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A Note on Minimal Hypersurfaces of an Odd Dimensional Sphere

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Abstract: We obtain the Wang-type integral inequalities for compact minimal hypersurfaces in the unit sphere S^{2n+1} with Sasakian structure and use these inequalities to find two characterizations of minimal Clifford hypersurfaces in the unit sphere S^{2n+1}.

Keywords: clifford minimal hypersurfaces; sasakian structure; integral inequalities; reeb function; contact vector field

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1. Introduction

Let M be a compact minimal hypersurface of the unit sphere S^{n+1} with shape operator A. In his pioneering work, Simons [1] has shown that on a compact minimal hypersurface M of the unit sphere S^{n+1} either $A = 0$ (totally geodesic), or $\|A\|^2 = n$, or $\|A\|^2(p) > n$ for some point $p \in M$, where $\|A\|$ is the length of the shape operator. This work was further extended in [2] and for compact constant mean curvature hypersurfaces in [3]. If for every point p in M, the square of the length of the second fundamental form of M is n, then it is known that M must be a subset of a Clifford minimal hypersurface

$$S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right),$$

where l, m are positive integers, $l + m = n$ (cf. Theorem 3 in [4]). Note that this result was independently proven by Lawson [2] and Chern, do Carmo, and Kobayashi [5]. One of the interesting questions in differential geometry of minimal hypersurfaces of the unit sphere S^{n+1} is to characterize minimal Clifford hypersurfaces. Minimal hypersurfaces have also been studied in (cf. [6–8]). In [2], bounds on Ricci curvature are used to find a characterization of the minimal Clifford hypersurfaces in the unit sphere S^4. Similarly in [3,9–11], the authors have characterized minimal Clifford hypersurfaces in the odd-dimensional unit spheres S^3 and S^5 using constant contact angle. Wang [12] studied compact minimal hypersurfaces in the unit sphere S^{n+1} with two distinct principal curvatures, one of them being simple and obtained the following integral inequality,

$$\int_M \|A\|^2 \leq nVol(M),$$

where $Vol(M)$ is the volume of M. Moreover, he proved that equality in the above inequality holds if and only if M is the Clifford hypersurface,

$$S^1\left(\sqrt{\frac{1}{n}}\right)\times S^m\left(\sqrt{\frac{n-1}{n}}\right).$$

In this paper, we are interested in studying compact minimal hypersurfaces of the unit sphere S^{2n+1} using the Sasakian structure (φ,ξ,η,g) (cf. [13]) and finding characterizations of minimal Clifford hypersurface of S^{2n+1}. On a compact minimal hypersurface M of the unit sphere S^{2n+1}, we denote by N the unit normal vector field and define a smooth function $f=g(\xi,N)$, which we call the *Reeb function* of the minimal hypersurface M. Also, on the hypersurface M, we have a smooth vector field $v=\varphi(N)$, which we call the *contact vector field* of the hypersurface (v being orthogonal to ξ belongs to contact distribution). Instead of demanding two distinct principal curvatures one being simple, we ask the contact vector field v of the minimal hypersurface in S^{2n+1} to be conformal vector field and obtain an inequality similar to Wang's inequality and show that the equality holds if and only if M is isometric to a Clifford hypersurface. Indeed we prove

Theorem 1. *Let M be a compact minimal hypersurface of the unit sphere S^{2n+1} with Reeb function f and contact vector field v a conformal vector field on M. Then,*

$$\int_M(1-f^2)\,\|A\|^2\leq 2n\int_M\left(1-f^2\right)$$

and the equality holds if and only if M is isometric to the Clifford hypersurface $S^l\left(\sqrt{\frac{l}{2n}}\right)\times S^m\left(\sqrt{\frac{m}{2n}}\right)$, where $l+m=2n$.

Also in [12], Wang studied embedded compact minimal non-totally geodesic hypersurfaces in S^{n+1} those are symmetric with respect to $n+2$ pair-wise orthogonal hyperplanes of R^{n+2}. If M is such a hypersurface, then it is proved that

$$\int_M\|A\|^2\geq nVol(M),$$

and the equality holds precisely if M is a Clifford hypersurface. Note that compact embedded hypersurface has huge advantage over the compact immersed hypersurface, as it divides the ambient unit sphere S^n into two connected components.

In our next result, we consider compact immersed minimal hypersurface M of the unit sphere S^{2n+1} such that the Reeb function f is a constant along the integral curves of the contact vector field v and show that above inequality of Wang holds, and we get another characterization of minimal Clifford hypersurface in the unit sphere S^{2n+1}. Precisely, we prove the following.

Theorem 2. *Let M be a compact minimal hypersurface of the unit sphere S^{2n+1} with Reeb function f a constant along the integral curves of the contact vector field v. Then,*

$$\int_M\|A\|^2\geq 2nVol(M)$$

and the equality holds if and only if M is isometric to the Clifford hypersurface $S^l\left(\sqrt{\frac{l}{2n}}\right)\times S^m\left(\sqrt{\frac{m}{2n}}\right)$, where $l+m=2n$.

2. Preliminaries

Recall that conformal vector fields play an important role in the geometry of a Riemannian manifolds. A conformal vector field v on a Riemannian manifold (M, g) has local flow consisting of conformal transformations, which is equivalent to

$$\pounds_v g = 2\rho g. \tag{1}$$

The smooth function ρ appearing in Equation (1) defined on M is called the potential function of the conformal vector field v. We denote by (φ, ξ, η, g) the Sasakian structure on the unit sphere S^{2n+1} as a totally umbilical real hypersurface of the complex space form $(C^{n+1}, \overline{J}, \langle, \rangle)$, where $\overline{J}$ is the complex structure and $\langle, \rangle$ is the Euclidean Hermitian metric. The Sasakian structure (φ, ξ, η, g) on S^{2n+1} consists of a $(1,1)$ skew symmetric tensor field φ, a smooth unit vector field ξ, a smooth 1-form η dual to ξ, and the induced metric g on S^{2n+1} as real hypersurface of C^{n+1} and they satisfy (cf. [13])

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta \circ \varphi = 0,\ \eta(\xi) = 1,\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

and

$$\left(\overline{\nabla}\varphi\right)(X, Y) = g(X, Y)\xi - \eta(Y)X, \quad \overline{\nabla}_X \xi = -\varphi X, \tag{3}$$

where X, Y are smooth vector fields, $\overline{\nabla}$ is Riemannian connection on S^{2n+1} and the covariant derivative

$$\left(\overline{\nabla}\varphi\right)(X, Y) = \overline{\nabla}_X \varphi Y - \varphi\left(\overline{\nabla}_X Y\right).$$

We dente by N and A the unit normal and the shape operator of the hypersurface M of the unit sphere S^{2n+1}. We denote the induced metric on the hypersurface M by the same letter g and denote by ∇ the Riemannian connection on the hypersurface M with respect to the induced metric g. Then, the fundamental equations of hypersurface are given by (cf. [14])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y),\ \ \overline{\nabla}_X N = -AX,\ \ X, Y \in \mathfrak{X}(M), \tag{4}$$

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \tag{5}$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in \mathfrak{X}(M), \tag{6}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields and $R(X, Y)Z$ is the curvature tensor field of the hypersurface M. The Ricci tensor of the minimal hypersurface M of the unit sphere S^{2n+1} is given by

$$Ric(X, Y) = (2n - 1)g(X, Y) - g(AX, AY), \quad X, Y \in \mathfrak{X}(M) \tag{7}$$

and

$$\sum_{i=1}^{2n} (\nabla A)(e_i, e_i) = 0 \tag{8}$$

holds for a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the minimal hypersurface M.

Using the Sasakian structure (φ, ξ, η, g) on the unit sphere S^{2n+1}, we analyze the induced structure on a hypersurface M of S^{2n+1}. First, we have a smooth function f on the hypersurface M defined by $f = g(\xi, N)$, which we call the *Reeb function* of the hypersurface M, where N is the unit normal vector field. As the operator φ is skew symmetric, we get a vector field $v = \varphi N$ defined on M, which we call the *contact vector field* of the hypersurface M. Note that the vector field v is orthogonal to ξ, and therefore lies in the contact distribution of the Sasakian manifold S^{2n+1}. We denote by $u = \xi^T$ the tangential component of ξ to the hypersurface M and, consequently, we have $\xi = u + fN$. Let α and β be smooth 1-forms on M dual to the vector fields u and v, respectively, that is, $\alpha(X) = g(X, u)$ and $\beta(X) = g(X, v)$, $X \in \mathfrak{X}(M)$. For $X \in \mathfrak{X}(M)$, we set $JX = (\varphi X)^T$ the tangential component of φX to the hypersurface, which gives a skew symmetric $(1,1)$ tensor field J on the hypersurface M. It follows that $\varphi X = JX - \beta(X)N$. Thus, we get a structure $(J, u, v, \alpha, \beta, f, g)$ on the hypersurface M and using properties in Equations (2) and (3) of the Sasakian structure (φ, ξ, η, g) on the unit sphere S^{2n+1} and Equation (4), it is straightforward to see that the structure $(J, u, v, \alpha, \beta, f, g)$ on the hypersurface M has the properties described in the following Lemma.

Lemma 1. *Let M be a hypersurface of the unit sphere S^{2n+1}. Then, M admits the structure $(J, u, v, \alpha, \beta, f, g)$ satisfying*

(i) $J^2 = -I + \alpha \otimes u + \beta \otimes v,$

(ii) $Ju = -fv, \quad Jv = fu,$

(iii) $g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y) - \beta(X)\beta(Y),$

(iv) $\nabla_X u = -JX + fAX, \quad \nabla_X v = -fX - JAX,$

(v) $(\nabla J)(X, Y) = g(X, Y)u - \alpha(Y)X + g(AX, Y)v - \beta(Y)AX,$

(vi) $\nabla f = -Au + v,$

(vii) $\|u\|^2 = \|v\|^2 = (1 - f^2), \quad g(u, v) = 0,$

where ∇f is the gradient of the Reeb function f.

Let Δf be the Laplacian of the Reeb function f of the minimal hypersurface M of the unit sphere S^{2n+1} defined by $\Delta f = \operatorname{div} \nabla f$. Then using Lemma 1 and $\frac{1}{2}\Delta f^2 = f\Delta f + \|\nabla f\|^2$ and Equations (6) and (8), we get the following:

Lemma 2. *Let M be a minimal hypersurface of the unit sphere S^{2n+1}. Then, the Reeb function f satisfies*

(i) $\Delta f = -\left(2n + \|A\|^2\right) f,$

(ii) $\frac{1}{2}\Delta f^2 = -\left(2n + \|A\|^2\right) f^2 + \|\nabla f\|^2.$

On the hypersurface M of the unit sphere S^{2n+1}, we define a $(1,1)$ tensor field $\Psi = JA - AJ$, then it follows that $g(\Psi X, Y) = g(X, \Psi Y)$, $X, Y \in \mathfrak{X}(M)$, that is, Ψ is symmetric and that $tr\Psi = 0$. Next, we prove the following:

Lemma 3. *Let M be a compact minimal hypersurface of the unit sphere S^{2n+1}. Then,*

$$\int_M \left(1 - f^2\right) \|A\|^2 = \int_M \left(2n - 2n(2n+1)f^2 + \frac{1}{2}\|\Psi\|^2\right).$$

Proof. Using Equation (7), we have $Ric(v, v) = (2n-1)\|v\|^2 - \|Av\|^2$. Now, using Lemma 1, we get

$$(\pounds_v g)(X, Y) = -2fg(X, Y) - g(\Psi X, Y),$$

which on using the fact that $tr\Psi = 0$, gives

$$|\pounds_v g|^2 = 8nf^2 + \|\Psi\|^2.$$

Also, using (iii) of Lemma 1, we have

$$\|JA\|^2 = \|A\|^2 - \|Au\|^2 - \|Av\|^2,$$

which together with second equation in (iv) of Lemma 1 and the fact that $trJA = 0$, implies

$$\|\nabla v\|^2 = 2nf^2 + \|A\|^2 - \|Au\|^2 - \|Av\|^2.$$

Note that second equation in (iv) of Lemma 1 also gives

$$\mathrm{div} v = -2nf.$$

Now, inserting above values in the following Yano's integral formula (cf. [15])

$$\int_M \left(Ric(v,v) + \frac{1}{2} |\pounds_v g|^2 - \|\nabla v\|^2 - (\mathrm{div} v)^2 \right) = 0,$$

we get

$$\int_M \left((2n-1) \|v\|^2 + 2nf^2 + \frac{1}{2} \|\Psi\|^2 - \|A\|^2 + \|Au\|^2 - 4n^2 f^2 \right) = 0. \tag{9}$$

Also, (vi) of Lemma 1, gives $Au = v - \nabla f$, that is, $\|Au\|^2 = \|v\|^2 + \|\nabla f\|^2 - 2v(f)$, which on using $\mathrm{div}(fv) = v(f) + f\mathrm{div} v = v(f) - 2nf^2$, gives

$$\|Au\|^2 = \|v\|^2 + \|\nabla f\|^2 - 2\mathrm{div}(fv) - 4nf^2.$$

Inserting above value of $\|Au\|^2$ in Equation (9), yields

$$\int_M \left(2n \|v\|^2 - 2nf^2 + \frac{1}{2} \|\Psi\|^2 - \|A\|^2 + \|\nabla f\|^2 - 4n^2 f^2 \right) = 0. \tag{10}$$

Integrating (ii) of Lemma 2, we get

$$\int_M \|\nabla f\|^2 = \int_M \left(2n + \|A\|^2 \right) f^2,$$

which together with $\|v\|^2 = 1 - f^2$ and Equation (10) proves the integral formula. □

Lemma 4. *Let M be a minimal hypersurface of the unit sphere S^{2n+1}. Then, the contact vector field v is a conformal vector field if and only if $JA = AJ$.*

Proof. Suppose that $AJ = JA$. Then, using Lemma 1 and symmetry of shape operator A and skew symmetry of the operator J, we have

$$(\pounds_v g)(X,Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = -2fg(X,Y), \qquad X \in \mathfrak{X}(M),$$

which proves that v is a conformal vector field with potential function $-f$. Conversely, suppose v is conformal vector field with potential function ρ. Then, using Equation (1), we have

$$(\pounds_v g)(X,Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = 2\rho g(X,Y),$$

which on using Lemma 1, gives

$$g(-JAX - fX, Y) + g(-JAY - fY, X) = 2\rho g(X,Y),$$

that is,

$$g(AJX - JAX, Y) = 2(\rho + f) g(X,Y).$$

Choosing a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the minimal hypersurface M and taking $X = Y = e_i$ in above equation and summing, we get $\rho = -f$. This gives $g(AJX - JAX, Y) = 0$, $X, Y \in \mathfrak{X}(M)$, that is, $AJ = JA$. □

Lemma 5. *Let M be a minimal hypersurface of the unit sphere S^{2n+1}. If the contact vector field v is a conformal vector field on M, then*

$$Au = \frac{\|A\|^2}{2n} v.$$

Proof. Suppose v is a conformal vector field. Then, by Lemma 4, we have $JA = AJ$. Note that for the Hessian operator A_f of the Reeb function f using Lemma 1, we have

$$A_f(X) = \nabla_X \nabla f = \nabla_X (v - Au) = -JAX - fX - \nabla_X Au, \qquad X \in \mathfrak{X}(M),$$

which on using (vi) of Lemma 1, gives

$$A_f(X) = -f(X + A^2X) - (\nabla A)(X, u).$$

Taking covariant derivative in above equation gives

$$\begin{aligned} \left(\nabla A_f\right)(X,Y) &= -X(f)((Y + A^2Y) - f\left(\nabla A^2\right)(X,Y) - \left(\nabla^2 A\right)(X,Y,u) \\ &\quad + (\nabla A)(Y, JX) - f(\nabla A)(Y, AX), \end{aligned}$$

where we used (iv) of Lemma 1. Now, on taking a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the minimal hypersurface M and taking $X = Y = e_i$ in above equation and summing, we get

$$\begin{aligned} \sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i) &= -\nabla f - A^2 \nabla f - f \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i) - \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i, e_i, u) \\ &\quad + \sum_{i=1}^{2n} (\nabla A)(e_i, Je_i) - f \sum_{i=1}^{2n} (\nabla A)(e_i, Ae_i). \end{aligned}$$

Note that for the minimal hypersurface, we have

$$\begin{aligned} \sum_{i=1}^{2n} (\nabla A)(e_i, Ae_i) &= \sum_{i=1}^{2n} \left(\nabla_{e_i} A^2 e_i - A\left((\nabla A)\right)(e_i, e_i) + A\left(\nabla_{e_i} e_i\right)\right) \\ &= \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i). \end{aligned}$$

Thus, the previous equation takes the form

$$\textstyle\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i) = -\nabla f - A^2 \nabla f - 2f \sum_{i=1}^{2n} \left(\nabla A^2\right)(e_i, e_i) - \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i, e_i, u) + \sum_{i=1}^{2n} (\nabla A)(e_i, Je_i). \quad (11)$$

Now, using the definition of Hessian operator, we have

$$R(X,Y)\nabla f = \left(\nabla A_f\right)(X,Y) - \left(\nabla A_f\right)(Y,X),$$

which gives

$$Ric(Y, \nabla f) = g\left(Y, \sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i)\right) - Y(\Delta f)$$

and we conclude

$$Q(\nabla f) = -\nabla(\Delta f) + \sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i, e_i), \qquad (12)$$

where Q is the Ricci operator defined by $Ric(X,Y) = g(QX,Y)$, $X,Y \in \mathfrak{X}(M)$. Using (i) of Lemma 2, we have

$$\nabla(\Delta f) = -2n\nabla f - \|A\|^2 \nabla f - f\nabla \|A\|^2$$

and, consequently, using $Q(X) = (2n-1)X - A^2X$ (outcome of Equation (7)), the Equation (12) takes the form

$$\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i,e_i) = (2n-1)\nabla f - A^2(\nabla f) - 2n\nabla f - \|A\|^2 \nabla f - f\nabla \|A\|^2,$$

that is,

$$\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i,e_i) = -\nabla f - A^2(\nabla f) - \|A\|^2 \nabla f - f\nabla \|A\|^2. \tag{13}$$

Also, note that

$$\begin{aligned} X\left(\|A\|^2\right) &= X\left(\sum_{i=1}^{2n} g(Ae_i, Ae_i)\right) = 2\sum_{i=1}^{2n} g\left((\nabla A)(X,e_i), Ae_i\right) \\ &= 2\sum_{i=1}^{2n} g\left(X, (\nabla A)(e_i, Ae_i)\right), \end{aligned}$$

where we have used Equation (6) and symmetry of the shape operator A. Therefore, the gradient of the function $\|A\|^2$ is

$$\nabla \|A\|^2 = 2\sum_{i=1}^{2n} (\nabla A)(e_i, Ae_i),$$

and, consequently, Equation (13), takes the form

$$\sum_{i=1}^{2n} \left(\nabla A_f\right)(e_i,e_i) = -\nabla f - A^2(\nabla f) - \|A\|^2 \nabla f - 2f\sum_{i=1}^{2n} (\nabla A)(e_i, Ae_i). \tag{14}$$

Using Equations (11) and (14), we conclude

$$-\|A\|^2 \nabla f = -\sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i,e_i,u) + \sum_{i=1}^{2n} (\nabla A)(e_i, Je_i). \tag{15}$$

Now, using Equations (6) and (8) and the Ricci identity, we have

$$\sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i,e_i,u) = \sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i,u,e_i) = \sum_{i=1}^{2n} \left(R(e_i,u)Ae_i - AR(e_i,u)e_i\right),$$

which on using Equation (5) and $trA = 0$ gives

$$\sum_{i=1}^{2n} \left(\nabla^2 A\right)(e_i,e_i,u) = -\|A\|^2 Au + 2nAu. \tag{16}$$

Also, using $JA = AJ$, we have

$$\begin{aligned} \sum_{i=1}^{2n} (\nabla A)(e_i, Je_i) &= \sum_{i=1}^{2n} \left(\nabla_{e_i} JAe_i - A\left((\nabla J)(e_i,e_i) + J\left(\nabla_{e_o} e_i\right)\right)\right) \\ &= \sum_{i=1}^{2n} \left((\nabla J)(e_i, Ae_i) - A\left((\nabla J)(e_i,e_i)\right)\right), \end{aligned}$$

which on using (v) of Lemma 1, yields

$$\sum_{i=1}^{2n} (\nabla A)(e_i, Je_i) = \|A\|^2 v - 2nAu. \tag{17}$$

Finally, using (vi) of Lemma 1 and Equations (16) and (17) in Equation (15), we get

$$-\|A\|^2(-Au + v) = \|A\|^2 Au - 2nAu + \|A\|^2 v - 2nAu$$

and this proves the Lemma. □

3. Proof of Theorem 1

As the contact vector field v is a conformal vector field by Lemma 4, we have $JA = AJ$, that is, $\Psi = 0$. Then Lemma 3 implies

$$\int_M \left(1 - f^2\right)\|A\|^2 = \int_M \left(2n - 2n(2n+1)f^2\right),$$

that is,

$$\int_M \left(1 - f^2\right)\|A\|^2 = \int_M \left(2n(1 - f^2) - 4nf^2\right). \tag{18}$$

Therefore, we get the inequality

$$\int_M \left(1 - f^2\right)\|A\|^2 \leq \int_M 2n(1 - f^2).$$

Moreover, if the equality holds, then by Equation (18), we get $f = 0$, which in view of (vi), (vii) of Lemma 1, we conclude that $Au = v$ and that the contact vector field v is a unit vector field. As v is a conformal vector field, combining $Au = v$ with Lemma 5, we get $\|A\|^2 v = 2nv$, that is, $\|A\|^2 = 2n$. Therefore, M is a Clifford hypersurface (cf. [5]).

The converse is trivial.

4. Proof of Theorem 2

As the Reeb function f is a constant along the integral curves of the contact vector field v, we have $v(f) = 0$. Note that $\mathrm{div}(fv) = v(f) + f\mathrm{div}v = -2nf^2$, which on integration gives $f = 0$, and consequently, the contact vector field v is a unit vector field. Then Lemma 3, implies

$$\int_M \|A\|^2 = \int_M \left(2n + \frac{1}{2}\|\Psi\|^2\right), \tag{19}$$

which proves the inequality

$$\int_M \|A\|^2 \geq 2nVol(M).$$

If the equality holds, then by Equation (4.1), we get that $\Psi = 0$, that is, $JA = AJ$. Thus, by Lemma 4, the contact vector field v is a conformal vector field. Using Lemma 5, we get $\|A\|^2 = 2n$. Therefore, M is a Clifford hypersurface (cf. [5]).

The converse is trivial.

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