mathematics

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Common Fixed Point and Endpoint Theorems for a Countable Family of Multi-Valued Mappings

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Received: 4 December 2019; Accepted: 24 December 2019; Published: 21 February 2020

Abstract: We prove some common fixed point and endpoint theorems for a countable infinite family of multi-valued mappings, as well as Allahyari et al. (2015) did for self-mappings. An example and an application to a system of integral equations are given to show the usability of the results.

Keywords: common fixed point; endpoint; infinite family; multi-valued mapping; (HS) property

1. Introduction

The study of common fixed point for a family of contraction mappings was initiated by Ćirić in [1]. Recently, in 2015, Allahyari et al. [2] introduced some new type of contractions for a countable family of contraction self-mappings and studied common fixed point for them.

On the other hand, existence of a fixed point for multi-valued mappings has been important for many mathematicians. In 1969, Nadler [3] extended the Banach contraction principle to multi-valued mappings. After that, many authors generalized Nadler's result in different ways (see, for instance [4–8]).

In 2012, Samet et al. [9] introduced the notion of α-admisssible mappings and a new type of contraction to a mapping $T : X \to X$ called α-ψ-contractive mapping, that is, $\alpha(x,y)d(Tx,Ty) \leqslant \psi(d(x,y))$ for all $x,y \in X$. This result generalized and improved many existing fixed point results. In the last few years, some authors have extended the notion of α-admisssibility and α-ψ-contraction to multi-valued mappings (see, [10,11]). In addition, common fixed point for a finite family or countable family of multi-valued mappings has been studied by some researchers (see, for example [12–16]).

The aim of this paper is to extend the new type of common contractivity for a family of mappings, introduced by Allahyari et al. (2015), to α-admisssible multi-valued mappings.

Let (X,d) be a metric space, 2^X the set of all nonempty subsets of X, and $\mathcal{CL}(X)$ the set of all nonempty closed subsets of X. Assume that $\mathcal{H}$ is the generalized Hausdorff metric on $\mathcal{CL}(X)$ defined by

$$\mathcal{H}(A,B) = \begin{cases} \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\}, & \text{if it exists,} \\ \infty, & \text{otherwise,} \end{cases} \quad (1)$$

for all $A,B \in \mathcal{CL}(X)$, where $D(x,B) = \inf_{y\in B} d(x,y)$. Let $T : X \to 2^X$ is a multi-valued mapping. An element $x \in X$ is said to be a fixed point of T if $x \in Tx$, and x is called an endpoint of T whenever $Tx = \{x\}$.

2. Main Results

Now, we are ready to state and prove the main results of this study.

Definition 1. *Let X be an arbitrary space and $\alpha : X \times X \to [0, \infty)$ be a function. Assume that $T_n : X \to 2^X$ ($n =$ 1,2,...) is a family of multi-valued mappings. We say that $\{T_n\}$ is α-admissible whenever for each $x \in X$ and $y \in T_n x$ with $\alpha(x, y) \geqslant 1$, we have $\alpha(y, z) \geqslant 1$ for all $z \in T_{n+1}y$.*

Theorem 1. *Let (X, d) be a complete metric space and $0 < a_{i,j}$ $(i, j = 1, 2, ...)$ with $a_{i,i+1} \neq 1$ for all $i = 1, 2, ...$ satisfy:*

(i) for each j, $\overline{\lim}_{i\to\infty} a_{i,j} < 1$;
(ii) $\sum_{n=1}^{\infty} A_n < \infty$, where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$.

Let $\alpha : X \times X \to [0, \infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ ($n =$ 1,2,...) such that

$$\alpha(x, y)\mathcal{H}(T_i x, T_j y) \leqslant a_{i,j}[D(x, T_j y) + D(y, T_i x)], \tag{2}$$

for all $x, y \in X$; $i, j = 1, 2, ...$ with $x \neq y$ and $i \neq j$. Moreover, assume that the following assertions hold:

(iii) there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $x_0 \neq x_1$ and $\alpha(x_0, x_1) \geqslant 1$;
(iv) $\{T_n\}$ is α-admissible;
(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.

Then each T_n have a common fixed point in X.

Proof. Using *(iii)* and (2), we have

$$\begin{aligned} D(x_1, T_2 x_1) &\leqslant \alpha(x_0, x_1)\mathcal{H}(T_1 x_0, T_2 x_1) \\ &\leqslant a_{1,2}[D(x_0, T_2 x_1) + D(x_1, T_1 x_0)] \\ &= a_{1,2} D(x_0, T_2 x_1) \\ &\leqslant a_{1,2}[d(x_0, x_1) + D(x_1, T_2 x_1)], \end{aligned}$$

which implies

$$D(x_1, T_2 x_1) \leqslant \frac{a_{1,2}}{1 - a_{1,2}} d(x_0, x_1) < \frac{a_{1,2}}{1 - a_{1,2}} p d(x_0, x_1),$$

where $p > 1$ is a fixed number. From the above inequality, there exists $x_2 \in T_2 x_1$ such that $d(x_1, x_2) < \frac{a_{1,2}}{1-a_{1,2}} p d(x_0, x_1)$. Since $\{T_n\}$ is α-admissible, we have $\alpha(x_1, x_2) \geqslant 1$. Similarly,

$$D(x_2, T_3 x_2) \leqslant \frac{a_{2,3}}{1 - a_{2,3}} d(x_1, x_2) < \frac{a_{2,3}}{1 - a_{2,3}} \frac{a_{1,2}}{1 - a_{1,2}} p d(x_0, x_1),$$

and so there exists $x_3 \in T_3 x_2$ such that $d(x_2, x_3) < \frac{a_{2,3}}{1-a_{2,3}} \frac{a_{1,2}}{1-a_{1,2}} p d(x_0, x_1)$. Continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in T_{n+1} x_n$, $\alpha(x_n, x_{n+1}) \geqslant 1$, and

$$d(x_n, x_{n+1}) < A_n p d(x_0, x_1), \quad \text{for all } n = 1, 2, ... \tag{3}$$

For any $n, m \in \mathbb{N}$ with $n < m$, from triangle inequality, we get

$$d(x_n, x_m) \leqslant \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leqslant \sum_{k=n}^{m-1} A_k p d(x_0, x_1) \to 0$$

as $n, m \to \infty$. Therefore, we have shown that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that $x_n \to x$. From *(v)*, we get $\alpha(x_n, x) \geqslant 1$ for all n. Now, we shall show that x is a common fixed point of T_n. Let m be an arbitrary positive integer. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, T_m x) &\leqslant d(x, x_n) + D(x_n, T_m x) \\ &\leqslant d(x, x_n) + \alpha(x_{n-1}, x)\mathcal{H}(T_n x_{n-1}, T_m x) \\ &\leqslant d(x, x_n) + a_{n,m}[D(x_{n-1}, T_m x) + D(x, T_n x_{n-1})] \\ &\leqslant d(x, x_n) + a_{n,m}[D(x_{n-1}, T_m x) + d(x, x_n)]. \end{aligned}$$

Taking $\overline{lim}$ in both sides of the above inequality, as $n \to \infty$, we get

$$D(x, T_m x) \leqslant (\overline{lim}_{n\to\infty} a_{n,m}) D(x, T_m x),$$

which implies $D(x, T_m x) = 0$ and so $x \in T_m x$. $\square$

Theorem 2. *Let (X, d) be a complete metric space and $0 < a_{i,j}$ $(i, j = 1, 2, ...)$ with $a_{i,i+1} \neq 1$ for all $i = 1, 2, ...$ satisfy:*

(i) for each (j), $\overline{lim}_{i\to\infty} a_{i,j} < 1$;
(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$.

Let $\alpha : X \times X \to [0, \infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ (n = 1,2,...) such that

$$\alpha(x, y)\mathcal{H}(T_i x, T_j y) \leqslant a_{i,j} \max\{d(x, y), D(x, T_i x), D(y, T_j y), D(x, T_j y), D(y, T_i x)\}, \tag{4}$$

for all $x, y \in X$; $i, j = 1, 2, ...$ with $x \neq y$ and $i \neq j$. Moreover, assume that the following assertions hold:

(iii) there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $x_0 \neq x_1$ and $\alpha(x_0, x_1) \geqslant 1$;
(iv) $\{T_n\}$ is α-admissible;
(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.

Then each T_n have a common fixed point in X.

Proof. By *(iii)* and (4), we have

$$\begin{aligned} D(x_1, T_2 x_1) &\leqslant \alpha(x_0, x_1)\mathcal{H}(T_1 x_0, T_2 x_1) \\ &\leqslant a_{1,2} \max\{d(x_0, x_1), D(x_0, T_1 x_0), D(x_1, T_2 x_1), D(x_0, T_2 x_1), D(x_1, T_1 x_0)\} \\ &\leqslant a_{1,2}[d(x_0, x_1) + D(x_1, T_2 x_1)], \end{aligned}$$

which implies

$$D(x_1, T_2 x_1) \leqslant \frac{a_{1,2}}{1 - a_{1,2}} d(x_0, x_1) < \frac{a_{1,2}}{1 - a_{1,2}} p d(x_0, x_1),$$

which $p > 1$ is a fixed number. From the above inequality, there exists $x_2 \in T_2 x_1$ such that $d(x_1, x_2) < \frac{a_{1,2}}{1-a_{1,2}} p d(x_0, x_1)$. Continuing in this manner and as in proof of Theorem 1, we obtain a sequence $\{x_n\}$ with

$\alpha(x_n, x_{n+1}) \geqslant 1$ and $x \in X$ such that $x_n \to x$. Using *(v)*, we get $\alpha(x_n, x) \geqslant 1$ for all n. Next, we show that x is a common fixed point of T_n. Let m be an arbitrary positive integer. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, T_m x) &\leqslant d(x, x_n) + D(x_n, T_m x) \\ &\leqslant d(x, x_n) + \alpha(x_{n-1}, x)\mathcal{H}(T_n x_{n-1}, T_m x) \\ &\leqslant d(x, x_n) + a_{n,m} \max\{d(x_{n-1}, x), D(x_{n-1}, T_n x_{n-1}), D(x, T_m x), \\ &\quad D(x_{n-1}, T_m x), D(x, T_n x_{n-1})\} \\ &\leqslant d(x, x_n) + a_{n,m} \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), D(x, T_m x), D(x_{n-1}, T_m x), d(x, x_n)\}. \end{aligned}$$

Taking $\overline{lim}$ as $n \to \infty$, we obtain $D(x, T_m x) \leqslant (\overline{lim}_{n\to\infty} a_{n,m}) D(x, T_m x)$, which implies $D(x, T_m x) = 0$. This means that $x \in T_m x$ and the proof is complete. □

Theorem 3. *Let (X, d) be a complete metric space and $0 \leqslant a_{i,j}, 0 < b_{i,j}$ $(i, j = 1, 2, ...)$ with $a_{i,i+1} \neq 1$ for all $i = 1, 2, ...$ satisfy:*

- *(i) for each j, $\overline{lim}_{i\to\infty} a_{i,j} < 1$ and $\overline{lim}_{i\to\infty} b_{i,j} < \infty$;*
- *(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{b_{i,i+1}}{1-a_{i,i+1}}$.*

Let $\alpha : X \times X \to [0, \infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ (n = 1,2,...) such that

$$\alpha(x, y)\mathcal{H}(T_i x, T_j y) \leqslant a_{i,j} D(y, T_j y)\varphi(D(x, T_i x), d(x, y)) + b_{i,j} d(x, y), \tag{5}$$

for all $x, y \in X$; $i, j = 1, 2, ...$ with $x \neq y$ and $i \neq j$, where $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0, \infty)$ and for any $t_1, s_1, t_2, s_2 \in [0, \infty)$,

$$t_1 \leqslant t_2, s_1 = s_2 \Longrightarrow \varphi(t_1, s_1) \leqslant \varphi(t_2, s_2).$$

Moreover, assume that the following assertions hold:

- *(iii) there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $x_0 \neq x_1$ and $\alpha(x_0, x_1) \geqslant 1$;*
- *(iv) $\{T_n\}$ is α-admissible;*
- *(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.*

Then each T_n have a common fixed point in X.

Proof. By *(iii)* and (5), we have

$$\begin{aligned} D(x_1, T_2 x_1) &\leqslant \alpha(x_0, x_1)\mathcal{H}(T_1 x_0, T_2 x_1) \\ &\leqslant a_{1,2} D(x_1, T_2 x_1)\varphi(D(x_0, T_1 x_0), d(x_0, x_1)) + b_{1,2} d(x_0, x_1) \\ &\leqslant a_{1,2} D(x_1, T_2 x_1)\varphi(d(x_0, x_1), d(x_0, x_1)) + b_{1,2} d(x_0, x_1) \\ &\leqslant a_{1,2} D(x_1, T_2 x_1) + b_{1,2} d(x_0, x_1), \end{aligned}$$

which gives us

$$D(x_1, T_2 x_1) \leqslant \frac{b_{1,2}}{1 - a_{1,2}} d(x_0, x_1) < \frac{b_{1,2}}{1 - a_{1,2}} p d(x_0, x_1),$$

where $p > 1$ is a fixed number. From the above inequality, there exists $x_2 \in T_2 x_1$ such that $d(x_1, x_2) < \frac{b_{1,2}}{1-a_{1,2}} p d(x_0, x_1)$. Similarly,

$$D(x_2, T_3 x_2) \leqslant \frac{b_{2,3}}{1 - a_{2,3}} d(x_1, x_2) < \frac{b_{2,3}}{1 - a_{2,3}} \frac{b_{1,2}}{1 - a_{1,2}} p d(x_0, x_1),$$

and so there exists $x_3 \in T_3x_2$ such that $d(x_2, x_3) < \frac{b_{2,3}}{1-a_{2,3}} \frac{b_{1,2}}{1-a_{1,2}} pd(x_0, x_1)$. Continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in T_{n+1}x_n$, $\alpha(x_n, x_{n+1}) \geqslant 1$, and

$$d(x_n, x_{n+1}) < A_n pd(x_0, x_1), \quad \text{for all } n = 1, 2, \ldots . \tag{6}$$

Again, as in the proof of Theorem 1, we conclude that $\{x_n\}$ is a Cauchy sequence, and so there exists $x \in X$ such that $x_n \to x$. From the assumption *(v)*, we get $\alpha(x_n, x) \geqslant 1$ for all n. To show that x is a common fixed point of T_n, let m be an arbitrary positive integer. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, T_mx) &\leqslant d(x, x_n) + D(x_n, T_mx) \leqslant d(x, x_n) + \alpha(x_{n-1}, x)\mathcal{H}(T_nx_{n-1}, T_mx) \\ &\leqslant d(x, x_n) + a_{n,m}D(x, T_mx)\varphi(D(x_{n-1}, T_nx_{n-1}), d(x_{n-1}, x)) + b_{n,m}d(x_{n-1}, x) \\ &\leqslant d(x, x_n) + a_{n,m}D(x, T_mx)\varphi(d(x_{n-1}, x_n), d(x_{n-1}, x)) + b_{n,m}d(x_{n-1}, x). \end{aligned}$$

Taking $\overline{lim}$ in both sides of the above inequality, as $n \to \infty$, we obtain

$$D(x, T_mx) \leqslant (\overline{lim}_{n\to\infty} a_{n,m})D(x, T_mx).$$

We conclude $D(x, T_mx) = 0$ and thus $x \in T_mx$. □

3. Common Endpoint Theorems

The notion of endpoints of multi-valued mappings has been studied by some researchers in the last decade (see for instance, [17–19]). In current section, we state and prove some common endpoint theorems for a sequence of multi-valued mappings with the contractions mentioned in Section 2. We need the following definition.

Definition 2. *Let $T_n : X \to \mathcal{CL}(X)$ (n = 1,2,...) be a sequence of multi-valued mappings. We say that $\{T_n\}$ has (HS) property whenever for each $x \in X$ there exists $y \in T_nx$ such that $\mathcal{H}(T_nx, T_{n+1}y) \geqslant \sup_{b\in T_{n+1}y} d(y, b)$.*

Theorem 4. *Let (X, d) be a complete metric space and $0 \leqslant a_{i,j}$ $(i, j = 1, 2, \ldots)$ with $a_{i,i+1} \neq 1$ for all $i = 1, 2, \ldots$ satisfy:*

- *(i) for each (j), $\overline{lim}_{i\to\infty} a_{i,j} < 1$;*
- *(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$.*

Let $\alpha : X \times X \to [0, \infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ (n = 1,2,...) satisfying (HS) property such that

$$\alpha(x, y)\mathcal{H}(T_ix, T_jy) \leqslant a_{i,j}[D(x, T_jy) + D(y, T_ix)], \tag{7}$$

for all $x, y \in X$; $i, j = 1, 2, \ldots$ with $x \neq y$ and $i \neq j$. Moreover, assume that the following assertions hold:

- *(iii) there exists $x_0 \in X$ such that for any $x \in T_1x_0$, we have $\alpha(x_0, x) \geqslant 1$;*
- *(iv) $\{T_n\}$ is α-admissible;*
- *(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.*

Then each T_n have a common endpoint in X.

Proof. Since $\{T_n\}$ has (HS) property, there exists $x_1 \in T_1x_0$ such that $\mathcal{H}(T_1x_0, T_2x_1) \geqslant \sup_{b\in T_2x_1} d(x_1, b)$. From *(iii)*, we have $\alpha(x_0, x_1) \geqslant 1$. Similarly, there exists $x_2 \in T_2x_1$ such that $\mathcal{H}(T_2x_1, T_3x_2) \geqslant \sup_{b\in T_3x_2} d(x_2, b)$. Since $\{T_n\}$ is α-admissible, so $\alpha(x_1, x_2) \geqslant 1$. If we continue this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in T_n x_{n-1}$, $\alpha(x_{n-1}, x_n) \geqslant 1$, and

$$\mathcal{H}(T_n x_{n-1}, T_{n+1}x_n) \geqslant \sup_{b\in T_{n+1}x_n} d(x_n, b), \tag{8}$$

for all $n \geqslant 1$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \sup_{b\in T_{n+1}x_n} d(x_n, b) \leqslant \alpha(x_{n-1}, x_n)\mathcal{H}(T_n x_{n-1}, T_{n+1}x_n) \\ &\leqslant a_{n,n+1}[D(x_{n-1}, T_{n+1}x_n) + D(x_n, T_n x_{n-1})] \\ &\leqslant a_{n,n+1}[d(x_{n-1}, x_{n+1})] \leqslant a_{n,n+1}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

From the above inequality, we get

$$d(x_n, x_{n+1}) \leqslant \frac{a_{n,n+1}}{1-a_{n,n+1}} d(x_{n-1}, x_n) \leqslant \ldots \leqslant A_n d(x_0, x_1).$$

Hence $\{x_n\}$ is a Cauchy sequence, and so there exists $x \in X$ such that $x_n \to x$. From *(v)* we deduce $\alpha(x_n, x) \geqslant 1$ for all n. Now we show that x is a common endpoint of T_n. Let $m \in \mathbb{N}$ be arbitrary. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{H}(\{x\}, T_m x) &\leqslant d(x, x_n) + \mathcal{H}(\{x_n\}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_m x) \\ &\leqslant d(x, x_n) + \alpha(x_{n-1}, x_n)\mathcal{H}(T_n x_{n-1}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_m x) \\ &\leqslant d(x, x_n) + a_{n,n+1}[D(x_{n-1}, T_{n+1}x_n) + D(x_n, T_n x_{n-1})] \\ &\quad + a_{n+1,m}[D(x_n, T_m x) + D(x, T_{n+1}x_n)] \\ &\leqslant d(x, x_n) + a_{n,n+1}[d(x_{n-1}, x_{n+1})] + a_{n+1,m}[D(x_n, T_m x) + d(x, x_{n+1})]. \end{aligned}$$

Taking $\overline{lim}$ as $n \to \infty$, we obtain

$$\mathcal{H}(\{x\}, T_m x) \leqslant (\overline{lim}_{n\to\infty} a_{n+1,m}) D(x, T_m x) \leqslant (\overline{lim}_{n\to\infty} a_{n+1,m}) \mathcal{H}(\{x\}, T_m x),$$

which implies $\mathcal{H}(\{x\}, T_m x) = 0$ and so $T_m x = \{x\}$. Since m was arbitrary, the proof is complete. □

Theorem 5. *In the statement of Theorem 4, if we add the extra condition $\alpha(x, y) \geqslant 1$ for any common endpoints x, y of T_n, then the common endpoint of T_n is unique.*

Proof. Let x, y be two common endpoints of T_n. Since $\sum_{n=1}^{\infty} A_n < \infty$, there exists $i_0 \in \mathbb{N}$ such that $\frac{a_{i_0,i_0+1}}{1-a_{i_0,i_0+1}} < 1$, which implies $a_{i_0,i_0+1} < \frac{1}{2}$. Then, using (7), we get

$$\begin{aligned} d(x, y) &= \mathcal{H}(T_i x, T_{i_0+1}y) \\ &\leqslant \alpha(x, y)\mathcal{H}(T_{i_0}x, T_{i_0+1}y) \\ &\leqslant a_{i_0,i_0+1}[D(x, T_{i_0+1}y) + D(y, T_{i_0}x)] \\ &= 2a_{i_0,i_0+1} d(x, y), \end{aligned}$$

which implies $d(x, y) = 0$ and so $x = y$. □

Example 1. *Consider the space $X = [0,1]$ with the usual metric $d(x,y) = |x - y|$. Define a sequence of mappings $T_n : X \to \mathcal{CL}(X)$ by*

$$T_n(x) = \begin{cases} \{1\}, & \frac{1}{2} \leqslant x \leqslant 1, \\ \left[\frac{2}{3} + \frac{1}{n+2}, 1\right], & x = 0, \\ \{0\}, & 0 < x < \frac{1}{2}. \end{cases}$$

Also consider the constants $a_{i,j} = \frac{1}{3} + \frac{1}{|i-j|+6}$. Then $\overline{lim}_{i\to\infty} a_{i,j} = \frac{1}{3} < 1$, for all $j \in \mathbb{N}$. $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}} = (\frac{10}{11})^n$. Thus $\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} (\frac{10}{11})^n < \infty$. Also let

$$\alpha(x,y) = \begin{cases} 1, & x, y \in \{0\} \cup \left[\frac{1}{2}, 1\right], \\ 0, & otherwise. \end{cases}$$

Now we show that $\alpha(x,y)\mathcal{H}(T_i x, T_j y) \leqslant a_{i,j}[D(x, T_j y) + D(y, T_i x)]$, for all $x, y \in X$. If $0 < x < \frac{1}{2}$ or $0 < y < \frac{1}{2}$, then $\alpha(x,y) = 0$ and we have nothing to prove. Therefore, we may assume $x, y \in \{0\} \cup \left[\frac{1}{2}, 1\right]$. We consider the following cases:

(1) $x, y \in \left[\frac{1}{2}, 1\right]$. In this case we have $\alpha(x,y)\mathcal{H}(T_i x, T_j y) = \mathcal{H}(\{1\}, \{1\}) = 0 \leqslant a_{i,j}[D(x, T_j y) + D(y, T_i x)]$, for all $x, y \in X$.

(2) $x \in \left[\frac{1}{2}, 1\right]$ and $y = 0$. In this case we have

$$\begin{aligned} \alpha(x,y)\mathcal{H}(T_i x, T_j y) &= \mathcal{H}(\{1\}, \left[\frac{2}{3} + \frac{1}{j+2}, 1\right]) \\ &= |1 - (\frac{2}{3} + \frac{1}{j+2})| = \frac{1}{3} - \frac{1}{j+2} \leqslant \frac{1}{3} \\ &\leqslant (\frac{1}{3} + \frac{1}{|i-j|+6})(|x - (\frac{2}{3} + \frac{1}{j+2})| + |0 - 1|) \\ &= a_{i,j}[D(x, T_j y) + D(y, T_i x). \end{aligned}$$

(3) $x = y = 0, i < j$. Then

$$\begin{aligned} \alpha(x,y)\mathcal{H}(T_i x, T_j y) &= |\frac{2}{3} + \frac{1}{j+2} - (\frac{2}{3} + \frac{1}{i+2})| = \frac{1}{i+2} - \frac{1}{j+2} \leqslant \frac{1}{i+2} \\ &\leqslant (\frac{1}{3} + \frac{1}{|i-j|+6})(\frac{2}{3} + \frac{1}{i+2} + (\frac{2}{3} + \frac{1}{j+2})) \\ &= a_{i,j}[D(x, T_j y) + D(y, T_i x)]. \end{aligned}$$

Also for $x_0 = 0$ and $x_1 = 1$, we have $x_1 \in \{1\} = \left[\frac{2}{3} + \frac{1}{1+2}, 1\right] = T_1 x_0$ and $\alpha(x,y) = 1 \geqslant 1$. It is easy to check that $\{T_n\}$ is α-admissible. Also, for any common endpoints x, y, we have $\alpha(x,y) \geqslant 1$. Thus, all of the conditions of Theorem 4 and Theorem 5 are satisfied. Therefore, the mappings T_n have a unique common endpoint. Here $x = 1$ is the unique common endpoint of T_n.

Theorem 6. *Let (X,d) be a complete metric space and $0 \leqslant a_{i,j}$ $(i,j = 1,2,...)$ with $a_{i,i+1} \neq 1$ for all $i = 1,2,...$ satisfy:*

(i) for each (j), $\overline{lim}_{i\to\infty} a_{i,j} < 1$;
(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$.

Let $\alpha : X \times X \to [0,\infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ ($n =$ 1,2,...) satisfying (HS) property such that

$$\alpha(x,y)\mathcal{H}(T_ix, T_jy) \leqslant a_{i,j}\max\{d(x,y), D(x, T_ix), D(y, T_jy), D(x, T_jy), D(y, T_ix)\}, \tag{9}$$

for all $x, y \in X$; $i, j = 1, 2, \ldots$ with $x \neq y$ and $i \neq j$. Moreover, assume that the following assertions hold:

(iii) there exists $x_0 \in X$ such that for any $x \in T_1x_0$, we have $\alpha(x_0, x) \geqslant 1$;
(iv) $\{T_n\}$ is α-admissible;
(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.

Then each T_n have a common endpoint in X.

Proof. As in the proof of Theorem 4, there exists a sequence $\{x_n\}$ in X such that $x_n \in T_nx_{n-1}$, $\alpha(x_{n-1}, x_n) \geqslant$ 1, and

$$\mathcal{H}(T_nx_{n-1}, T_{n+1}x_n) \geqslant \sup_{b\in T_{n+1}x_n} d(x_n, b),$$

for all $n \geqslant 1$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \sup_{b\in T_{n+1}x_n} d(x_n, b) \leqslant \alpha(x_{n-1}, x_n)\mathcal{H}(T_nx_{n-1}, T_{n+1}x_n) \\ &\leqslant a_{n,n+1}\max\{d(x_{n-1}, x_n), D(x_{n-1}, T_nx_{n-1}), D(x_n, T_{n+1}x_n), \\ &\quad D(x_{n-1}, T_{n+1}x_n), D(x_n, T_nx_{n-1})\} \\ &\leqslant a_{n,n+1}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

From the above inequality, we get

$$d(x_n, x_{n+1}) \leqslant \frac{a_{n,n+1}}{1 - a_{n,n+1}} d(x_{n-1}, x_n) \leqslant \ldots \leqslant A_n d(x_0, x_1).$$

Thus, $\{x_n\}$ is a Cauchy sequence and so there exists $x \in X$ such that $x_n \to x$ and $\alpha(x_n, x) \geqslant 1$ for all n. Now, we show that x is a common endpoint of T_n. Let $m \in \mathbb{N}$ be arbitrary. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{H}(\{x\}, T_mx) &\leqslant d(x, x_n) + \mathcal{H}(\{x_n\}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_mx) \\ &\leqslant d(x, x_n) + \alpha(x_{n-1}, x_n)\mathcal{H}(T_nx_{n-1}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_mx) \\ &\leqslant d(x, x_n) + a_{n,n+1}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad + a_{n+1,m}\max\{d(x_n, x), D(x_n, T_{n+1}x_n), D(x, T_mx), D(x_n, T_mx), D(x, T_{n+1}x_n)\} \\ &\leqslant d(x, x_n) + a_{n,n+1}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad + a_{n+1,m}\max\{d(x_n, x), D(x_n, x_{n+1}), D(x, T_mx), D(x_n, T_mx), D(x, x_{n+1})\}. \end{aligned}$$

Taking $\overline{lim}$ in both sides of the above inequality, as $n \to \infty$, we obtain

$$\mathcal{H}(\{x\}, T_mx) \leqslant (\overline{lim}_{n\to\infty} a_{n+1,m})D(x, T_mx) \leqslant (\overline{lim}_{n\to\infty} a_{n+1,m})\mathcal{H}(\{x\}, T_mx),$$

which implies $\mathcal{H}(\{x\}, T_mx) = 0$ and so $T_mx = \{x\}$. □

Theorem 7. *With the conditions of Theorem 6, if we add the extra condition $\alpha(x, y) \geqslant 1$ for any common endpoints x, y of T_n, then the common endpoint of T_n is unique.*

Proof. Let x, y be two common endpoints of T_n. Using (9), we get

$$\begin{aligned} d(x,y) &= \mathcal{H}(T_i x, T_j y) \leqslant \alpha(x,y)\mathcal{H}(T_i x, T_j y) \\ &\leqslant a_{i,j} \max\{d(x,y), D(x, T_i x), D(y, T_j y), D(x, T_j y), D(y, T_i x)\} \\ &= a_{i,j} d(x,y). \end{aligned}$$

Thus, $d(x,y) \leqslant \overline{lim}_{i\to\infty} a_{i,j} d(x,y)$, which means that $d(x,y) = 0$ and hence $x = y$. □

Theorem 8. *Let (X, d) be a complete metric space and $0 \leqslant a_{i,j}, 0 \leqslant b_{i,j}$ $(i, j = 1, 2, ...)$ with $a_{i,i+1} \neq 1$ for all $i = 1, 2, ...$ satisfy:*

(i) for each (j), $\overline{lim}_{i\to\infty} a_{i,j} < 1$, $\overline{lim}_{i\to\infty} b_{i,j} < 1$;
(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{b_{i,i+1}}{1-a_{i,i+1}}$.

Let $\alpha : X \times X \to [0, \infty)$ be a given function and $\{T_n\}$ be a sequence of multi-valued operators $T_n : X \to \mathcal{CL}(X)$ (n = 1,2,...) satisfying (HS) property such that

$$\alpha(x,y)\mathcal{H}(T_i x, T_j y) \leqslant a_{i,j} D(y, T_j y)\varphi(D(x, T_i x), d(x,y)) + b_{i,j} d(x,y), \tag{10}$$

for all $x, y \in X$; $i, j = 1, 2, ...$ with $x \neq y$ and $i \neq j$, where φ is as in Theorem 3. Moreover, assume that the following assertions hold:

(iii) there exists $x_0 \in X$ such that for any $x \in T_1 x_0$, we have $\alpha(x_0, x) \geqslant 1$;
(iv) $\{T_n\}$ is α-admissible;
(v) for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, we have $\alpha(x_n, x) \geqslant 1$ for all n.

Then each T_n have a common endpoint in X.

Proof. As in the proof of Theorem 4, there exists a sequence $\{x_n\}$ in X such that $x_n \in T_n x_{n-1}$, $\alpha(x_{n-1}, x_n) \geqslant 1$, and

$$\mathcal{H}(T_n x_{n-1}, T_{n+1} x_n) \geqslant \sup_{b \in T_{n+1} x_n} d(x_n, b),$$

for all $n \geqslant 1$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \sup\nolimits_{b \in T_{n+1} x_n} d(x_n, b) \\ &\leqslant \alpha(x_{n-1}, x_n)\mathcal{H}(T_n x_{n-1}, T_{n+1} x_n) \\ &\leqslant a_{n,n+1} D(x_n, T_{n+1} x_n)\varphi(D(x_{n-1}, T_n x_{n-1}), d(x_{n-1}, x_n)) + b_{n,n+1} d(x_{n-1}, x_n) \\ &\leqslant a_{n,n+1} d(x_n, x_{n+1}) + b_{n,n+1} d(x_{n-1}, x_n). \end{aligned}$$

From the above inequality, we get

$$d(x_n, x_{n+1}) \leqslant \frac{b_{n,n+1}}{1 - a_{n,n+1}} d(x_{n-1}, x_n) \leqslant ... \leqslant A_n d(x_0, x_1).$$

As in proof of Theorem 1, we conclude that $\{x_n\}$ is a Cauchy sequence, and so there exists $x \in X$ such that $x_n \to x$ and $\alpha(x_n, x) \geqslant 1$ for all n. To show that x is a common endpoint of T_n, consider an arbitrary natural number m. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{H}(\{x\}, T_m x) &\leqslant d(x, x_n) + \mathcal{H}(\{x_n\}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_m x) \\
&\leqslant d(x, x_n) + \alpha(x_{n-1}, x_n)\mathcal{H}(T_n x_{n-1}, T_{n+1}x_n) + \alpha(x_n, x)\mathcal{H}(T_{n+1}x_n, T_m x) \\
&\leqslant d(x, x_n) + a_{n,n+1}D(x_n, T_{n+1}x_n)\varphi(D(x_{n-1}, T_n x_{n-1}), d(x_{n-1}, x_n)) \\
&\quad + b_{n,n+1}d(x_{n-1}, x_n) \\
&\quad + a_{n+1,m}D(x, T_m x)\varphi(D(x_n, T_{n+1}x_n), d(x_n, x)) + b_{n+1,m}d(x_n, x) \\
&\leqslant d(x, x_n) + a_{n,n+1}d(x_n, x_{n+1}) + b_{n,n+1}d(x_{n-1}, x_n) \\
&\quad + a_{n+1,m}D(x, T_m x)\varphi(d(x_n, x_{n+1}), d(x_n, x)) + b_{n+1,m}d(x_n, x).
\end{aligned}$$

Taking $\overline{lim}$ as $n \to \infty$, we obtain

$$\begin{aligned}
\mathcal{H}(\{x\}, T_m x) &\leqslant (\overline{lim}_{n\to\infty} a_{n+1,m})D(x, T_m x) \\
&\leqslant (\overline{lim}_{n\to\infty} a_{n+1,m})\mathcal{H}(\{x\}, T_m x),
\end{aligned}$$

which shows $\mathcal{H}(\{x\}, T_m x) = 0$. Thus $T_m x = \{x\}$. □

Theorem 9. *In the statement of Theorem 8, if we add the extra condition $\alpha(x, y) \geqslant 1$ for any common endpoints x, y of T_n, then the common endpoint of T_n is unique.*

Proof. Let x, y be two common endpoints of T_n. Using (10), we have

$$\begin{aligned}
d(x, y) &= \mathcal{H}(T_i x, T_j y) \\
&\leqslant \alpha(x, y)\mathcal{H}(T_i x, T_j y) \\
&\leqslant a_{i,j}D(y, T_j y)\varphi(D(x, T_i x), d(x, y)) + b_{i,j}d(x, y) \\
&= b_{i,j}d(x, y).
\end{aligned}$$

Therefore, $d(x, y) \leqslant \overline{lim}_{i\to\infty} b_{i,j}d(x, y)$. Hence $d(x, y) = 0$, which means that $x = y$. □

4. Application to Integral Equations

Take $I = [0, T]$. Let $X := C(I, \mathbb{R})$ be the set of all real valued continuous functions with domain I. Define the meric d on X with

$$d(x, y) = \sup_{t\in I}(|x(t) - y(t)|) = ||x - y||.$$

Consider the system of integral equation:

$$x(t) = p(t) + \int_0^T G(t, s)F_n(s, x(s))ds, \quad t \in I, \ n = 1, 2, 3, \ldots. \tag{11}$$

Our hypotheses on the data are the following:

(A) $p : I \to \mathbb{R}$ and $F_n : I \times \mathbb{R} \to \mathbb{R}$ are continuous, for all $n \in \mathbb{N}$;
(B) $G : I \times I \to \mathbb{R}$ is continuous and measurable at $s \in I$ for all $t \in I$;
(C) $G(t, s) \geqslant 0$ for all $t, s \in I$ and $\int_0^T G(t, s)ds \leqslant 1$ for all $t \in I$;
(D) there exists $x_0 \in X$ such that $x_0(t) \leqslant \int_0^T G(t, s)F_1(s, x_0(s))ds$, for all $t \in I$;
(E) for any $x \in X$ with $x(t) \leqslant \int_0^T G(t, s)F_n(s, x(s))ds$, for all $t \in I$, then we have $\int_0^T G(t, s)F_n(s, x(s))ds \leqslant \int_0^T G(t, s)F_{n+1}(s, \int_0^T G(s, \tau)F_n(\tau, x(\tau))d\tau)ds$, for all $t \in I$.

Let $0 \leqslant a_{i,j}$ $(i,j = 1,2,...)$ with $a_{i,i+1} \neq 1$ for all $i = 1,2,...$ satisfy:

(F) for each (j), $\overline{lim}_{i\to\infty} a_{i,j} < 1$;
(G) $\Sigma_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$;

(H) for each $t \in I$, $x, y \in X$ with $x \leqslant y$, and $i \neq j$, we have

$$\begin{aligned} |F_i(t,x(t)) - F_j(t,y(t))| \quad &\leqslant a_{i,j}(|x(t) - \textstyle\int_0^T G(t,s)F_j(s,y(s))ds| \\ &+|y(t) - \textstyle\int_0^T G(t,s)F_i(s,x(s))ds|). \end{aligned} \tag{12}$$

Theorem 10. *Under the assumptions (A)–(H), the system of integral Equation (11) has a solution in X.*

Proof. Define $\Upsilon_n : X \to X$ as

$$(\Upsilon_n x)(t) = p(t) + \int_0^T G(t,s)F_n(s,x(s))ds, \quad t \in I$$

for all $n \in \mathbb{N}$. In addition, define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x(t) \leqslant y(t) \ for\ all\ t \in I, \\ 0, & otherwise. \end{cases}$$

Let x, y be two arbitrary elements of X. If $x \nleqslant y$, then $\alpha(x,y) = 0$ and so inequality (2) holds, obviously. Now, let $x \leqslant y$. Then

$$\begin{aligned} |(\Upsilon_i x)(t) - (\Upsilon_j y)(t)| &= |\int_0^T G(t,s)(F_i(s,x(s)) - F_j(s,y(s))ds| \\ &\leqslant \int_0^T G(t,s)|F_i(s,x(s)) - F_j(s,y(s))|ds \\ &\leqslant \int_0^T G(t,s)a_{i,j}(|x(s) - \int_0^T G(s,\tau)F_j(\tau,y(\tau))d\tau| \\ &\quad + |y(s) - \int_0^T G(s,\tau)F_i(\tau,x(\tau))d\tau|)ds \\ &\leqslant \int_0^T G(t,s)a_{i,j}(|x(s) - (\Upsilon_j y)(s)| + |y(s) - \Upsilon_i x)(s)|)ds \\ &\leqslant \int_0^T G(t,s)a_{i,j}(||x - (\Upsilon_j y)|| + ||y - \Upsilon_i x)(s)||)ds \\ &\leqslant a_{i,j}(||x - \Upsilon_j y|| + ||y - \Upsilon_i x||) \end{aligned}$$

for every $t \in I$. Take sup in the above inequality to find that

$$\begin{aligned} \alpha(x,y)d(\Upsilon_i x, \Upsilon_j y) &= ||\Upsilon x - \Upsilon y|| \\ &\leqslant a_{i,j}(||x - \Upsilon_j y|| + ||y - \Upsilon_i x||) = a_{i,j}(d(x,\Upsilon_j y) + d(y,\Upsilon_i x)). \end{aligned}$$

The properties (D) and (E) yield that properties (iii) and (iv) of Theorem 1 are satisfied. Obviously, the property (v) of Theorem 1 holds. Thus, by that theorem, $\{\Upsilon_n\}$ have a common fixed point, that is, the system of integral Equation (11) having a solution. □

Author Contributions: Writing–original draft, H.I., B.M., C.P. and V.P.; Writing–review and editing, H.I., B.M., C.P. and V.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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