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A Spectral Conjugate Gradient Method with Descent Property

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Abstract: Spectral conjugate gradient method (SCGM) is an important generalization of the conjugate gradient method (CGM), and it is also one of the effective numerical methods for large-scale unconstrained optimization. The designing for the spectral parameter and the conjugate parameter in SCGM is a core work. And the aim of this paper is to propose a new and effective alternative method for these two parameters. First, motivated by the strong Wolfe line search requirement, we design a new spectral parameter. Second, we propose a hybrid conjugate parameter. Such a way for yielding the two parameters can ensure that the search directions always possess descent property without depending on any line search rule. As a result, a new SCGM with the standard Wolfe line search is proposed. Under usual assumptions, the global convergence of the proposed SCGM is proved. Finally, by testing 108 test instances from 2 to 1,000,000 dimensions in the CUTE library and other classic test collections, a large number of numerical experiments, comparing with both SCGMs and CGMs, for the presented SCGM are executed. The detail results and their corresponding performance profiles are reported, which show that the proposed SCGM is effective and promising.

Keywords: unconstrained optimization; spectral conjugate gradient method; Wolfe line search; global convergence

1. Introduction

Conjugate gradient method (CGM) is one class of the prevailing methods commonly used for solving large-scale optimization problems, since it possesses simple iterations, fast convergence properties and low memory requirements. In this paper, we consider the following unconstrained optimization problem:

$$\min\{f(x)|\ x \in R^n\}, \tag{1}$$

where $f : R^n \to R$ is a continuously differentiable function and its gradient is denoted by $g(x) = \nabla f(x)$. The iterates of the classical CGM can be formulated as

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

and

$$d_k = \begin{cases} -g_k, & \text{if } k = 1, \\ -g_k + \beta_k d_{k-1}, & \text{if } k > 1, \end{cases} \tag{3}$$

where $g_k = g(x_k)$, d_k is the search direction, β_k is the so-called conjugate parameter, and α_k is the steplength which obtained by a suitable exact or inexact line search. However, as the high cost of the exact line search, α_k is usually generated by an inexact line search, such as the Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k, \end{cases} \tag{4}$$

or the strong Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ |g(x_k + \alpha_k d_k)^T d_k| \leq \sigma |g_k^T d_k|. \end{cases} \tag{5}$$

The parameters δ and σ above generally are required to satisfy $0 < \delta < \sigma < 1$. As we know, different choice for β_k would generate different CGM. The most well-known CGMs are the Hestenes-Stiefel (HS, 1952) method [1], Fletcher-Reeves (FR, 1964) method [2], Polak-Ribière-Polyak (PRP, 1969) method [3,4] and the Dai-Yuan (DY, 1999) method [5], where the corresponding formulas for β_k are

$$\beta_k^{\mathrm{HS}} = \frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T (g_k - g_{k-1})}, \quad \beta_k^{\mathrm{FR}} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_k^{\mathrm{PRP}} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2}, \quad \beta_k^{\mathrm{DY}} = \frac{\|g_k\|^2}{d_{k-1}^T (g_k - g_{k-1})},$$

respectively, where $\|\cdot\|$ is the standard Euclidean norm. Generally, these four methods are often referred to the classical CGMs. Under the corresponding assumptions, the authors analysed the convergence properties and tested the numerical performance of the four CGMs in References [2–6], respectively.

It is well known that the FR CGM and DY CGM possess nice convergence properties. However, their numerical performances are not so good. Inversely, the PRP and HS methods have excellent performance in practical computation. But their convergence properties are hardly to be obtained. Thus, to overcome these shortcomings of the classical CGMs, many researchers pay great attention to improve the CGMs. As a result, many improvements with excellent theoretical properties and numerical performance of the CGMs results were proposed, for example, References [7–20]. Where, the spectral conjugate gradient method (SCGM) proposed by Birgin and Martinez [12] can be seen as an important development of CGM. The main difference between SCGM and CGM lies in the computation of the search direction. The search direction of SCGM is usually yielded as follows:

$$d_k = \begin{cases} -g_k, & \text{if } k = 1, \\ -\theta_k g_k + \beta_k d_{k-1}, & \text{if } k > 1, \end{cases} \tag{6}$$

where θ_k is a spectral parameter. Obviously, for a SCGM, the selection techniques for the spectral parameter θ_k and the conjugate parameter β_k are core work and very important.

In Reference [12], after giving the concrete conjugate parameter β_k, Birgin and Martinez required the spectral search direction yielded by (6) satisfying $g_k^T d_k = -\|g_k\|^2$ (a special sufficient descent property), and then obtained the corresponding spectral parameter:

$$\theta_k = \frac{s_k^T s_k}{s_k^T y_k}, \text{ where } s_k = x_k - x_{k-1},\ y_k = g_k - g_{k-1}. \tag{7}$$

Under suitable line search, the SCGM yielded by (2) and (6) as well as (7) performs superiorly to the PRP CGM, the FR CGM and the Perry method [21].

Yielding the conjugate parameter β_k by a modified DY formula, based on the Newton's direction and the quasi-Newton equation as well as the conjugate conditions, respectively, Andrei [13] considered two approaches to generate the spectral parameter θ_k, namely,

$$\theta_k^{\mathrm{N}} = \frac{1}{y_k^T g_k}(\|g_k\|^2 - \frac{\|g_k\|^2 s_k^T g_k}{y_k^T s_k} + s_k^T g_k),$$

and

$$\theta_k^{\mathrm{C}} = \frac{1}{y_k^T g_k}(\|g_k\|^2 - \frac{\|g_k\|^2 s_k^T g_k}{y_k^T s_k}).$$

The two SCGMs associated with θ_k^{N} and θ_k^{C} are all sufficient descent without depending on any line search, and are global convergent with the Wolfe line search (4). Also, the numerical results show that the SCGM associated with θ_k^{N} is more encouraging.

Recently, by requiring the spectral direction d_k defined by (6) satisfying the special sufficient descent condition $g_k^T d_k = -\|g_k\|^2$ for general β_k, Liu et al. [15] proposed a class of choice for θ_k as follows:

$$\theta_k^{\mathrm{LFZ}} = -\frac{g_{k-1}^T d_{k-1}}{\|g_{k-1}\|} + \beta_k \frac{g_k^T d_{k-1}}{\|g_k\|^2}.$$

Under the conventional assumptions and request $|\beta_k| \leq |\beta_k^{\mathrm{FR}}|$ as well as the Wolfe line search (4), the SCGM developed by θ_k^{LFZ} is global convergent, and implemented with good computation performance.

On the other hand, Jiang et al. [19] considered to improve both the FR method and the DY method by utilizing the strong Wolfe line search (5), and achieved their good numerical effect. As a result, two schemes for the conjugate parameter are proposed, namely,

$$\beta_k^{\mathrm{IFR}} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_k^{\mathrm{IDY}} = \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \cdot \frac{\|g_k\|^2}{d_{k-1}^T (g_k - g_{k-1})}.$$

Interestingly, it is found, from formulas (3) and (6), that the SCGM can lead to more decrease than the classical CGM for a same β_k and any $\theta_k > 1$. Therefore, in this work, motivated by the ideas of the modified FR method and DY method [19], and making full use of the second condition of the strong Wolfe line search (5), we first introduce a new approach for yielding the spectral parameter as follows:

$$\theta_k^{\mathrm{JYJLL}} = 1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}. \tag{8}$$

Obviously, $\theta_k^{\mathrm{JYJLL}} \geq 1$ if the previous d_{k-1} is a descent direction.

Secondly, based on the scheme of the conjugate parameter in Reference [20] with the form

$$\beta_k^{\mathrm{JYJ}} = \frac{\|g_k\|^2 - \frac{(g_k^T d_{k-1})^2}{\|d_{k-1}\|^2}}{\|g_{k-1}\|^2 + u|d_{k-1}^T g_k|},$$

and fully absorbing the hybrid idea of Reference [11], we propose a new conjugate parameter in the following manner

$$\beta_k^{\mathrm{JYJLL}} = \frac{\|g_k\|^2 - \frac{(g_k^T d_{k-1})^2}{\|d_{k-1}\|^2}}{\max\{\|g_{k-1}\|^2, d_{k-1}^T (g_k - g_{k-1})\}}. \tag{9}$$

So far, a basic conception of our SCGM has been formed. As a result, a new SCGM is proposed, and the theoretical features and numerical performance is analysed and reported.

2. Algorithm and the Descent Property

Based on the formulas (2) and (6) as well as (9), we establish the new SCGM as follows (Algorithm 1).

Algorithm 1: JYJLL-SCGM

Step 0. Given any initial point $x_1 \in R^n$, parameters δ and σ satisfying $0 < \delta < \sigma < 1$, and an accuracy tolerance $\epsilon > 0$. Let $d_1 = -g_1$, set $k := 1$.

Step 1. If $\|g_k\| \leq \epsilon$, terminate. Otherwise, go to Step 2.

Step 2. Determine a steplength α_k by an inexact line search.

Step 3. Generate the next iterate by $x_{k+1} = x_k + \alpha_k d_k$, compute gradient $g_{k+1} := g(x_{k+1})$ and the spectral parameter $\theta_{k+1} := \theta_{k+1}^{\mathrm{JYJLL}}$ by (8), as well as the conjugate parameter $\beta_{k+1} := \beta_{k+1}^{\mathrm{JYJLL}}$ by (9).

Step 4. Let $d_{k+1} = -\theta_{k+1} g_{k+1} + \beta_{k+1} d_k$. Set $k := k + 1$, repeat Step 1.

The following lemma indicates that the JYJLL-SCGM always satisfies the descent condition without depending on any line search, and the conjugate parameter β_k^{JYJLL} has the similar properties as the DY formula.

Lemma 1. *Suppose that the search direction d_k is generated by JYJLL-SCGM. Then, we have $g_k^T d_k < 0$ for $k \geq 1$, that is, the search direction satisfies the descent condition. Furthermore, we obtain $0 \leq \beta_k^{\mathrm{JYJLL}} \leq \frac{g_k^T d_k}{g_{k-1}^T d_{k-1}}$.*

Proof. We first prove the former claim by induction. For $k = 1$, it is easy to see that $g_1^T d_1 = -\|g_1\|^2 < 0$. Assume that $g_{k-1}^T d_{k-1} < 0$ holds for $k - 1$ ($k \geq 2$). Now, we prove that $g_k^T d_k < 0$ holds for k. Let $\hat{\theta}_k$ be the angle between g_k and d_{k-1}. The proof is divided into two cases as follows.

(a) If $d_{k-1}^T (g_k - g_{k-1}) > \|g_{k-1}\|^2$, then $d_{k-1}^T (g_k - g_{k-1}) > 0$. It follows from (8) and (9) that

$$\begin{aligned} g_k^T d_k =& g_k^T (-\theta_k^{\mathrm{JYJLL}} g_k + \beta_k^{\mathrm{JYJLL}} d_{k-1}) \\ =& -(1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}) \|g_k\|^2 + \frac{\|g_k\|^2 (1 - \cos^2 \hat{\theta}_k)}{d_{k-1}^T (g_k - g_{k-1})} g_k^T d_{k-1} \\ =& -\frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}} \|g_k\|^2 - \|g_k\|^2 \cos^2 \hat{\theta}_k + \frac{\|g_k\|^2 (1 - \cos^2 \hat{\theta}_k)}{d_{k-1}^T (g_k - g_{k-1})} g_{k-1}^T d_{k-1} \\ =& -(\cos^2 \hat{\theta}_k + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}) \|g_k\|^2 + \beta_k^{\mathrm{JYJLL}} g_{k-1}^T d_{k-1} < 0. \end{aligned} \tag{10}$$

(b) If $d_{k-1}^T (g_k - g_{k-1}) \leq \|g_{k-1}\|^2$, then $g_k^T d_{k-1} \leq \|g_{k-1}\|^2 + g_{k-1}^T d_{k-1}$, and hence by (8) and (9), we have

$$\begin{aligned} g_k^T d_k =& g_k^T (-\theta_k^{\mathrm{JYJLL}} g_k + \beta_k^{\mathrm{JYJLL}} d_{k-1}) \\ =& -\theta_k^{\mathrm{JYJLL}} \|g_k\|^2 + \frac{\|g_k\|^2 (1 - \cos^2 \hat{\theta}_k)}{\|g_{k-1}\|^2} g_k^T d_{k-1} \\ \leq& -(1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}) \|g_k\|^2 + \frac{\|g_k\|^2 (1 - \cos^2 \hat{\theta}_k)}{\|g_{k-1}\|^2} (\|g_{k-1}\|^2 + g_{k-1}^T d_{k-1}) \\ =& -(\cos^2 \hat{\theta}_k + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}) \|g_k\|^2 + \beta_k^{\mathrm{JYJLL}} g_{k-1}^T d_{k-1} < 0. \end{aligned} \tag{11}$$

Thus, $g_k^T d_k < 0$ holds for $k \geq 1$.

Now we prove the second assertion. From (10) and (11), it follows that $g_k^T d_k \leq \beta_k^{\mathrm{JYJLL}} g_{k-1}^T d_{k-1}$. This together with $g_k^T d_k < 0$ implies that $\beta_k^{\mathrm{JYJLL}} \leq \frac{g_k^T d_k}{g_{k-1}^T d_{k-1}}$, furthermore, we deduce that $0 \leq \beta_k^{\mathrm{JYJLL}}$ from (9), and the proof is complete. □

3. Convergence Analysis

To analyze and ensure the global convergence of the JYJLL-SCGM, we choose the Wolfe line search (4) to yield the steplength α_k. Further, a basic assumption about the objective function as follows is needed.

Assumption 1. (H1) *For any initial point $x_1 \in R^n$, the level set $\Lambda = \{x \in R^n \mid f(x) \leq f(x_1)\}$ is bounded;* (H2) *$f(x)$ is continuously differentiable in a neighborhood U of Λ, and its gradient $g(x)$ is Lipschitz continuous, namely, there exists a constant $L > 0$ such that $\|g(x) - g(y)\| \leq L\|x - y\|, \forall\, x, y \in U$.*

In the following lemma, we review the well-known Zoutendijk condition [6], which plays an important role in the convergence analysis of CGMs. Also, the Zoutendijk condition is suitable for the convergence analysis of the JYJLL-SCGM.

Lemma 2. *Suppose that Assumption 1 holds. Consider a general iterative method $x_{k+1} = x_k + \alpha_k d_k$, where d_k is a descent direction such that $g_k^T d_k < 0$, and the steplength α_k satisfies the Wolfe line search condition (4). Then $\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty$.*

Based on Lemmas 1 and 2, we can establish the global convergence of the JYJLL-SCGM.

Theorem 1. *Suppose that Assumption 1 holds, and let the sequence $\{x_k\}$ be generated by the JYJLL-SCGM with the wolfe line search (4). Then $\liminf_{k\to\infty} \|g_k\| = 0$.*

Proof. By contradiction, suppose that the conclusion is not true. Then there exists a positive constant $\gamma > 0$ such that $\|g_k\|^2 \geq \gamma, \forall\, k$. Again, from (8), we obtain $d_k + \theta_k^{\mathrm{JYJLL}} g_k = \beta_k^{\mathrm{JYJLL}} d_{k-1}$. Combining this equation and Lemma 1, we have

$$\begin{aligned} \|d_k\|^2 &= (\beta_k^{\mathrm{JYJLL}})^2 \|d_{k-1}\|^2 - 2\theta_k^{\mathrm{JYJLL}} g_k^T d_k - (\theta_k^{\mathrm{JYJLL}})^2 \|g_k\|^2 \\ &\leq (\tfrac{g_k^T d_k}{g_{k-1}^T d_{k-1}})^2 \|d_{k-1}\|^2 - 2\theta_k^{\mathrm{JYJLL}} g_k^T d_k - (\theta_k^{\mathrm{JYJLL}})^2 \|g_k\|^2. \end{aligned}$$

Next, dividing both sides of the above inequality by $(g_k^T d_k)^2$, we obtain

$$\begin{aligned} \tfrac{\|d_k\|^2}{(g_k^T d_k)^2} &\leq \tfrac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} - \tfrac{2\theta_k^{\mathrm{JYJLL}}}{g_k^T d_k} - \tfrac{(\theta_k^{\mathrm{JYJLL}})^2 \|g_k\|^2}{(g_k^T d_k)^2} \\ &= \tfrac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \tfrac{1}{\|g_k\|^2} - (\tfrac{1}{\|g_k\|} + \tfrac{\theta_k^{\mathrm{JYJLL}} \|g_k\|}{g_k^T d_k})^2 \\ &\leq \tfrac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \tfrac{1}{\|g_k\|^2}. \end{aligned}$$

In terms of $\frac{\|d_1\|^2}{(g_1^T d_1)^2} = \frac{1}{\|g_1\|^2}$, together with the above relations and $\|g_k\|^2 \geq \gamma$, we have

$$\begin{aligned} \tfrac{\|d_k\|^2}{(g_k^T d_k)^2} &\leq \tfrac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \tfrac{1}{\|g_k\|^2} \\ &\leq \tfrac{\|d_{k-2}\|^2}{(g_{k-2}^T d_{k-2})^2} + \tfrac{1}{\|g_k\|^2} + \tfrac{1}{\|g_{k-1}\|^2} \\ &\leq \cdots \leq \textstyle\sum_{i=1}^{k} \tfrac{1}{\|g_i\|^2} \leq \tfrac{k}{\gamma}, \end{aligned}$$

that is, $\frac{(g_k^T d_k)^2}{\|d_k\|^2} \geq \frac{\gamma}{k}$. Hence $\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty$, which contradicts Lemma 2. Therefore, the proof is complete. □

4. Numerical Results

In this section, we test the numerical performance of our method (denoted by JYJLL for short) via 108 test problems, and compare it with the four methods HZ [7], KD [8], AN1 [13] and LFZ [15]. The HZ and KD methods belong to the CGMs with excellent effect, and the AN1 and LFZ methods are the SCGMs with more efficient performance. The first 53 (from bard to woods) test problems are taken from the CUTE library in N. I. M. Gould et al. [22], and the last 55 are from References [23,24], their dimensions ranging from 2 to 1,000,000. All codes were written in Matlab 2016a and run on a DELL PC with 4GB of memory and windows 10 operating system. All the steplength α_k is generated by the Wolfe line search with $\sigma = 0.1$ and $\delta = 0.01$.

In the experiments, notations Itr, NF, NG, and Tcpu and $\|g_*\|$ denote the number of iteration, function evaluation, gradient evaluation, computing time of CPU and gradient values, respectively. We stop the iteration, if one of the following two cases is satisfied: (i) $\|g_k\| \leq 10^{-6}$; (ii) Itr > 2000. When case (ii) appears, the method is deemed to be invalid and is denoted by "F".

To show the numerical performance of the tested methods, we observed and reported the values of Itr, NF, NG, Tcpu and $\|g_*\|$ generated by the five tested methods for each test instance, see Tables 1 and 2 below. On the other hand, to visually characterize and compare the numerical results in Tables 1 and 2, we use the performance profiles introduced by Dolan and Morè [25] to describe the performance of the five tested methods according to Itr, NF, NG and Tcpu, respectively, see Figures 1–4 below.

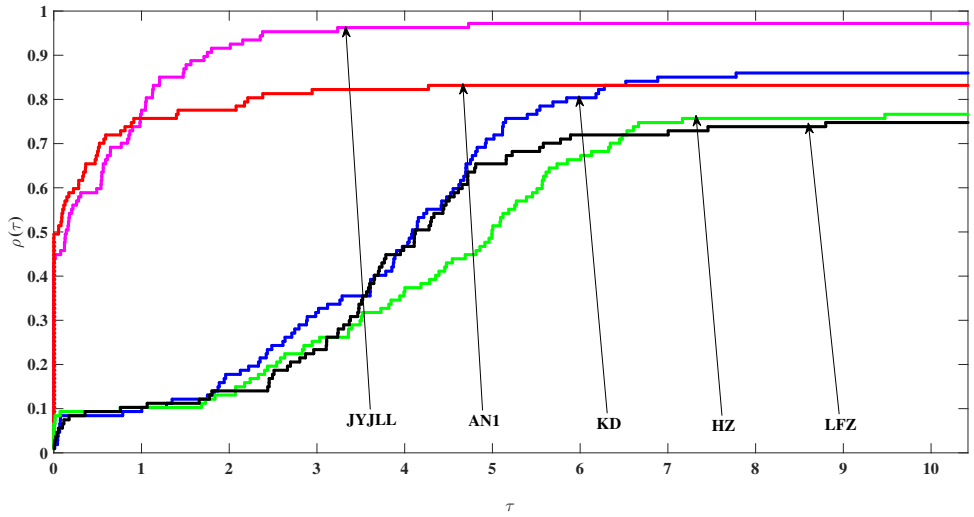


Figure 1. Performance profiles on Tcpu.

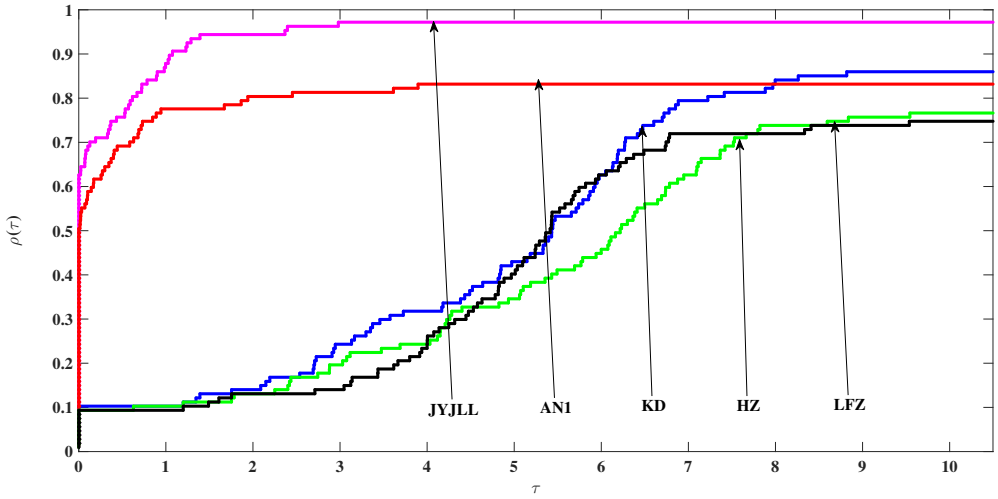


Figure 2. Performance profiles on NF .

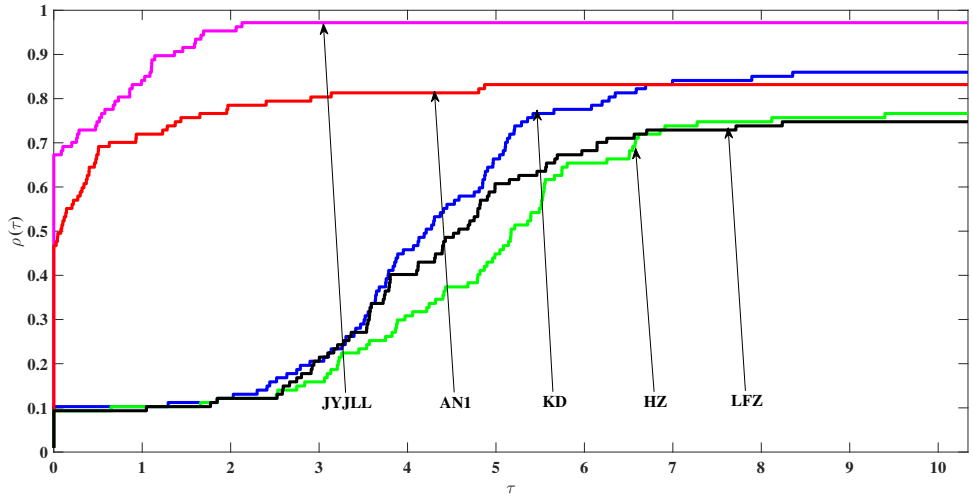


Figure 3. Performance profiles on NG

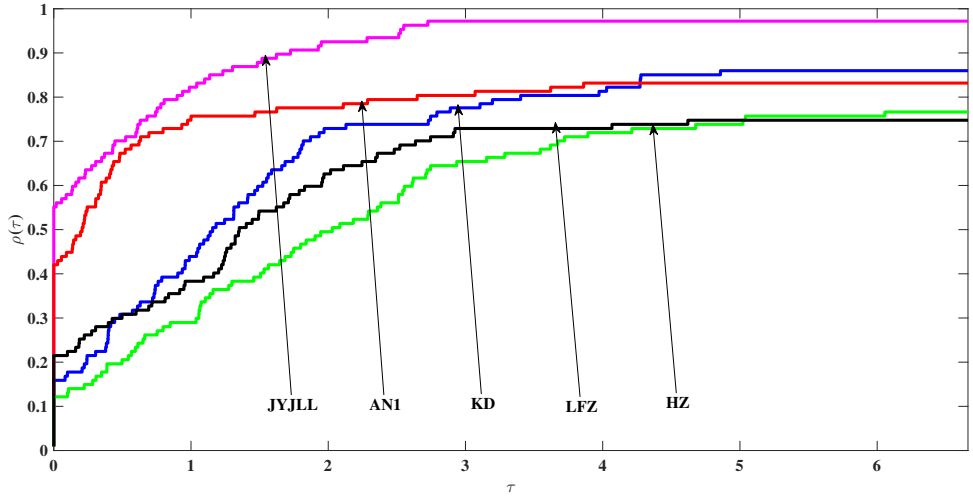


Figure 4. Performance profiles on Itr.

Table 1. Numerical test reports for the five tested methods.

Problems	**JYJLL**	**KD**	**AN1**	**HZ**	**LFZ**
Name/n	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$
bard 3	178/218/259/0.171/8.74 $\times 10^{-7}$	1565/48,617/19,915/4.657/8.01 $\times 10^{-7}$	81/206/156/0.051/8.35 $\times 10^{-7}$	620/18,622/7190/1.732/3.95 $\times 10^{-7}$	265/6647/3284/0.691/1.72 $\times 10^{-7}$
beale 2	67/93/87/0.015/9.28 $\times 10^{-7}$	326/9668/4385/0.429/7.71 $\times 10^{-7}$	44/150/92/0.012/9.17 $\times 10^{-7}$	251/7325/3130/0.318/6.17 $\times 10^{-7}$	171/4757/2347/0.214/4.31 $\times 10^{-8}$
box 3	139/193/207/0.026/6.64 $\times 10^{-7}$	330/10,108/4330/0.504/5.98 $\times 10^{-7}$	36/104/64/0.009/7.52 $\times 10^{-7}$	475/14,246/5816/0.702/8.94 $\times 10^{-7}$	68/1569/764/0.083/3.17 $\times 10^{-7}$
cosine 2000	17/55/19/0.104/2.48 $\times 10^{-7}$	493/13,783/6221/3.593/8.15 $\times 10^{-7}$	247/673/556/0.438/9.68 $\times 10^{-7}$	435/11,148/5281/3.126/4.62 $\times 10^{-7}$	F/F/F/F/1.88 $\times 10^{3}$
cosine 4000	20/61/24/0.050/9.50 $\times 10^{-8}$	F/F/F/F/6.04 $\times 10^{-3}$	246/908/671/0.964/5.96 $\times 10^{-7}$	178/4537/2189/3.495/8.83 $\times 10^{-7}$	F/F/F/F/7.43 $\times 10^{-4}$
cosine 20,000	26/65/32/0.228/6.76 $\times 10^{-7}$	F/F/F/F/3.13 $\times 10^{-3}$	163/207/240/1.029/7.51 $\times 10^{-7}$	1730/48,495/21,578/161.976/6.59 $\times 10^{-7}$	F/F/F/F/1.85 $\times 10^{4}$
dixmaana 3000	21/63/22/0.318/8.90 $\times 10^{-7}$	17/266/117/0.683/3.09 $\times 10^{-7}$	20/60/22/0.178/9.35 $\times 10^{-7}$	26/520/206/1.282/1.33 $\times 10^{-7}$	24/393/169/1.012/1.85 $\times 10^{-7}$
dixmaanb 3000	12/59/12/0.159/3.75 $\times 10^{-7}$	12/150/49/0.370/3.26 $\times 10^{-7}$	14/59/14/0.145/8.48 $\times 10^{-7}$	49/1246/565/3.231/9.61 $\times 10^{-8}$	14/180/72/0.458/1.48 $\times 10^{-7}$
dixmaanc 3000	15/64/15/0.197/8.85 $\times 10^{-7}$	25/458/217/1.207/5.54 $\times 10^{-7}$	19/61/19/0.174/9.87 $\times 10^{-7}$	33/678/323/1.829/1.92 $\times 10^{-7}$	35/659/315/1.718/8.03 $\times 10^{-7}$
dixmaand 3000	19/68/19/0.194/1.05 $\times 10^{-8}$	25/389/167/0.977/3.58 $\times 10^{-7}$	21/67/21/0.191/3.53 $\times 10^{-7}$	54/1304/627/3.484/2.30 $\times 10^{-7}$	45/822/409/2.179/6.68 $\times 10^{-7}$
dixmaane 3000	564/525/797/3.316/8.67 $\times 10^{-7}$	787/22,182/9876/60.145/9.62 $\times 10^{-7}$	406/249/500/2.203/9.08 $\times 10^{-7}$	1239/35,277/14,918/96.673/7.67 $\times 10^{-7}$	369/9463/4622/26.732/7.85 $\times 10^{-7}$
dixmaanf 3000	486/470/691/3.060/5.26 $\times 10^{-7}$	720/20,789/8946/56.355/7.80 $\times 10^{-7}$	395/238/483/2.094/7.12 $\times 10^{-7}$	1387/40,775/17,299/111.617/9.92 $\times 10^{-7}$	283/7443/3631/21.630/8.83 $\times 10^{-7}$
dixmaang 3000	291/286/403/1.872/8.98 $\times 10^{-7}$	557/15,559/6999/43.071/8.50 $\times 10^{-7}$	430/266/531/2.370/6.61 $\times 10^{-7}$	1492/43,815/18,382/120.388/6.52 $\times 10^{-7}$	441/11,502/5595/32.348/6.66 $\times 10^{-7}$
dixmaanh 3000	501/499/719/3.300/8.91 $\times 10^{-7}$	933/26,886/11971/73.607/9.35 $\times 10^{-7}$	428/245/518/2.273/9.72 $\times 10^{-7}$	F/F/F/F/2.23 $\times 10^{-5}$	478/12,359/6015/35.150/9.39 $\times 10^{-7}$
dixmaanj 3000	874/829/1258/5.304/7.87 $\times 10^{-7}$	1201/34,580/15,144/93.734/9.58 $\times 10^{-7}$	1038/575/1295/5.617/7.20 $\times 10^{-7}$	F/F/F/F/3.16 $\times 10^{-6}$	642/16,203/7988/45.658/9.24 $\times 10^{-7}$
dixmaank 3000	891/876/1297/5.757/9.74 $\times 10^{-7}$	1158/33,535/14,587/89.969/7.82 $\times 10^{-7}$	978/551/1222/5.291/6.98 $\times 10^{-7}$	F/F/F/F/2.66 $\times 10^{-5}$	1899/49,488/24,316/139.479/8.81 $\times 10^{-7}$
dixon3dq 5	51/54/54/0.011/5.58 $\times 10^{-7}$	136/3776/1687/0.156/4.24 $\times 10^{-7}$	52/58/58/0.007/5.73 $\times 10^{-7}$	77/1803/786/0.072/7.91 $\times 10^{-7}$	168/4217/2066/0.165/4.14 $\times 10^{-7}$
dixon3dq 40	631/600/908/0.076/9.98 $\times 10^{-7}$	1164/34,979/14,945/1.418/5.17 $\times 10^{-7}$	394/114/427/0.041/7.07 $\times 10^{-7}$	1789/52,134/21,597/1.999/7.37 $\times 10^{-7}$	477/12,218/5965/0.492/3.93 $\times 10^{-7}$
dqdrtic 10,000	105/159/144/0.174/8.38 $\times 10^{-7}$	275/7800/3415/3.965/2.96 $\times 10^{-7}$	78/124/99/0.159/7.25 $\times 10^{-7}$	911/28,054/11,907/15.545/7.25 $\times 10^{-7}$	191/5281/2547/3.215/3.10 $\times 10^{-7}$
dqdrtic 100,000	118/176/164/1.740/9.70 $\times 10^{-7}$	455/13,662/6077/53.011/3.67 $\times 10^{-7}$	104/232/178/2.033/8.42 $\times 10^{-7}$	685/20,674/8804/82.583/5.80 $\times 10^{-7}$	293/7575/3761/30.040/3.73 $\times 10^{-7}$
dqdrtic 1,000,000	67/143/96/13.649/2.81 $\times 10^{-7}$	256/7211/3161/354.291/7.07 $\times 10^{-7}$	84/185/132/19.420/9.45 $\times 10^{-7}$	854/26,185/11,120/1264.582/5.93 $\times 10^{-7}$	214/5856/2872/297.431/3.91 $\times 10^{-7}$
dqrtic 480	33/89/33/0.043/6.19 $\times 10^{-7}$	F/F/F/F/4.77 $\times 10^{9}$	43/100/51/0.037/2.83 $\times 10^{-7}$	F/F/F/F/4.77 $\times 10^{9}$	F/F/F/F/4.77 $\times 10^{9}$
dqrtic 510	28/89/28/0.028/2.08 $\times 10^{-7}$	F/F/F/F/5.90 $\times 10^{9}$	F/F/F/F/1.16 $\times 10^{9}$	F/F/F/F/5.90 $\times 10^{9}$	F/F/F/F/5.90 $\times 10^{9}$
edensch 4000	39/326/155/0.683/9.42 $\times 10^{-7}$	45/855/380/1.666/6.67 $\times 10^{-7}$	F/F/F/F/7.59 $\times 10^{-6}$	51/1123/490/2.195/8.67 $\times 10^{-7}$	F/F/F/F/1.22 $\times 10^{-5}$
edensch 5000	34/196/83/0.573/6.12 $\times 10^{-7}$	66/1491/715/3.760/6.48 $\times 10^{-7}$	F/F/F/F/1.99 $\times 10^{-6}$	48/1035/475/2.590/6.56 $\times 10^{-7}$	F/F/F/F/2.25 $\times 10^{-5}$
edensch 7500	35/259/96/0.956/2.94 $\times 10^{-7}$	68/1673/797/4.458/3.76 $\times 10^{-7}$	F/F/F/F/2.64 $\times 10^{-6}$	F/F/F/F/2.13 $\times 10^{-6}$	F/F/F/F/9.86 $\times 10^{-6}$
eg2 30	66/84/79/0.036/6.82 $\times 10^{-7}$	F/F/F/F/3.05 $\times 10^{-1}$	62/138/102/0.016/7.12 $\times 10^{-7}$	F/F/F/F/3.20 $\times 10^{-3}$	F/F/F/F/2.61 $\times 10^{-5}$
eg2 100	111/156/157/0.025/7.73 $\times 10^{-7}$	156/4392/1951/0.320/5.40 $\times 10^{-7}$	45/289/160/0.031/5.13 $\times 10^{-7}$	F/F/F/F/7.42 $\times 10^{-6}$	F/F/F/F/2.74 $\times 10^{-5}$
fletchcr 50	51/107/81/0.028/4.96 $\times 10^{-7}$	88/2309/1016/0.239/4.70 $\times 10^{-7}$	63/108/89/0.025/7.14 $\times 10^{-7}$	140/3679/1711/0.383/1.92 $\times 10^{-7}$	181/4599/2284/0.539/3.65 $\times 10^{-7}$
fletchcr 100	59/75/69/0.016/5.93 $\times 10^{-7}$	157/4341/1989/0.306/4.38 $\times 10^{-7}$	F/F/F/F/4.08 $\times 10^{-6}$	86/2125/978/0.127/5.68 $\times 10^{-7}$	67/1635/810/0.093/9.77 $\times 10^{-7}$
fletchcr 200	59/75/72/0.012/7.79 $\times 10^{-7}$	117/3079/1414/0.205/7.30 $\times 10^{-7}$	75/288/195/0.032/4.41 $\times 10^{-7}$	F/F/F/F/8.57 $\times 10^{-6}$	F/F/F/F/1.00 $\times 10^{-5}$

Table 1. *Cont.*

Problems	JYJLL	KD	AN1	HZ	LFZ
freuroth 5	269/390/421/0.061/9.04 × 10^{-7}	941/28,582/12,159/1.563/3.39 × 10^{-8}	F/F/F/F/1.56 × 10^{-5}	1580/48,118/19,484/2.562/6.79 × 10^{-7}	F/F/F/F/5.26 × 10^{-6}
genrose 5000	283/292/397/0.537/7.01 × 10^{-7}	371/10,726/47,41/6.456/3.94 × 10^{-7}	219/194/283/0.441/7.91 × 10^{-7}	1070/31,988/13,201/16.842/4.83 × 10^{-7}	616/16,065/7855/5.567/8.05 × 10^{-7}
genrose 100,000	142/170/193/2.068/9.01 × 10^{-7}	416/12,351/5418/62.603/9.70 × 10^{-7}	166/203/233/2.965/9.99 × 10^{-7}	494/14,196/6016/71.947/9.73 × 10^{-7}	361/9600/4616/54.420/9.84 × 10^{-7}
genrose 1000	233/266/335/0.077/8.18 × 10^{-7}	295/8492/3729/0.942/8.13 × 10^{-7}	224/197/291/0.087/7.28 × 10^{-7}	701/20,368/8502/2.625/4.17 × 10^{-7}	393/10,221/5053/1.526/9.81 × 10^{-7}
gulf 3000	2/1/2/0.008/0.00 × 10^{0}	2/1/2/0.001/0.00 × 10^{0}	2/1/2/0.000/0.00 × 10^{0}	2/1/2/0.000/0.00 × 10^{0}	2/1/2/0.000/0.00 × 10^{0}
helix 20,000	133/203/199/0.064/7.70 × 10^{-7}	282/8168/3481/0.733/4.20 × 10^{-7}	119/258/212/0.044/8.96 × 10^{-7}	731/21,643/8941/2.214/6.74 × 10^{-7}	F/F/F/F/4.06 × 10^{1}
himmelbg 30,000	3/6/7/0.051/8.71 × 10^{-28}	3/6/7/0.043/7.27 × 10^{-28}	3/6/7/0.043/8.73 × 10^{-28}	3/6/7/0.042/6.57 × 10^{-28}	3/6/7/0.043/8.75 × 10^{-28}
himmelbg 50,000	3/6/7/0.075/1.12 × 10^{-27}	3/6/7/0.067/9.38 × 10^{-28}	3/6/7/0.081/1.13 × 10^{-27}	3/6/7/0.066/8.48 × 10^{-28}	3/6/7/0.072/1.13 × 10^{-27}
himmelbg 10	3/6/7/0.001/1.59 × 10^{-29}	3/6/7/0.001/1.33 × 10^{-29}	3/6/7/0.001/1.59 × 10^{-29}	3/6/7/0.001/1.20 × 10^{-29}	3/6/7/0.001/1.60 × 10^{-29}
kowosb 10	449/470/668/0.097/9.29 × 10^{-7}	817/24,298/10,391/1.768/4.66 × 10^{-7}	118/285/243/0.042/6.03 × 10^{-7}	F/F/F/F/5.22 × 10^{-4}	312/8413/4176/0.666/7.72 × 10^{-7}
liarwhd 15	59/95/73/0.018/9.69 × 10^{-7}	72/1725/753/0.127/4.53 × 10^{-7}	35/129/68/0.013/7.67 × 10^{-7}	229/6565/2858/0.417/5.51 × 10^{-7}	47/1028/474/0.072/6.32 × 10^{-7}
liarwhd 1000	258/421/426/0.094/6.91 × 10^{-7}	F/F/F/F/1.33 × 10^{-4}	39/211/102/0.027/7.23 × 10^{-7}	1267/38,967/15,799/3.867/9.84 × 10^{-7}	224/5956/2888/0.612/5.39 × 10^{-7}
nondquar 10	385/461/586/0.085/9.98 × 10^{-7}	F/F/F/F/2.49 × 10^{-4}	F/F/F/F/2.76 × 10^{-3}	F/F/F/F/1.28 × 10^{-3}	F/F/F/F/1.99 × 10^{1}
penalty1 1000	14/75/14/0.279/6.45 × 10^{-8}	15/253/94/1.084/1.21 × 10^{-7}	14/75/14/0.279/7.70 × 10^{-8}	15/253/94/0.996/1.21 × 10^{-7}	15/253/94/0.954/1.21 × 10^{-7}
penalty1 8000	17/79/17/13.954/3.44 × 10^{-7}	114/3407/1633/545.410/1.79 × 10^{-7}	17/79/17/11.996/3.00 × 10^{-7}	114/3407/1624/568.693/1.79 × 10^{-7}	114/3407/1607/573.841/1.77 × 10^{-7}
quartc 50	23/67/23/0.010/5.57 × 10^{-7}	28/516/226/0.034/1.42 × 10^{-8}	21/68/23/0.006/1.03 × 10^{-7}	26/449/192/0.028/8.61 × 10^{-7}	85/2319/1093/0.157/6.36 × 10^{-7}
quartc 300	25/86/26/0.019/4.33 × 10^{-7}	34/562/249/0.100/3.74 × 10^{-7}	29/87/32/0.020/2.30 × 10^{-7}	37/633/284/0.116/8.92 × 10^{-8}	108/2821/1345/0.494/4.26 × 10^{-7}
tridia 400	1196/1131/1723/0.225/8.86 × 10^{-7}	F/F/F/F/1.07 × 10^{-3}	865/482/1067/0.152/6.41 × 10^{-7}	F/F/F/F/3.76 × 10^{-5}	832/22,141/10,798/1.437/5.76 × 10^{-7}
tridia 500	1163/1073/1661/0.230/9.65 × 10^{-7}	1655/47,555/20,693/3.330/4.97 × 10^{-7}	F/F/F/F/1.52 × 10^{3}	F/F/F/F/3.97 × 10^{-3}	991/25,621/12,586/1.797/5.03 × 10^{-7}
sinquad 3	202/284/320/0.032/6.77 × 10^{-8}	272/7998/3343/0.334/8.95 × 10^{-7}	86/220/173/0.018/6.56 × 10^{-8}	501/14,834/5870/0.639/2.96 × 10^{-7}	166/4182/2052/0.201/9.27 × 10^{-7}
vardim 2	10/54/10/0.014/5.74 × 10^{-7}	10/124/36/0.005/9.89 × 10^{-8}	10/54/10/0.003/5.74 × 10^{-7}	10/124/36/0.009/9.89 × 10^{-8}	10/124/36/0.006/9.89 × 10^{-8}
woods 10,000	662/760/998/1.323/9.70 × 10^{-7}	1851/54,675/25,702/47.380/9.00 × 10^{-7}	215/321/331/0.640/5.16 × 10^{-7}	1122/32,157/13,925/29.130/8.10 × 10^{-7}	515/13,196/6550/12.415/1.79 × 10^{-7}

Table 2. Numerical test reports for the five tested methods (Continue).

Problems	**JYJLL**	**KD**	**AN1**	**HZ**	**LFZ**
Name/n	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$	Itr/ NF/ NG/ Tcpu/ $\|g_*\|$
bdexp 50,000	3/2/3/0.079/1.33 × 10^{-109}	3/2/3/0.075/1.28 × 10^{-109}	3/2/3/0.072/1.32 × 10^{-109}	3/2/3/0.073/1.25 × 10^{-109}	3/2/3/0.074/1.33 × 10^{-109}
bdexp 100,000	3/2/3/0.134/1.73 × 10^{-109}	3/2/3/0.129/1.70 × 10^{-109}	3/2/3/0.132/1.73 × 10^{-109}	3/2/3/0.123/1.68 × 10^{-109}	3/2/3/0.123/1.73 × 10^{-109}
bdexp 1,000,000	3/2/3/1.245/5.09 × 10^{-109}	3/2/3/1.276/5.08 × 10^{-109}	3/2/3/1.341/5.09 × 10^{-109}	3/2/3/1.239/5.08 × 10^{-109}	3/2/3/1.259/5.09 × 10^{-109}
exdenschnf 100,000	24/84/25/0.751/8.51 × 10^{-7}	69/1886/922/16.019/6.00 × 10^{-7}	74/130/97/2.004/8.07 × 10^{-7}	59/1490/692/12.027/2.21 × 10^{-7}	123/3284/1568/26.773/8.35 × 10^{-7}
exdenschnf 1,000,000	29/86/30/8.772/3.60 × 10^{-7}	72/1974/940/155.062/9.90 × 10^{-8}	57/114/73/16.513/3.88 × 10^{-7}	123/3517/1677/281.678/5.75 × 10^{-8}	220/6391/3128/520.133/6.92 × 10^{-7}
mccormak 2	23/58/30/0.024/7.24 × 10^{-7}	19/364/163/0.021/8.11 × 10^{-7}	33/55/39/0.005/7.39 × 10^{-7}	39/900/404/0.039/1.21 × 10^{-7}	F/F/F/F/2.89 × 10^{4}
exdenschnb 15,000	24/62/25/0.067/3.34 × 10^{-7}	28/600/263/0.496/8.71 × 10^{-7}	32/61/33/0.101/7.18 × 10^{-7}	38/787/367/0.764/9.24 × 10^{-8}	134/3726/1766/3.688/8.57 × 10^{-7}
exdenschnb 120,000	23/62/24/0.497/7.31 × 10^{-7}	30/632/282/3.333/2.86 × 10^{-7}	33/63/34/0.638/8.21 × 10^{-7}	28/523/223/2.688/2.98 × 10^{-7}	89/2344/1134/12.149/4.31 × 10^{-7}
genquartic 120,000	35/78/41/0.793/9.37 × 10^{-7}	46/929/437/6.281/1.34 × 10^{-7}	47/83/57/1.178/5.73 × 10^{-7}	59/1355/643/9.081/9.61 × 10^{-7}	166/4182/2080/28.307/9.70 × 10^{-7}
genquartic 100,000	28/67/31/0.556/5.69 × 10^{-7}	37/736/306/4.120/5.17 × 10^{-7}	39/74/44/0.786/6.59 × 10^{-7}	81/2048/984/11.667/8.80 × 10^{-7}	174/4464/2191/25.852/9.57 × 10^{-7}
biggsb1 110	790/738/1136/0.118/7.29 × 10^{-7}	F/F/F/F/2.37 × 10^{-4}	522/143/571/0.065/9.63 × 10^{-7}	1868/55,264/23,430/2.526/9.82 × 10^{-7}	594/15,762/7618/0.728/4.96 × 10^{-7}
biggsb1 200	1717/1618/2504/0.255/8.47 × 10^{-7}	1809/52,361/22,385/2.674/9.04 × 10^{-7}	1148/205/1228/0.163/9.09 × 10^{-7}	F/F/F/F/2.38 × 10^{-3}	866/22,409/11,027/1.192/7.31 × 10^{-7}
sine 750,000	111/147/156/31.089/8.79 × 10^{-7}	141/3661/1775/379.752/3.73 × 10^{-11}	F/F/F/F/1.65 × 10^{-3}	101/2714/1330/318.993/5.72 × 10^{-7}	F/F/F/F/5.77 × 10^{-3}
sine 1,000,000	238/407/412/110.321/2.67 × 10^{-7}	72/2052/994/297.446/2.56 × 10^{-7}	89/198/160/47.856/2.98 × 10^{-7}	150/3798/1858/538.100/9.23 × 10^{-8}	F/F/F/F/4.35 × 10^{1}
fletcbv3 55	667/488/895/0.408/2.07 × 10^{-7}	1381/26,076/12,787/1.507/8.07 × 10^{-8}	F/F/F/F/2.03 × 10^{-4}	1388/26,927/13,220/1.712/8.48 × 10^{-7}	F/F/F/F/1.59 × 10^{-4}
fletcbv3 85	1654/911/2080/0.348/5.69 × 10^{-7}	F/F/F/F/7.89 × 10^{-4}	F/F/F/F/7.43 × 10^{-4}	F/F/F/F/8.15 × 10^{-4}	772/13,968/7106/1.214/2.55 × 10^{-7}
nonscomp 30,000	F/F/F/F/2.14 × 10^{-4}	147/3574/1604/6.135/8.57 × 10^{-7}	95/124/119/0.409/4.82 × 10^{-7}	927/27600/11616/41.504/5.89 × 10^{-7}	F/F/F/F/4.08 × 10^{-3}
nonscomp 25000	60/78/62/0.217/6.47 × 10^{-7}	1164/35,210/14,738/47.678/6.09 × 10^{-7}	93/125/118/0.381/9.82 × 10^{-7}	199/4988/2234/6.920/4.66 × 10^{-7}	1006/26,503/13,011/38.180/4.78 × 10^{-7}
power1 100	1588/1500/2295/0.277/7.70 × 10^{-7}	F/F/F/F/8.27 × 10^{-5}	1496/908/1907/0.181/9.52 × 10^{-7}	F/F/F/F/2.49 × 10^{-1}	1178/31,067/15,281/1.268/7.06 × 10^{-7}
power1 90	1614/1548/2346/0.201/9.42 × 10^{-7}	F/F/F/F/5.18 × 10^{-5}	F/F/F/F/1.03 × 10^{-2}	F/F/F/F/5.28 × 10^{-2}	1043/27,237/13,510/1.104/8.22 × 10^{-7}
raydan1 1000	320/362/470/0.141/6.35 × 10^{-7}	527/14,585/6962/1.225/8.97 × 10^{-7}	F/F/F/F/1.91 × 10^{-5}	1029/29,815/13,278/2.239/7.75 × 10^{-7}	F/F/F/F/4.19 × 10^{-5}
raydan1 1200	404/391/568/0.101/2.46 × 10^{-7}	601/17,005/7837/1.484/7.66 × 10^{-7}	1971/2141/3002/0.527/8.77 × 10^{-7}	850/24,008/10,762/1.970/9.80 × 10^{-7}	F/F/F/F/1.31 × 10^{-4}
raydan2 5000	17/52/17/0.067/9.00 × 10^{-7}	21/456/230/0.265/7.22 × 10^{-8}	17/52/17/0.033/9.35 × 10^{-7}	38/972/472/0.461/2.67 × 10^{-7}	21/458/205/0.207/8.34 × 10^{-7}
raydan2 10,000	17/56/20/0.069/1.63 × 10^{-7}	18/364/149/0.374/1.55 × 10^{-7}	23/60/28/0.089/6.20 × 10^{-7}	17/303/143/0.326/3.56 × 10^{-7}	33/780/348/0.819/8.82 × 10^{-7}
raydan2 20,000	13/50/14/0.100/7.46 × 10^{-7}	29/662/305/1.224/3.55 × 10^{-7}	20/83/44/0.185/6.68 × 10^{-7}	14/238/81/0.602/8.97 × 10^{-7}	21/413/163/0.867/5.83 × 10^{-7}
diagonal1 50	69/67/72/0.047/6.89 × 10^{-7}	152/4438/2120/0.247/8.73 × 10^{-7}	79/244/169/0.017/7.22 × 10^{-7}	204/5701/2669/0.239/9.74 × 10^{-7}	188/4914/2287/0.218/5.44 × 10^{-7}
diagonal1 80	100/321/228/0.063/8.82 × 10^{-7}	227/6698/3133/0.355/5.49 × 10^{-7}	F/F/F/F/1.81 × 10^{-5}	215/5707/2704/0.267/8.70 × 10^{-7}	F/F/F/F/1.11 × 10^{-5}
diagonal2 5000	821/775/1183/1.415/7.04 × 10^{-7}	1614/47,941/22,089/25.459/9.28 × 10^{-7}	470/415/652/0.717/7.50 × 10^{-7}	F/F/F/F/1.33 × 10^{-1}	619/15,686/7753/8.985/9.65 × 10^{-7}
diagonal2 10,000	1043/974/1505/3.503/5.89 × 10^{-7}	F/F/F/F/1.93 × 10^{-4}	873/775/1235/5.941/9.42 × 10^{-7}	F/F/F/F/2.20 × 10^{-1}	819/21,318/10,576/30.265/7.97 × 10^{-7}
diagonal3 150	164/575/422/0.058/7.53 × 10^{-7}	194/5161/2419/0.294/7.94 × 10^{-7}	F/F/F/F/2.95 × 10^{-5}	F/F/F/F/5.99 × 10^{-6}	F/F/F/F/2.13 × 10^{-5}
diagonal3 200	144/655/421/0.056/9.54 × 10^{-7}	F/F/F/F/3.24 × 10^{-6}	F/F/F/F/2.96 × 10^{-5}	F/F/F/F/2.85 × 10^{-5}	F/F/F/F/6.48 × 10^{-5}
bv 1000	19/13/23/0.152/8.69 × 10^{-7}	29/805/356/3.956/6.69 × 10^{-7}	82/25/90/0.705/7.51 × 10^{-7}	33/931/383/4.416/9.42 × 10^{-7}	143/4195/1754/19.534/8.70 × 10^{-7}
bv 2000	5/5/5/0.125/5.84 × 10^{-7}	14/385/172/5.830/3.11 × 10^{-7}	42/9/44/0.964/8.45 × 10^{-7}	9/225/93/3.519/4.59 × 10^{-7}	123/3714/1509/55.763/5.77 × 10^{-7}

Table 2. *Cont.*

Problems	JYJLL	KD	AN1	HZ	LFZ
ie 215	$13/40/13/0.977/8.36 \times 10^{-7}$	$13/183/64/3.470/7.39 \times 10^{-7}$	$15/42/15/1.083/9.11 \times 10^{-7}$	$17/294/120/6.959/9.65 \times 10^{-8}$	$40/889/394/21.664/3.31 \times 10^{-7}$
singx 900	$F/F/F/F/1.46 \times 10^{-5}$	$1479/44741/16{,}996/180.509/2.36 \times 10^{-7}$	$223/580/477/3.497/8.63 \times 10^{-7}$	$F/F/F/F/4.08 \times 10^{-5}$	$366/9279/4616/38.993/4.12 \times 10^{-7}$
band 3	$20/64/22/0.024/1.48 \times 10^{-7}$	$67/1838/860/0.089/6.29 \times 10^{-7}$	$40/96/57/0.007/4.87 \times 10^{-7}$	$77/2156/1006/0.097/6.95 \times 10^{-7}$	$46/1025/467/0.048/8.94 \times 10^{-8}$
gauss 3	$23/38/28/0.029/4.72 \times 10^{-7}$	$47/1263/533/0.085/6.39 \times 10^{-7}$	$12/30/13/0.003/6.97 \times 10^{-7}$	$14/245/114/0.018/8.13 \times 10^{-7}$	$30/743/361/0.052/7.35 \times 10^{-7}$
jensam 2	$69/113/98/0.024/8.86 \times 10^{-7}$	$119/3545/1697/0.155/6.69 \times 10^{-7}$	$37/139/82/0.009/2.89 \times 10^{-7}$	$211/6152/2894/0.271/6.99 \times 10^{-7}$	$188/5415/2603/0.242/3.45 \times 10^{-8}$
lin 200	$2/2/2/0.012/1.71 \times 10^{-13}$	$2/2/2/0.006/1.71 \times 10^{-13}$	$2/2/2/0.006/1.71 \times 10^{-13}$	$2/2/2/0.006/1.71 \times 10^{-13}$	$2/2/2/0.007/1.71 \times 10^{-13}$
lin 500	$2/2/2/0.021/9.93 \times 10^{-14}$	$2/2/2/0.021/9.93 \times 10^{-14}$	$2/2/2/0.020/9.93 \times 10^{-14}$	$2/2/2/0.021/9.93 \times 10^{-14}$	$2/2/2/0.020/9.93 \times 10^{-14}$
lin 500	$2/2/2/0.023/9.93 \times 10^{-14}$	$2/2/2/0.021/9.93 \times 10^{-14}$	$2/2/2/0.022/9.93 \times 10^{-14}$	$2/2/2/0.020/9.93 \times 10^{-14}$	$2/2/2/0.020/9.93 \times 10^{-14}$
osb2 11	$1266/1180/1833/0.415/9.27 \times 10^{-7}$	$F/F/F/F/2.54 \times 10^{-3}$	$615/483/831/0.200/5.98 \times 10^{-7}$	$F/F/F/F/6.43 \times 10^{-4}$	$F/F/F/F/5.58 \times 10^{-2}$
pen1 50	$121/233/192/0.035/8.32 \times 10^{-7}$	$222/6619/2668/0.414/7.03 \times 10^{-7}$	$58/152/89/0.016/2.20 \times 10^{-7}$	$691/21{,}027/8471/1.298/7.18 \times 10^{-7}$	$21/427/184/0.027/7.53 \times 10^{-7}$
pen1 90	$260/411/417/0.069/9.04 \times 10^{-7}$	$340/10{,}061/4115/0.750/8.46 \times 10^{-7}$	$104/284/194/0.035/2.08 \times 10^{-7}$	$348/10{,}351/4128/0.768/4.68 \times 10^{-7}$	$93/2475/1171/0.190/5.09 \times 10^{-9}$
pen2 105	$135/360/265/0.116/5.37 \times 10^{-7}$	$224/6471/3041/1.603/7.59 \times 10^{-7}$	$F/F/F/F/4.85 \times 10^{-5}$	$F/F/F/F/7.28 \times 10^{-5}$	$F/F/F/F/2.24 \times 10^{-4}$
pen2 100	$F/F/F/F/1.09 \times 10^{-5}$	$274/7792/3688/1.809/6.18 \times 10^{-7}$	$F/F/F/F/8.89 \times 10^{-6}$	$430/12{,}042/5778/3.055/8.95 \times 10^{-7}$	$F/F/F/F/7.94 \times 10^{-6}$
rose 2	$166/261/263/0.063/4.08 \times 10^{-7}$	$911/28{,}228/11{,}815/1.219/8.56 \times 10^{-7}$	$58/239/145/0.016/4.35 \times 10^{-7}$	$1075/32{,}570/13{,}545/1.405/9.49 \times 10^{-7}$	$148/3833/1917/0.174/8.42 \times 10^{-8}$
rosex 100	$377/471/577/0.089/1.30 \times 10^{-7}$	$1077/33{,}420/13{,}972/2.948/8.43 \times 10^{-7}$	$66/317/190/0.041/8.67 \times 10^{-7}$	$811/23{,}841/10{,}191/2.349/2.55 \times 10^{-7}$	$148/4021/1954/0.423/8.91 \times 10^{-8}$
rosex 500	$292/415/463/2.101/3.95 \times 10^{-7}$	$1160/36{,}016/15{,}090/88.764/7.03 \times 10^{-7}$	$60/241/146/0.751/6.09 \times 10^{-7}$	$891/26{,}552/11{,}144/65.695/7.84 \times 10^{-7}$	$150/4066/1960/10.359/4.00 \times 10^{-7}$
sing 4	$896/1018/1373/0.142/7.31 \times 10^{-7}$	$380/11052/4591/0.502/6.68 \times 10^{-7}$	$153/388/314/0.032/5.24 \times 10^{-7}$	$F/F/F/F/5.50 \times 10^{-6}$	$360/8971/4386/0.427/9.01 \times 10^{-7}$
trid 50	$82/79/93/0.029/9.78 \times 10^{-7}$	$201/5551/2510/0.475/5.26 \times 10^{-7}$	$81/89/96/0.019/9.39 \times 10^{-7}$	$196/5383/2389/0.451/4.84 \times 10^{-7}$	$317/8411/4103/0.723/9.60 \times 10^{-7}$
trid 90	$87/87/101/0.026/3.19 \times 10^{-7}$	$240/6330/2957/0.737/9.36 \times 10^{-7}$	$103/105/126/0.033/8.56 \times 10^{-7}$	$182/4501/2012/0.531/4.11 \times 10^{-7}$	$207/5447/2632/0.654/8.16 \times 10^{-7}$
wood 4	$351/391/509/0.074/8.86 \times 10^{-7}$	$631/17{,}914/8171/0.865/7.97 \times 10^{-7}$	$210/342/341/0.037/7.14 \times 10^{-7}$	$1284/36{,}519/15{,}935/1.768/7.91 \times 10^{-7}$	$375/9721/4715/0.479/4.73 \times 10^{-7}$

5. Discussion of Results

First, it is known, from the characters of the performance profiles, that the higher the curve in the figures, the better the associated method. Second, by summarizing the convergence analysis and numerical reports in Tables 1 and 2 and Figures 1–4, the proposed JYJLL-SCGM shows the following three advantages.

(i) It has good global convergence under mild assumptions.

(ii) It is practically effective, at least for the 108 tested instances.

(iii) It is the most effective in the five tested methods. In addition, the numerical performance of AN1 [13] method and the JYJLL-SCGM is relatively stable.

Of cause, the advantages above are attributed to the choice techniques (8) and (9) for the spectral parameter and the conjugate parameter.

6. Conclusions

The contributions of this work are two aspects. The one is to design a new computing schemes for the spectral parameter which ensures the $\theta_k > 1$. The other is to propose a new computing method for the conjugate parameter. These two techniques such that the search directions always possess descent property independent of the line search technique. As a result, the presented JYJLL-SCGM possesses global convergence if using the Wolfe line search to yield the steplength. A lot of numerical experiments in comparison with relative methods show that our SCGM is promising.

As further works, we think the following two problems are interesting and worth studying. The one is to design new approaches for the spectral parameter to guarantee $\theta_k > 1$, such as, combining the Newton direction and some new quasi-Newton equations, or the new conjugate conditions. The other is to find new computation techniques for the conjugate parameter with the help of the existing approaches, for example, the hybrid parameter, the three–term conjugate parameter et al.

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