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Clones of Terms of a Fixed Variable

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Abstract: Let τ be a type of algebras. This paper introduces special terms of type τ called terms of a fixed variable. It turns out that the set of all terms of a fixed variable of type τ forms the many-sorted clone satisfying the superassociative law as identity, under the many-sorted superposition operations. Moreover, based on terms of a fixed variable of type τ, hypersubstitutions of a fixed variable and the related closed identities of a fixed variable and closed variety of a fixed variable are introduced and studied.

Keywords: clone; term; terms of a fixed variable; hypersubstitutions of a fixed variable; closed identities of a fixed variable; closed variety of a fixed variable

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1. The Many-Sorted Clone of Terms of a Fixed Variable

Term is one of an important concepts in universal algebras. In particular, the structures called clones and partial clones of terms of a given type have been widely studied ([1–6]). For clones of terms, we refer to [7], and, for universal algebras, we refer to [8].

Let τ be a type of algebra. In this paper, we introduce special terms of type τ called terms of a fixed variable. We first prove that the set of all terms of a fixed variable of type τ forms the many-sorted clone. Moreover, under the many-sorted superposition operations, the many-sorted clone obtained satisfies the superassociative law as identity. Using the notion of terms of a fixed variable of type τ, we introduce hypersubstitutions mapping operation symbols to terms of a fixed variable and study the related closed identities of a fixed variable and closed variety of a fixed variable.

Let $\tau = (n_i)_{i\in I}$ be a type of algebras with n_i-ary operation symbols f_i indexed by some non-empty set I (Here, let us consider $n_i \in \mathbb{N}^+$ for all $i \in I$; let $\mathbb{N}^+ = \{1, 2, \ldots\}$). Let $X_n = \{x_1, \ldots, x_n\}$ be an n-elements alphabet of *variables*, disjoint from the set of all operation symbols $\{f_i \mid i \in I\}$. An *n-ary term* of type τ is inductively defined by:

(i) $x_i \in X_n$ is an n-ary term of type τ; and
(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ, and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The set of all n-ary terms of type τ will be denoted by $W_\tau(X_n)$. Indeed, $W_\tau(X_n)$ contains the variables $x_1, \ldots, x_n$ and is closed under finite application of (ii). We then have the many-sorted set of all terms of type τ:

$$W_\tau(X) = (W_\tau(X_n))_{n\in\mathbb{N}^+}.$$

For $m, n \in \mathbb{N}^+$, define *many-sorted superposition operation*

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

by:

(i) $S^n_m(x_i, t_1, \ldots, t_n) \ = \ t_i$ if $x_i \in X_n$; and
(ii) $S^n_m(f_i(t'_1, \ldots, t'_{n_i}), t_1, \ldots, t_n) \ = \ f_i(S^n_m(t'_1, t_1, \ldots, t_n), \ldots, S^n_m(t'_{n_i}, t_1, \ldots, t_n))$.

This leads to form the many-sorted algebra or the clone of all terms of type τ:

$$clone(\tau) \ = \ ((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}).$$

The $clone(\tau)$ satisfies the following identities:

(C1) $\tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n) \approx$
$\tilde{S}^p_m(\tilde{Z}, \tilde{S}^n_m(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n_m(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n)), m, n, p \in \mathbb{N}^+$;
(C2) $\tilde{S}^n_m(\lambda_i, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_i, n, m \in \mathbb{N}^+, 1 \leq i \leq n$;
(C3) $\tilde{S}^n_n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}, n \in \mathbb{N}^+$.

Here, $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n$ are variables, $\tilde{S}^n_m$ are operation symbols, and λ_i are nullary operation symbols. It is observed that, for $m \ = \ n \ = \ 1$, (C1) is the associative law where $\tilde{S}^n_m$ acts as the associative binary operation.

Many sorted algebras of the same kind as the $clone(\tau)$ are said to be abstract clones if they satisfy the identities (C1)–(C3). The following is an example of an abstract clone. Let V be a variety of one-sorted total algebras of type τ, and let $Id_n(V)$ denote the set of all n-ary identities $s \approx t$ satisfied in V (i.e., $s, t \in W_\tau(X_n)$). We then have the many-sorted quotient algebra:

$$clone(V) \ = \ clone(\tau)/Id(V)$$

where $Id(V)$ is a congruence generated by $(Id_n(V))_{n\in\mathbb{N}^+}$; this, $clone(V)$, is an abstract clone.

A hypersubstitution of type τ is defined as a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ such that, for each $i \in I$, $\sigma(f_i) \in W_\tau(X_{n_i})$. A hypersubstitution $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ can be uniquely extended to the mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ by:

(i) $\hat{\sigma}[x_i] \ = \ x_i$ if $x_i \in X$; and
(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] \ = \ S^{n_i}_m(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_m)$.

The set of all hypersubstitutions of type τ, denoted by $Hyp(\tau)$, forms a monoid under the associative binary operation defined by:

$$\sigma \circ_h \sigma' \ = \ \hat{\sigma} \circ \sigma'$$

for all $\sigma, \sigma' \in Hyp(\tau)$; a hypersubstitution $\sigma_{id} : \{f_i \mid i \in I\} \to W_\tau(X)$ defined by $\sigma_{id}(f_i) \ = \ f_i(x_1, \ldots, x_{n_i})$ for all $i \in I$ acts as an identity; the operation $\circ$ is composition of functions. The monoid $Hyp(\tau)$ can be regarded as many-sorted mappings: For $n \in \mathbb{N}^+$, let $I_n \subseteq I$ be the set of all indexes such that f_j with $j \in I_n$ is an n-ary. Let $F^n_\tau \ = \ \{f_j \mid j \in I_n\}$. Then, for $\sigma \in Hyp(\tau)$, $\sigma \ = \ (\sigma_n)_{n\in\mathbb{N}^+}$, where $\sigma_n(F^n_\tau) \subseteq W_\tau(X_n)$. In addition, let

$$Hyp(\tau) \ = \ (Hyp_n(\tau))_{n\in\mathbb{N}^+}$$

such that, for each $n \in \mathbb{N}^+$, $Hyp_n(\tau)$ denotes the set of all σ_n. Finally, it would be remarked here that, for $\sigma \in Hyp(\tau)$,

$$\hat{\sigma} \ = \ (\hat{\sigma}_n)_{n\in\mathbb{N}^+}.$$

In order to introduce the main concept of this paper, we need the following. For $t \in W_\tau(X_n)$, the set of all variables of the term t will be denoted by $var(t)$. Hence, we introduce terms of a fixed variable of type τ as follows:

Definition 1. *An n-*ary terms of a fixed variable *of type τ is inductively defined by:*

(i) $x_i \in X_n$ *are n-ary terms of a fixed variable; and*

(ii) *if* $t_1, \ldots, t_{n_i}$ *are n-ary terms of a fixed variable, and if* $var(t_j) = var(t_k)$ *for all* $1 \leq j \leq k \leq n_i$*, then* $f_i(t_1, \ldots, t_{n_i})$ *is an n-ary term of a fixed variable.*

Let $W^{fv}_\tau(X_n)$ be the set of all n-ary terms of a fixed variable of type τ; i.e., $W^{fv}_\tau(X_n)$ contains $x_1, \ldots, x_n$ and is closed under finite applications of (ii). Analogous to the case of terms, we have the many-sorted set of terms of a fixed variable of type τ:

$$W^{fv}_\tau(X) = (W^{fv}_\tau(X_n))_{n\in\mathbb{N}^+}.$$

For example, let us consider the type $\tau = (2)$ with a binary operation symbol f. Then,

$$x_1, x_2, f(x_1, x_1), f(x_2, x_2), f(x_1, f(x_1, x_1)), f(f(x_2, x_2), x_2) \in W^{fv}_\tau(X_2);$$

$$x_1, x_2, x_3, f(x_1, x_1), f(x_2, x_2), f(x_3, x_3), f(x_1, f(x_1, x_1)), f(f(x_1, x_1), x_1),$$
$$f(x_2, f(x_2, x_2)), f(x_3, f(x_3, x_3)), f(f(x_3, x_3), f(x_3, x_3)) \in W^{fv}_\tau(X_3).$$

If we consider the type $\tau = (3)$ with 3-ary operation symbol f, then, for $W^{fv}_\tau(X_2)$,

$$x_1, x_2, f(x_1, x_1, x_1), f(x_2, x_2, x_2), f(x_1, x_1, f(x_1, x_1, x_1)),$$

$$f(f(x_2, x_2, x_2), x_2, x_2) \in W^{fv}_\tau(X_2).$$

For $W^{fv}_\tau(X_3)$, we have

$$x_1, x_2, x_3, f(x_1, x_1, x_1), f(x_2, x_2, x_2), f(x_3, x_3, x_3),$$

$$f(x_1, x_1, f(x_1, x_1, x_1)), f(x_2, x_2, f(x_2, x_2, x_2)), f(x_3, x_3, f(x_3, x_3, x_3)),$$
$$f(f(x_3, x_3, x_3), f(x_3, x_3, x_3), f(x_3, x_3, x_3)) \in W^{fv}_\tau(X_3).$$

Remark 1. *Observe that, for* $t \in W^{fv}_\tau(X_n)$*,* $var(t) = \{x_j\}$ *for some* $x_j \in X_n$*.*

The following lemma shows that $(W^{fv}_\tau(X_n))_{n\in\mathbb{N}^+}$ is closed under the superposition operations.

Lemma 1. *For* $n, m \in \mathbb{N}^+$*, if* $f_i(t_1, \ldots, t_{n_i}) \in W^{fv}_\tau(X_n)$*, and if* $t'_1, \ldots, t'_n \in W^{fv}_\tau(X_m)$*, then*

$$S^n_m(f_i(t_1, \ldots, t_{n_i}), t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m).$$

Proof. We have, by definition of operation S^n_m, that

$$S^n_m(f_i(t_1, \ldots, t_{n_i}), t'_1, \ldots, t'_n) = f_i(S^n_m(t_1, t'_1, \ldots, t'_n), \ldots, S^n_m(t_{n_i}, t'_1, \ldots, t'_n)).$$

Then, we must show that Equations (1) and (2) hold:

(1) $S^n_m(t_1, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m), \ldots, S^n_m(t_{n_i}, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m)$; and

(2) $var(S^n_m(t_1, t'_1, \ldots, t'_n)) = \cdots = var(S^n_m(t_{n_i}, t'_1, \ldots, t'_n))$.

To show Equation (1), we will only show that $S^n_m(t_1, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m)$; for $S^n_m(t_2, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m)$, …, $S^n_m(t_{n_i}, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m)$ can be proved similarly. If $t_1 \in X_n$, put $t_1 = x_1$ (without loss of generality), then

$$S^n_m(t_1, t'_1, \ldots, t'_n) = t'_1 \in W^{fv}_\tau(X_m).$$

Assume that $t_1 = f_i(t''_1, \ldots, t''_{n_i})$ such that

$$S^n_m(t''_1, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m), \ldots, S^n_m(t''_{n_i}, t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m).$$

Since

$$S^n_m(t_1, t'_1, \ldots, t'_n) = f_i(S^n_m(t''_1, t'_1, \ldots, t'_n), \ldots, S^n_m(t''_{n_i}, t'_1, \ldots, t'_n))$$

and

$$var(t''_1) = \cdots = var(t''_{n_i}),$$

then

$$var(S^n_m(t''_1, t'_1, \ldots, t'_n)) = \cdots = var(S^n_m(t''_{n_i}, t'_1, \ldots, t'_n)).$$

Thus,

$$S^n_m(f_i(t''_1, \ldots, t''_{n_i}), t'_1, \ldots, t'_n) \in W^{fv}_\tau(X_m).$$

From

$$var(t_1) = \cdots = var(t_{n_i}),$$

it follows that

$$var(S^n_m(t_1, t'_1, \ldots, t'_n)) = \cdots = var(S^n_m(t_{n_i}, t'_1, \ldots, t'_n)).$$

□

Using Lemma 1, we then have many-sorted mappings

$$S^n_m : W^{fv}_\tau(X_n) \times (W^{fv}_\tau(X_m))^n \to W^{fv}_\tau(X_m)$$

for $m, n \in \mathbb{N}^+$. Hence, we obtain another many-sorted algebra:

$$clone_{fv}(\tau) = ((W^{fv}_\tau(X_n))_{n \in \mathbb{N}^+}, (S^n_m)_{m,n \in \mathbb{N}^+}, (x_i)_{i \le n \in \mathbb{N}^+}).$$

Moreover, sine $(W^{fv}_\tau(X_n))_{n \in \mathbb{N}^+} \subseteq (W_\tau(X_n))_{n \in \mathbb{N}^+}$, the following theorem follows.

Theorem 1. *$clone_{fv}(\tau)$ satisfies (C1)–(C3).*

2. Variable Fixed Hypersubstitutions

Based on the set of terms of a fixed variable, we introduce fv-hypersubstitutions as follows:

Definition 2. *A hypersubstitution $\sigma \in Hyp(\tau)$ is called an fv-*hypersubstitution *of type τ if, for all $i \in I$, $\sigma(f_i) \in W^{fv}_\tau(X_{n_i})$.*

For example, let consider the type $\tau = (2)$ with a binary operation symbol f. A hypersubstitution $\sigma : \{f\} \to W_\tau(X_2)$ of type τ such that $\sigma(f) = f(x_1, f(x_1, x_1))$ is an fv-hypersubstitution. However, $\sigma' : \{f\} \to W_\tau(X_2)$ of type τ such that $\sigma'(f) = f(x_1, x_2)$ is not an fv-hypersubstitution.
Let $Hyp^{fv}(\tau)$ denote the set of all fv-hypersubstitutions of type τ.

In order to prove the next results, the following lemma is needed.

Lemma 2. *If $\sigma \in Hyp^{fv}(\tau)$, then $\hat{\sigma} : W^{fv}_\tau(X) \to W^{fv}_\tau(X)$.*

Proof. Let $\sigma \in Hyp^{fv}(\tau)$, and let $t \in W_\tau^{fv}(X)$. We must show that $\hat{\sigma}[t] \in W_\tau^{fv}(X)$. It is clear, for $t \in X$, that $\hat{\sigma}[t] \in W_\tau^{fv}(X)$. Assume that $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau^{fv}(X)$ such that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}] \in W_\tau^{fv}(X)$. Since $var(t_1) = \cdots = var(t_{n_i})$,

$$var(\hat{\sigma}[t_1]) = \cdots = var(\hat{\sigma}[t_{n_i}]).$$

Hence, since $\sigma(f_i) \in W_\tau^{fv}(X)$,

$$\hat{\sigma}[t] = S_m^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$$

is a term of a fixed variable. □

For $n \in \mathbb{N}^+$, let $\sigma_n : F_\tau^n \to W_\tau^{fv}(X_n)$. Define

(i) $\hat{\sigma}_n[x_i] = x_i$ if $x_i \in X_n$; and
(ii) $\hat{\sigma}_n[f_i(t_1, \ldots, t_{n_i})] = S_n^{n_i}(\sigma_n(f_i), \hat{\sigma}_n[t_1], \ldots, \hat{\sigma}_n[t_{n_i}])$.

Thus, an fv-hypersubstitution σ of type τ can be regarded as $(\sigma_n)_{n\in\mathbb{N}^+}$. As $Hyp(\tau)$, we consider

$$Hyp^{fv}(\tau) = (Hyp_n^{fv}(\tau))_{n\in\mathbb{N}^+}.$$

By Lemma 2, we have that $(Hyp_n^{fv}(\tau))_{n\in\mathbb{N}^+}$ is a subsemigroup of $(Hyp_n(\tau))_{n\in\mathbb{N}^+}$.

The section will be closed with the following useful result.

Theorem 2. *The extension of $\sigma \in (Hyp_n^{fv}(\tau))_{n\in\mathbb{N}^+}$ is an endomorphism on $clone_{fv}(\tau)$.*

Proof. Let $\sigma \in (Hyp_n^{fv}(\tau))_{n\in\mathbb{N}^+}$. By Lemma 2,

$$\hat{\sigma} : (W_\tau^{fv}(X_n))_{n\in\mathbb{N}^+} \to (W_\tau^{fv}(X_n))_{n\in\mathbb{N}^+}.$$

Let $t_0 \in W_\tau^{fv}(X_n)$, and let $t_1, \ldots, t_n \in W_\tau^{fv}(X_m)$. We have to show, by induction on the complexity of t_0 that

$$\hat{\sigma}_m[S_m^n(t_0, t_1, \ldots, t_n)] = S_m^n(\hat{\sigma}_n[t_0], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]).$$

Let $t_0 \in X_n$; put $t_0 = x_1$; then,

$$\begin{aligned}
\hat{\sigma}_m[S_m^n(t_0, t_1, \ldots, t_n)] &= \hat{\sigma}_m[S_m^n(x_1, t_1, \ldots, t_n)] \\
&= \hat{\sigma}_m[t_1]; \\
S_m^n(\hat{\sigma}_n[t_0], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) &= S_m^n(\hat{\sigma}_n[x_1], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \\
&= S_m^n(x_1, \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \\
&= \hat{\sigma}_m[t_1].
\end{aligned}$$

Let $t_0 = f_i(t_1', \ldots, t_{n_i}')$ such that

$$\hat{\sigma}_m[S_m^n(t_k', t_1, \ldots, t_n)] = S_m^n(\hat{\sigma}_n[t_k'], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n])$$

for all $1 \leq k \leq n_i$. Consider

$$\begin{aligned}
& \hat{\sigma}_m[S^n_m(t_0, t_1, \ldots, t_n)] \\
= \;& \hat{\sigma}_m[S^n_m(f_i(t'_1, \ldots, t'_{n_i}), t_1, \ldots, t_n)] \\
= \;& \hat{\sigma}_m[f_i(S^n_m(t'_1, t_1, \ldots, t_n), \ldots, S^n_m(t'_{n_i}, t_1, \ldots, t_n)] \\
= \;& S^{n_i}_m(\sigma_m(f_i), \hat{\sigma}_m[S^n_m(t'_1, t_1, \ldots, t_n)], \ldots, \hat{\sigma}_m[S^n_m(t'_{n_i}, t_1, \ldots, t_n)]) \\
= \;& S^{n_i}_m(\sigma_m(f_i), S^n_m(\hat{\sigma}_n[t'_1], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]), \ldots, S^n_m(\hat{\sigma}_n[t'_{n_i}], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n])) \\
= \;& S^n_m(S^{n_i}_m(\sigma_m(f_i), \hat{\sigma}_n[t'_1], \ldots, \hat{\sigma}_n[t'_{n_i}]), \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \\
= \;& S^n_m(\hat{\sigma}_m[t_0], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]).
\end{aligned}$$

This completes the proof. □

3. Variable Fixed Closure

We begin this section with the following definition.

Definition 3. *Let V be a variety of algebras of type τ. An identity $s \approx t$ in $Id(V)$ is said to be an fv-*identity *of V if $s, t \in W^{fv}_\tau(X_n)$ for some $n \in \mathbb{N}^+$.*

For example, the identity

$$f(x, x) \approx x$$

in the variety of bands (semigroups with the elements are idempotent).

Let $Id^{fv}(V)$ denote the set of all fv-identities of the variety V. Consider

$$Id^{fv}_n(V) \;=\; \{s \approx t \mid s \approx t \in Id(V), s, t \in W^{fv}_\tau(X_n)\}.$$

Then,

$$Id^{fv}(V) \;=\; (Id^{fv}_n(V))_{n \in \mathbb{N}^+}$$

is a many-sorted equivalence on $(W^{fv}_\tau(X_n))_{n \in \mathbb{N}^+}$. However, this is not an equational theory of type τ because it is not closed under substitution.

Now, the definition of fv-identities was already defined. The purpose of next theorem is to show that the set of such identity is a congruence on the clone of terms of a fixed variable.

Theorem 3. *Let V be a variety of (one-sorted algebras) of type τ. Then, $(Id^{fv}_n(V))_{n \in \mathbb{N}^+}$ is a congruence on $clone_{fv}(\tau)$.*

Proof. That $(Id^{fv}_n(V))_{n \in \mathbb{N}^+}$ is preserved by the constant fundamental operations of $clone_{fv}(\tau)$ is clear.

Let $s \approx t \in Id^{fv}_n(V)$, and let $s_1 \approx t_1, \ldots, s_n \approx t_n \in Id^{fv}_m(V)$. We must show that

$$S^n_m(s, s_1, \ldots, s_n) \approx S^n_m(t, t_1, \ldots, t_n) \in Id^{fv}_m(V).$$

This follows from

$$S^n_m(s, s_1, \ldots, s_n), S^n_m(t, t_1, \ldots, t_n) \in W^{fv}_\tau(X_m)$$

and

$$S^n_m(s, s_1, \ldots, s_n) \approx S^n_m(t, t_1, \ldots, t_n) \in Id_m(V).$$

□

Using Theorem 3, for a variety V of type τ, we have the many-sorted algebra:

$$clone_{fv}(V) \ = \ clone_{fv}(\tau)/Id^{fv}(V).$$

Definition 4. *Let V be a variety of one-sorted total algebras of type τ. Let $(Hyp_n^{fv}(\tau))_{n\in\mathbb{N}^+}$ be a many-sorted semigroup of fv-hypersubstitutions of type τ. An fv-identity $s \approx t$ of the variety V is said to be an fv-*closed identity *of V if*

$$\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V)$$

for $s,t \in W_\tau^{fv}(X_n)$, $\sigma_n \in Hyp_n^{fv}(\tau)$, and $n \in \mathbb{N}^+$.

In addition, we call V variable fixed closed *if*

$$\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V)$$

for all $s,t \in W_\tau^{fv}(X_n)$, $\sigma_n \in Hyp_n^{fv}(\tau)$, and $n \in \mathbb{N}^+$.

The necessary condition implies that the variety V is variable fixed closed will be given in the theorem as follows:

Theorem 4. *Let V be a variety of one-sorted total algebras of type τ. If the congruence $(Id_n^{vf}(V))_{n\in\mathbb{N}^+}$ is fully invariant, then V is variable fixed closed.*

Proof. Assume that $(Id_n^{vf}(V))_{n\in\mathbb{N}^+}$ is fully invariant; then, $(Id_n^{vf}(V))_{n\in\mathbb{N}^+}$ is a fully invariant congruence on $clone_{vf}(\tau)$. Let $(\sigma_n)_{n\in\mathbb{N}^+} \in (Hyp_n^{vf}(\tau))_{n\in\mathbb{N}^+}$. By Theorem 2, $(\sigma_n)_{n\in\mathbb{N}^+}$ is an endomorphism on $clone_{vf}(\tau)$. Then, by assumption,

$$\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V)$$

for all $n \in \mathbb{N}^+$. Thus, $s \approx t$ is a variable fixed closed identity in V. Hence, V is variable fixed closed. □

Recall that, for a variety V of one-sorted total algebras of type τ, $Id^{vf}(V)$ is a congruence on $clone_{vf}(\tau)$ by Theorem 3. We then form the quotient algebra

$$clone_{vf}(V) \ = \ clone_{vf}(\tau)/Id^{vf}(V).$$

Note that we have a many-sorted natural homomorphism

$$nat_{n,Id^{vf}(V)} : clone_{vf}(\tau) \to clone_{vf}(\tau)(V)$$

such that

$$nat_{n,Id^{vf}(V)}(t) \ = \ [t]_{Id_n^{vf}(V)}.$$

Applying the result of Theorem 2, then we obtain the following theorems.

Theorem 5. *Let V be a variety of one-sorted total algebras of type τ. If $s \approx t \in Id^{vf}(V)$ is an identity in $clone_{vf}(V)$, then $s \approx t$ is variable fixed closed identity of V.*

Proof. Assume that $s \approx t \in Id^{vf}(V)$ is an identity in $clone_{vf}(V)$. Let $\sigma \in Hyp^{vf}(\tau)$; then, $\hat{\sigma}$: $clone_{vf}(\tau) \to clone_{vf}(\tau)$ is an endomorphism by Theorem 2. Thus,

$$nat_{n,Id^{vf}(V)} \circ \hat{\sigma}_n : clone_{vf}(\tau) \to clone_{vf}(V)$$

is a homomorphism. By assumption,

$$nat_{n,Id^{vf}(V)} \circ \hat{\sigma}_n(s) \ = \ nat_{n,Id^{vf}(V)} \circ \hat{\sigma}_n(t).$$

That is,

$$nat_{n,Id^{vf}(V)}(\hat{\sigma}_n[s]) \ = \ nat_{n,Id^{vf}(V)}(\hat{\sigma}_n[t]).$$

Thus,

$$[\hat{\sigma}_n[s]]_{Id^{vf}(V)} \;=\; [\hat{\sigma}_n[t]]_{Id^{vf}(V)}.$$

That is,

$$\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id^{vf}(V).$$

Hence, $s \approx t$ is a variable fixed closed identity of V. □

Let V be a variety of one-sorted total algebras of type τ. Define

$$HId_n^{vf}(V) \;=\; \{s \approx t \mid s, t \in W_\tau^{vf}(X_n), \hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V) \text{ for all } \sigma_n \in Hyp_n^{vf}(\tau)\}.$$

Thus,

$$(HId_n^{vf}(V))_{n\in\mathbb{N}^+}$$

is an equivalence on $(W_\tau^{vf}(X_n))_{n\in\mathbb{N}^+}$.

Theorem 6. *Let V be a variety of one-sorted total algebras of type τ. Then, $(HId_n^{vf}(V))_{n\in\mathbb{N}^+}$ is a congruence on $clone_{vf}(\tau)$.*

Proof. Let $s \approx t \in HId_n^{vf}(V)$; and let $s_1 \approx t_1, \ldots, s_n \approx t_n \in HId_m^{vf}(V)$. Then, $\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n^{vf}(V)$; and $\hat{\sigma}_m[s_1] \approx \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[s_n] \approx \hat{\sigma}_m[t_n] \in Id_m^{vf}(V)$ for all $\sigma \in (Hyp_n^{vf}(\tau))_{n\in\mathbb{N}^+}$.

We have to show that

$$\hat{\sigma}_m[S_m^n(s, s_1, \ldots, s_n)] \approx \hat{\sigma}_m[S_m^n(t, t_1, \ldots, t_n)] \in Id_m^{vf}(V)$$

for all $\sigma \in (Hyp_n^{vf}(\tau))_{n\in\mathbb{N}^+}$. In addition, hence $(HId_n^{vf}(V))_{n\in\mathbb{N}^+}$ is a congruence on $clone_{vf}(\tau)$. Let $\sigma \in (Hyp_n^{vf}(\tau))_{n\in\mathbb{N}^+}$. Since $\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n^{vf}(V)$,

$$\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V) \text{ and } \hat{\sigma}_n[s], \hat{\sigma}_n[t] \in W_\tau^{vf}(X_n).$$

Similarly, since $\hat{\sigma}_m[s_1] \approx \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[s_n] \approx \hat{\sigma}_m[t_n] \in Id_m^{vf}(V)$,

$$\hat{\sigma}_m[s_1] \approx \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[s_n] \approx \hat{\sigma}_m[t_n] \in Id_n(V)$$

and

$$\hat{\sigma}_m[s_1], \ldots, \hat{\sigma}_m[s_n], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n] \in W_\tau^{vf}(X_m).$$

By $\hat{\sigma}_n[s] \approx \hat{\sigma}_n[t] \in Id_n(V)$ and $\hat{\sigma}_m[s_1] \approx \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[s_n] \approx \hat{\sigma}_m[t_n] \in Id_n(V)$,

$$S_m^n(\hat{\sigma}_n[s], \hat{\sigma}_m[s_1], \ldots, \hat{\sigma}_m[s_n]) \approx S_m^n(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \in Id_m(V).$$

From $\hat{\sigma}_n[s], \hat{\sigma}_n[t] \in W_\tau^{vf}(X_n)$ and $\hat{\sigma}_m[s_1], \ldots, \hat{\sigma}_m[s_n], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n] \in W_\tau^{vf}(X_m)$, it follows that

$$S_m^n(\hat{\sigma}_n[s], \hat{\sigma}_m[s_1], \ldots, \hat{\sigma}_m[s_n]), S_m^n(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \in W_\tau^{vf}(X_m).$$

Thus,

$$S_m^n(\hat{\sigma}_n[s], \hat{\sigma}_m[s_1], \ldots, \hat{\sigma}_m[s_n]) \approx S_m^n(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \ldots, \hat{\sigma}_m[t_n]) \in Id_m^{vf}(V).$$

Since $\hat{\sigma}_m$ is an endomorphism (Theorem 2),

$$\hat{\sigma}_m[S_m^n(s, s_1, \ldots, s_n)] \approx \hat{\sigma}_m[S_m^n(t, t_1, \ldots, t_n)] \in Id_m^{vf}(V).$$

This completes the proof. □

4. Conclusions

In this paper, the concept of terms of a fixed variable is introduced. The set of such terms together with the superposition operation forms the clone since it satisfied the superassociative law. Furthermore, the notion of hypersubstitution which maps operation symbols to the term of a fixed variable is investigated. Finally, the relationship between closed identities of a fixed variable and closed variety of a fixed variable is considered.

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References

1. Denecke, K. Menger Algebras and Clones of Terms. *East-West J. Math.* **2013**, *5*, 179–193.
2. Denecke, K. The Partial Clone of Linear Terms. *Sib. Math. J.* **2016**, *57*, 589–598. [CrossRef]
3. Denecke, K.; Freiberg, L. The Algebra of Strongly Full Terms. *Novi. Sad. J. Math.* **2014**, *34*, 87–98.
4. Denecke, K.; Jampaclon, P. Clones of Full Terms. *Algebra Discrete Math.* **2004**, *4*, 1–11.
5. Puapong, S.; Leeratanawalee, S. The Algebra of Generalized Full Terms. *Int. J. Open Problems Compt. Math.* **2011**, *4*, 54–65.
6. Sarawut, P. Some Algebraic Properties of Generalized Clone Automorphisms. *Acta Univ. Apulensis* **2015**, *41*, 165–175.
7. Denecke, K.; Wismath, S.L. *Hyperidentities and Clones*; Gordon and Breach Science Publisher: London, UK, 2000.
8. Denecke, K.; Wismath, S.L. *Universal Algebra and Applications in Theoretical Computer Science*; Chapman & Hall/RCR: Boca Raton, FL, USA; London, UK; New York, NY, USA, 2002.