b_j e^{-a_j t} \right) e^{-\sqrt{-1}\omega t}\, dt = \int_{-\infty}^\infty \left\{ Y(t) \sum_{j=0}^{n-1} b_j G_j(t) \right\} e^{-\sqrt{-1}\omega t}\, dt. \end{aligned} $$

The only solution of (2) is given by

$$ u(t) = \int_{-\infty}^\infty G(t,s) f(s)\, ds = \int_{-\infty}^\infty G(t-s) f(s)\, ds \quad (t \in \mathbb{R}), $$

where $G(t,s) = G(t-s)$ is the Green function. For $\widehat{G}(\omega)$, using Proposition 1 and (7), we have (6). (5) follows immediately from

$$ \widehat{G}(\omega) = \prod_{j=0}^{n-1} \widehat{G}_j(\omega), $$

which completes the proof of Theorem 1. □

Since $G_j(t) = Y(t)e^{-a_j t} \in L^p(\mathbb{R})$, the Green function $G(t)$, which is a linear combination of $\{G_j(t)\}$, belongs to $L^p(\mathbb{R})$.

3. The Sharp Constant and the Best Function of Sobolev Type Inequality

In this section, we derive the Sobolev type inequality. This inequality is a special case of Young's inequality, for which sharp constants are given in [11]. However, we prove the theorem for the sake of self containedness. The main conclusion of this paper is as follows.

Theorem 2. *For any function u satisfying $u^{(i)} \in L^q(\mathbb{R})$ $(0 \le i \le n)$, there exists a positive constant C, which is independent of u, such that the following Sobolev type inequality holds:*

$$ \|u\|_\infty \le C \|P(d/dt)u\|_q. \tag{8} $$

Among such C, the sharp constant $C(n;\mathbf{a})$ is equal to the L^p norm of the Green function $G(t)$ and can be expressed as

$$C(n;\mathbf{a}) = C(n;a_0,a_1,\cdots,a_{n-1}) = \|G\|_p = \left\{ \int_{-\infty}^{\infty} |G(t)|^p \, dt \right\}^{\frac{1}{p}}. \tag{9}$$

Let $u(t) = U(t)$ be a solution of (2) for $f(t) = \{G(-t)\}^{\frac{p}{q}}$ $(t \in \mathbb{R})$. Then, if C is replaced by $C(n;\mathbf{a})$ in (8), the inequality holds for

$$u(t) = c\,U(t) \quad (t \in \mathbb{R}),$$

where c is an arbitrary complex number and $U(t)$ is given by

$$U(t) = \int_{-\infty}^{\infty} G(t-s)\{G(-s)\}^{\frac{p}{q}} \, ds \quad (-\infty < t < \infty).$$

The physical meaning of a Sobolev type inequality is then that the maximum absolute value of a solution for (2) is estimated by the constant multiple of the L^q norm of $P(d/dt)u(t)$.

Proof of Theorem 2. For any function u satisfying $u^{(i)} \in L^q(\mathbb{R})$ $(0 \le i \le n)$, we define $f \in L^q(\mathbb{R})$ by the following relation:

$$f(t) = P(d/dt)u(t) \quad (t \in \mathbb{R}).$$

Applying Hölder's inequality to the solution formula (4), we obtain

$$\left| \int_{-\infty}^{\infty} G(s-t) f(t) \, dt \right| \le \left(\int_{-\infty}^{\infty} |G(s-t)|^p \, dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |f(t)|^q \, dt \right)^{\frac{1}{q}},$$

hence

$$|u(s)| \le \left(\int_{-\infty}^{\infty} |G(t)|^p \, dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |f(t)|^q \, dt \right)^{\frac{1}{q}}.$$

Taking the maximum with respect to s on the inequality, we obtain the Sobolev type inequality as follows:

$$\sup_{-\infty<s<\infty} |u(s)| \le \|G\|_p \left(\int_{-\infty}^{\infty} |f(t)|^q \, dt \right)^{\frac{1}{q}} dt = \|G\|_p \left(\int_{-\infty}^{\infty} |P(d/dt)u(t)|^q \, dt \right)^{\frac{1}{q}}.$$

Taking the solution $u(t) = U(t)$ of (2) for a particular function $f(t) = \{G(-t)\}^{\frac{p}{q}}$ $(t \in \mathbb{R})$, we obtain the following relation:

$$U(s) = \int_{-\infty}^{\infty} G(s-t)\{G(-t)\}^{\frac{p}{q}} \, dt \quad (s \in \mathbb{R}).$$

From the above equality, we obtain the relation:

$$U(0) = \int_{-\infty}^{\infty} |G(-t)|^{1+\frac{p}{q}} \, dt = \int_{-\infty}^{\infty} |G(-t)|^p \, dt = \int_{-\infty}^{\infty} |G(t)|^p \, dt = \|G\|_p^p.$$

We also have:

$$\|G\|_p^p = U(0) \le \sup_{-\infty<s<\infty} |U(s)| \le \|G\|_p \left(\int_{-\infty}^{\infty} |P(d/dt)U(t)|^q \, dt \right)^{\frac{1}{q}} = \|G\|_p \left(\int_{-\infty}^{\infty} \left| \{G(-t)\}^{\frac{p}{q}} \right|^q dt \right)^{\frac{1}{q}}$$
$$= \|G\|_p \left(\int_{-\infty}^{\infty} |G(-t)|^p \, dt \right)^{\frac{1}{q}} = \|G\|_p \left\{ \left(\int_{-\infty}^{\infty} |G(t)|^p \, dt \right)^{\frac{1}{p}} \right\}^{\frac{p}{q}} = \|G\|_p^{1+\frac{p}{q}} = \|G\|_p^p.$$

This means that

$$\sup_{-\infty<s<\infty} |U(s)| = \|G\|_p \left(\int_{-\infty}^{\infty} |P(d/dt)U(t)|^q \, dt \right)^{\frac{1}{q}},$$

which completes the proof of Theorem 2. □

4. The Important Property of the Best Function $U(t)$

In this section, we show an important property of the best function of the Sobolev type inequality.

Lemma 1. *A particular solution $U(t)$ to (2) for $f(t) = \{G(-t)\}^{\frac{p}{q}}$ satisfies also (3), that is, is an element of $L^q(\mathbb{R})$, i.e.,*

$$U^{(i)} \in L^q(\mathbb{R}) \quad (0 \le i \le n).$$

Proof of Lemma 1. Since the best function is a special solution of (2) for $f(t) = \{G(-t)\}^{\frac{p}{q}}$,

$$U(s) = \int_{-\infty}^{\infty} G(s-t)\{G(-t)\}^{\frac{p}{q}} \, dt.$$

Specifically,

$$U(s) = \left\{ \begin{array}{ll} \int_{-\infty}^{0} G(s-t)\{G(-t)\}^{\frac{p}{q}} \, dt & (t \le 0 \le s) \\ \int_{-\infty}^{s} G(s-t)\{G(-t)\}^{\frac{p}{q}} \, dt & (t \le s < 0) \end{array} \right\} = \left\{ \begin{array}{ll} \int_{0}^{\infty} G(s+t)\{G(t)\}^{\frac{p}{q}} \, dt & (-t \le 0 \le s), \\ \int_{|s|}^{\infty} G(s+t)\{G(t)\}^{\frac{p}{q}} \, dt & (-t \le s < 0). \end{array} \right.$$

(i) $s \ge 0$
Applying the ith derivative of the best function $U(s)$ with respect to s, we obtain

$$U^{(i)}(s) = \int_0^{\infty} \frac{\partial^i}{\partial s^i} \left[G(s+t)\{G(t)\}^{\frac{p}{q}} \right] dt = \int_0^{\infty} \left\{ \sum_{j=0}^{n-1} (-1)^i a_j^i b_j e^{-a_j(s+t)} \right\} \left\{ \sum_{k=0}^{n-1} b_k e^{-a_k t} \right\}^{\frac{p}{q}} dt.$$

Subsequently,

$$|U^{(i)}(s)| \le \int_0^{\infty} \left| \left\{ \sum_{j=0}^{n-1} (-1)^i a_j^i b_j e^{-a_j(s+t)} \right\} (nb_{max})^{\frac{p}{q}} e^{-\frac{a_0 p}{q} t} \right| dt$$
$$\le (nb_{max})^{\frac{p}{q}} \sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j s} \int_0^{\infty} e^{-(a_j + \frac{a_0 p}{q})t} \, dt = (nb_{max})^{\frac{p}{q}} \sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j s} \frac{q}{a_0 p + a_j q}$$
$$\le \frac{(nb_{max})^{\frac{p}{q}}}{a_0 p} \sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j s}.$$

From this inequality, we obtain the following evaluation:

$$|U^{(i)}(s)| \leq \frac{(nb_{max})^{\frac{p}{q}}}{a_0 p} \sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j s} \quad (0 \leq i \leq n,\ s \geq 0).$$

Therefore,

$$\begin{aligned}\int_0^\infty |U^{(i)}(s)|^q\, ds &= \frac{(nb_{max})^p}{(a_0 p)^q} \int_0^\infty \left(\sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j s} \right)^q ds \\ &\leq \frac{(nb_{max})^p}{(a_0 p)^q} a_{n-1}^{iq} b_{max}^q n^q \int_0^\infty e^{-a_0 q s}\, ds = \frac{(nb_{max})^{p+q} a_{n-1}^{iq}}{a_0^{q+1} p^q q} < \infty.\end{aligned}$$

(ii) $s < 0$

Similarly, applying the ith $(1 \leq i \leq n)$ derivative of the best function $U(s)$ with respect to s, we obtain

$$U^{(i)}(s) = -\sum_{k=1}^{i} G^{(k-1)}(0) \left(\frac{p}{q}\right)_{i-k} (-1)^{i-k} \{G(|s|)\}^{\frac{p}{q}-(i-k)} + \int_{|s|}^\infty \frac{\partial^i}{\partial s^i} \left[G(s+t)\{G(t)\}^{\frac{p}{q}} \right] dt.$$

When $i = 0$, the above inequality does not have the first term on the right side, such that it is sufficient to prove the case of $i = 1, 2, \cdots, n$. Subsequently,

$$|U^{(i)}(s)| \leq \sum_{k=1}^{i} \left| G^{(k-1)}(0) \left(\frac{p}{q}\right)_{i-k} \{G(|s|)\}^{\frac{p}{q}-(i-k)} \right| + \int_{|s|}^\infty \left| \frac{\partial^i}{\partial s^i} \left[G(s+t)\{G(t)\}^{\frac{p}{q}} \right] \right| dt$$

where $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1)$ $(n \geq 1)$, and 1 $(n = 0)$ is the Pochhammer symbol. Since

$$G^{(k-1)}(0) = \sum_{j=0}^{n-1} (-a_j)^{k-1} b_j e^{-a_j s} \Big|_{s=0} = \sum_{j=0}^{n-1} (-1)^{k-1} a_j^{k-1} b_j,$$

$$G(|s|) = \sum_{j=0}^{n-1} b_j e^{-a_j |s|} \leq nb_{max} e^{-a_0 |s|}$$

and

$$\frac{\partial^i}{\partial s^i} G(s+t) = \sum_{j=0}^{n-1} (-1)^i a_j^i b_j e^{-a_j (s+t)},$$

$|U^{(i)}(s)|$ is similarly estimated from above as follows:

$$\begin{aligned}|U^{(i)}(s)| &\leq \sum_{k=1}^{i} \left\{ \sum_{j=0}^{n-1} a_j^{k-1} |b_j| \right\} \left| \left(\frac{p}{q}\right)_{i-k} \right| (nb_{max} e^{-a_0|s|})^{\frac{p}{q}-(i-k)} \\ &\quad + \int_{|s|}^\infty \left\{ \sum_{j=0}^{n-1} a_j^i |b_j| e^{-a_j(s+t)} \right\} (nb_{max} e^{-a_0 t})^{\frac{p}{q}}\, dt \\ &\leq \sum_{k=1}^{i} \left\{ n a_{n-1}^{k-1} b_{max} |(p-1)_{i-k}| (nb_{max} e^{-a_0|s|})^{\frac{p}{q}} \right\} + \frac{(nb_{max})^p a_{n-1}^i}{a_0 p} e^{-\frac{a_0 p}{q}|s|}.\end{aligned}$$

In summary, the following evaluation can be obtained:

$$
\begin{aligned}
&|U^{(i)}(s)| \leq A_i(s) + B_i(s), \\
&A_i(s) = \sum_{k=1}^{i} \left\{ n a_{n-1}^{k-1} b_{max} |(p-1)_{i-k}| (n b_{max} e^{-a_0 |s|})^{\frac{p}{q}} \right\}, \\
&B_i(s) = \frac{(n b_{max})^p a_{n-1}^i}{a_0 p} e^{-\frac{a_0 p}{q}|s|} \quad (0 \leq i \leq n,\ s < 0).
\end{aligned}
$$

From the above estimation, it is possible to obtain the following inequality using Jensen's inequality:

$$
\begin{aligned}
\int_{-\infty}^{0} |U^{(i)}(s)|^q \, ds &= \int_t^0 |U^{(i)}(s)|^q \, ds \\
&\leq 2^{q-1} \int_t^0 \{ |A_i(s)|^q + |B_i(s)|^q \} \, ds \\
&= 2^{q-1} \int_0^{|t|} \{ |A_i(-s)|^q + |B_i(-s)|^q \} \, ds.
\end{aligned}
$$

Finally, we confirm the boundedness of each term on the right side:

$$
\begin{aligned}
\int_0^{|t|} |A_i(-s)|^q \, ds &\leq \int_0^{|t|} \left| i n a_{n-1}^{i-1} b_{max} |(p-1)_{i-1}| (n b_{max} e^{-a_0 s})^{\frac{p}{q}} \right|^q ds \\
&= \{ i n a_{n-1}^{i-1} b_{max} |(p-1)_{i-1}| (n b_{max})^{\frac{p}{q}} \}^q \int_0^{|t|} e^{-a_0 p s} \, ds \\
&< \frac{1}{a_0 p} \{ i n a_{n-1}^{i-1} b_{max} |(p-1)_{i-1}| (n b_{max})^{\frac{p}{q}} \}^q < \infty
\end{aligned}
$$

and

$$
\int_0^{|t|} |B_i(-s)|^q \, ds = \left\{ \frac{(n b_{max})^p a_{n-1}^i}{a_0 p} \right\}^q \int_0^{|t|} e^{-a_0 p s} \, ds < \frac{\{ (n b_{max})^p a_{n-1}^i \}^q}{(a_0 p)^{q+1}} < \infty.
$$

Since the boundedness of each term is shown in the above evaluation formula, the following formula is obtained:

$$
\int_{-\infty}^{0} |U^{(i)}(s)|^q \, ds < \infty \quad (0 \leq i \leq n).
$$

Therefore, the boundedness of the L^q norm for $U^{(i)}(s)$ is shown using (i) and (ii).

$$
\|U^{(i)}\|_q^q = \int_{-\infty}^{\infty} |U^{(i)}(s)|^q \, ds = \int_{-\infty}^{0} |U^{(i)}(s)|^q \, ds + \int_0^{\infty} |U^{(i)}(s)|^q \, ds < \infty.
$$

This completes the proof of Lemma 1. □

5. Examples of Sharp Constant

In this section, we calculate the sharp constants for each of the two special cases.

5.1. In the Case of $p = 2, 3, 4 \cdots$

From Assumption 2, the exponent p is a real number greater than or equal to 2, but the following lemma holds when p is an integer greater than or equal to 2:

Lemma 2. *For $p = 2, 3, 4, \cdots$, the sharp constant $C_0(n; \mathbf{a})$ is specifically calculated as follows:*

$$C_0(n;\mathbf{a}) = \left[\sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} \left(\sum_{k=0}^{n-1} a_k j_k \right)^{-1} \prod_{l=0}^{n-1} \prod_{k=0, k\neq l}^{n-1} (-a_l + a_k)^{-j_l} \right]^{\frac{1}{p}}.$$

Proof of Lemma 2. From the sharp constant formula (9)

$$C_0(n;\mathbf{a}) = \|G\|_p = \left\{ \int_0^\infty \left(\sum_{j=0}^{n-1} b_j e^{-a_j t} \right)^p dt \right\}^{\frac{1}{p}},$$

we focus on only the integrand of the sharp constant, and calculate the function specifically.

From the multinomial expansion to the Green function $G(t)$ $(t \geq 0)$, we get

$$\begin{aligned} \{G(t)\}^p &= \left(\sum_{j=0}^{n-1} b_j e^{-a_j t} \right)^p = \sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} (b_0 e^{-a_0 t})^{j_0} \cdots (b_{n-1} e^{-a_{n-1} t})^{j_{n-1}} \\ &= \sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} b_0^{j_0} \cdots b_{n-1}^{j_{n-1}} e^{-(a_0 j_0 + \cdots + a_{n-1} j_{n-1})t}. \end{aligned}$$

From the above equation, the L^p norm of the Green function is calculated as follows:

$$\begin{aligned} \|G\|_p^p &= \sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} b_0^{j_0} \cdots b_{n-1}^{j_{n-1}} \int_0^\infty e^{-(a_0 j_0 + \cdots + a_{n-1} j_{n-1})t}\, dt \\ &= \sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} b_0^{j_0} \cdots b_{n-1}^{j_{n-1}} \left(\sum_{k=0}^{n-1} a_k j_k \right)^{-1}. \end{aligned}$$

Since the definition of the coefficients b_j $(j = 0, 1, \cdots, n-1)$ is

$$b_j = \frac{1}{P'(-a_j)} = \prod_{k=0, k\neq j}^{n-1} (-a_j + a_k)^{-1} \quad (0 \leq j \leq n-1),$$

the pth power of the L^p norm of the Green function is obtained as follows:

$$\|G\|_p^p = \sum_{j_0+\cdots+j_{n-1}=p} \frac{p!}{j_0!\cdots j_{n-1}!} \left(\sum_{k=0}^{n-1} a_k j_k \right)^{-1} \prod_{l=0}^{n-1} \prod_{k=0, k\neq l}^{n-1} (-a_l + a_k)^{-j_l}.$$

This completes the proof of Lemma 2. □

In addition, when $p = 1$, the L^1 norm of the Green function is expressed as follows:

$$\|G\|_1 = \int_0^\infty \left(\sum_{j=0}^{n-1} b_j e^{-a_j t} \right) dt = \sum_{j=0}^{n-1} b_j \left(-\frac{1}{a_j} \right) e^{-a_j t} \Big|_0^\infty = \sum_{j=0}^{n-1} \frac{b_j}{a_j} = \frac{1}{P(0)} = \prod_{k=0}^{n-1} a_k^{-1} < \infty.$$

5.2. In the Case of $a_j = j + 1$

For real coefficients $a_j = j + 1$ $(j = 0, 1, \cdots, n-1)$ that satisfy Assumption 1, the sharp constants may be calculated in a concretely closed form.

Lemma 3. *For $p \geq 2$ and $a_j = j+1$ $(j = 0, 1, \cdots, n-1)$, the sharp constant $C_0(n; \mathbf{a})$ is specifically calculated as follows:*

$$C_0(n; \mathbf{a}) = \frac{1}{\Gamma(n)} \left\{ \frac{\Gamma(p)\Gamma((n-1)p+1)}{\Gamma(np+1)} \right\}^{\frac{1}{p}}.$$

Proof of Lemma 3. Calculating the value from the definition of the coefficient b_j, we obtain

$$\begin{aligned} b_j &= \frac{1}{P'(-a_j)} = \frac{1}{\prod_{k=0,k\neq j}^{n-1} (-a_j + a_k)} = \frac{1}{\prod_{k=0,k\neq j}^{n-1} (-j+k)} \\ &= \frac{1}{\prod_{k=0}^{j-1}(-j+k) \prod_{k=j+1}^{n-1} (-j+k)} = \frac{(-1)^j}{j!(n-1-j)!} = \frac{(-1)^j}{\Gamma(n)} \binom{n-1}{j}. \end{aligned}$$

From the above equality, the Green function is composed as follows:

$$\begin{aligned} G(t) &= \sum_{j=0}^{n-1} b_j e^{-a_j t} = \sum_{j=0}^{n-1} \frac{(-1)^j}{\Gamma(n)} \binom{n-1}{j} e^{-a_j t} \\ &= \frac{e^{-t}}{\Gamma(n)} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} e^{-jt} = \frac{1}{\Gamma(n)} e^{-t}(1-e^{-t})^{n-1} \quad (t \geq 0). \end{aligned}$$

Subsequently,

$$\begin{aligned} \|G\|_p^p &= \int_{-\infty}^{\infty} |G(t)|^p \, dt = \frac{1}{\{\Gamma(n)\}^p} \int_0^{\infty} e^{-pt}(1-e^{-t})^{p(n-1)} \, dt \\ &= \frac{1}{\{\Gamma(n)\}^p} B(p, p(n-1)+1) = \frac{\Gamma(p)\Gamma((n-1)p+1)}{\{\Gamma(n)\}^p \Gamma(np+1)}. \end{aligned}$$

Therefore, we obtain the sharp constant

$$C_0 = \|G\|_p = \frac{1}{\Gamma(n)} \left\{ \frac{\Gamma(p)\Gamma((n-1)p+1)}{\Gamma(np+1)} \right\}^{\frac{1}{p}}.$$

This completes the proof of Lemma 3. □

Author Contributions: Y.K. wrote the first draft, K.T. corrected, and improved it. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by J. S. P. S. Grant-in-Aid for Scientific Research (C) No. 17K05374.

Conflicts of Interest: The authors declare no conflict of interest.

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