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Integral Domains in Which Every Nonzero w -Flat Ideal Is w -Invertible

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Abstract: Let D be an integral domain and w be the so-called w -operation on D . We define D to be a w -FF domain if every w -flat w -ideal of D is of w -finite type. This paper presents some properties of w -FF domains and related domains. Among other things, we study the w -FF property in the polynomial extension, the t -Nagata ring and the pullback construction.

Keywords: w -flat; w -FF domain; w -FP domain; Prüfer v -multiplication domain; FF domain; FP domain

1. Introduction

1.1. Motivations and Results

Let D be an integral domain. In order to study some questions of divisibility and finite generation, Sally and Vasconcelos first studied an integral domain in which every flat ideal is finitely generated [1]. Later, El Baghdadi et al. called a domain D an *FF domain* if every flat ideal of D is finitely generated [2]. In the same paper, as a weaker version, the authors also defined a weakly FF property on D such that each faithfully flat ideal of D is finitely generated. It is well known that a nonzero finitely generated ideal in an integral domain is flat if and only if it is invertible. Therefore an FF domain is an integral domain in which each nonzero flat ideal is invertible, and hence a weakly FF domain can be regarded as an integral domain in which each faithfully flat ideal is invertible. In [3], Anderson and Zafrullah introduced the notion of an *LPI domain*; that is, an integral domain in which every nonzero locally principal ideal is invertible. Since an ideal is faithfully flat if and only if it is locally principal, the concept of weakly FF domains is precisely the same as that of LPI domains. Hence an FF domain is always an LPI domain.

Recently, in [4], the authors introduced and studied the notion of a w -LPI domain; that is, an integral domain in which every nonzero w -locally principal ideal is w -invertible. (Relevant definitions and notations will be given in Section 1.2; and for more on w -LPI domains, the readers can refer to [5].) This is, the w -operation version of LPI domains. In the view of FF domains, we can define a weakly w -FF domain; that is, an integral domain in which each w -faithfully flat w -ideal is of w -finite type. Then the notion of w -LPI domains coincides with that of weakly w -FF domains because an ideal is w -faithfully flat if and only if it is w -locally principal [6] (Theorem 2.7). Thus it is natural and reasonable to study an integral domain in which every w -flat w -ideal is of w -finite type. We call an integral domain with this property a w -FF domain. Then this is both a w -operation version of FF domains and a stronger domain than a weakly w -FF domain. As an important class of FF domains, we call D an *FP domain* if every flat ideal of D is principal.

In this paper, we investigate w -FF domains and some related domains. Among other things, we study the w -FF property in the polynomial extension, the t -Nagata ring and the pullback construction. More precisely, we prove that D is a w -FF domain if and only if the polynomial ring $D[X]$ is a w -FF domain, if and only if the t -Nagata ring $D[X]_{N_v}$ is an FF domain (Theorem 1); if $K + M$ is t -local, then $D + M$ is a w -FF domain if and only if D is a field and $K + M$ is a w -FF domain, where K is a field containing D as a subring and M is a nonzero maximal ideal of $K + M$ (Corollary 4); and the power series ring over an FP domain is an FP domain (Theorem 3). Finally, we give several examples which represent relationships among integral domains related to w -FF domains (Example 1).

1.2. Preliminaries

For the reader's better understanding, we review some definitions and notations. Let D be an integral domain with quotient field $qf(D)$, and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D . For an $I \in \mathbf{F}(D)$, set $I^{-1} := \{a \in qf(D) \mid aI \subseteq D\}$. The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$ is called the v -operation on D ; the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t := \bigcup \{J_v \mid J \text{ is a nonzero finitely generated fractional subideal of } I\}$ called the t -operation on D ; and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w := \{a \in qf(D) \mid aJ \subseteq I \text{ for some finitely generated ideal } J \text{ of } D \text{ such that } J_v = D\}$ is called the w -operation on D . We say that an $I \in \mathbf{F}(D)$ is a v -ideal (or *divisorial ideal*) (respectively, t -ideal, w -ideal) if $I_v = I$ (respectively, $I_t = I$, $I_w = I$). It is well known that $I \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$. Clearly, if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. Let $*$ be t or w . Then a *maximal $*$ -ideal* means a $*$ -ideal which is maximal among proper integral $*$ -ideals. Let $*\text{-Max}(D)$ be the set of maximal $*$ -ideals of D . It was shown in [7] (Corollary 2.17) that the notion of maximal t -ideals coincides with that of maximal w -ideals. Additionally, it is well known that $t\text{-Max}(D) \neq \emptyset$ if D is not a field; and a maximal t -ideal is a prime ideal. A $*$ -ideal $I \in \mathbf{F}(D)$ is said to be of *$*$ -finite type* if $I = J_*$ for some finitely generated ideal J of D . An $I \in \mathbf{F}(D)$ is said to be *$*$ -invertible* if $(II^{-1})_* = D$ (or equivalently, $II^{-1} \not\subseteq M$ for all maximal $*$ -ideals M of D). If $*$ is d , then we just say "invertible" instead of " d -invertible." It was shown that the notion of t -invertibility is precisely the same as that of w -invertibility [7] (Theorem 2.18); and an $I \in \mathbf{F}(D)$ is t -invertible if and only if I_t is of t -finite type and ID_M is principal for all maximal t -ideals M of D [8] (Corollary 2.7). A nonzero ideal I of D is said to be *w -flat* if I is t -locally flat; i.e., ID_M is flat for all maximal t -ideals M of D . It was shown in [9] (Corollary 3.6(4)) that an $I \in \mathbf{F}(D)$ is w -flat if and only if I_w is w -flat.

Let D be an integral domain, M a D -module, and $E(M)$ the injective envelope of M . Then the w -closure of M is defined by $M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some finitely generated ideal } J \text{ of } D \text{ such that } J_v = D\}$. Let A and B be D -modules. In [6], the w -tensor product of A and B is defined as $A \hat{\otimes}_D B := (A \otimes_D B)_w$. We call M a *w -flat module* if for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of D -modules, $0 \rightarrow M \hat{\otimes}_D A \rightarrow M \hat{\otimes}_D B \rightarrow M \hat{\otimes}_D C \rightarrow 0$ is also exact. It was shown that M is w -flat if and only if M_P is a flat D_P -module for every maximal w -ideal P of D .

2. Main Results

Throughout this section, we always assume that D is an integral domain. It is well known that an ideal I of D is flat if and only if ID_P is flat for all prime ideals P of D [10] (Proposition 3.10); so a nonzero flat ideal is w -flat. Additionally, it was shown that a nonzero flat ideal is a t -ideal [11] (Theorem 1.4). Hence we have the following lemma.

Lemma 1. *A w -flat w -ideal is a t -ideal.*

Proof. Let I be a w -flat w -ideal of D . Then ID_M is a flat ideal of D_M for all maximal t -ideals M of D ; so we have

$$\begin{aligned} I_t &= \bigcap_{M \in t\text{-Max}(D)} (I_t)D_M \\ &\subseteq \bigcap_{M \in t\text{-Max}(D)} (ID_M)_t \\ &= \bigcap_{M \in t\text{-Max}(D)} ID_M \\ &\subseteq \bigcap_{M \in t\text{-Max}(D)} (I_w)D_M \\ &= I_w \\ &= I, \end{aligned}$$

where the second equality follows from the fact that a nonzero flat ideal is a t -ideal. Thus I is a t -ideal. $\square$

Our next result is the w -flat analogue of Zafrullah's result which states that a t -finite type t -ideal is flat if and only if it is invertible [12] (Proposition 1). This shows that if a w -ideal I is w -invertible but not invertible, then I is w -flat (see Lemma 2) but not flat.

Lemma 2. Let I be a w -finite type w -ideal of D . Then I is w -flat if and only if I is w -invertible.

Proof. ($\Rightarrow$) Since I is a w -finite type w -ideal of D , $I = J_w$ for some finitely generated ideal J of D . Let M be any maximal t -ideal of D . Then $ID_M = JD_M$; so ID_M is a finitely generated flat ideal of D_M because I is w -flat in D . Hence ID_M is invertible and so ID_M is principal. Thus I is w -invertible [8] (Corollary 2.7).

($\Leftarrow$) Let M be a maximal t -ideal of D . Since I is w -invertible, ID_M is principal; so ID_M is flat. Thus I is w -flat. $\square$

We define D to be a w -FF domain if every w -flat w -ideal of D is of w -finite type (equivalently, w -invertible by Lemma 2). Clearly, if every nonzero ideal of D is a w -ideal, then D is an FF domain if and only if D is a w -FF domain. A simple example of w -FF domains is a Krull domain, because D is a Krull domain if and only if every nonzero ideal of D is w -invertible [13] (Theorem 3.6). (Recall that D is a Krull domain if there exists a family $\{V_\alpha\}_{\alpha \in \Lambda}$ of rank-one essential discrete valuation overrings of D such that $D = \bigcap_{\alpha \in \Lambda} V_\alpha$ and this intersection has finite character, i.e., each nonzero nonunit in D is a nonunit in only finitely many of valuation overrings V_α .) Additionally, we say that D is of finite t -character if every nonzero nonunit in D is contained in only finitely many maximal t -ideals of D , and D is a Mori domain if it satisfies the ascending chain condition on integral v -ideals. It is well known that in a Mori domain, $v = t$, so every t -ideal of a Mori domain is of t -finite type.

Proposition 1. The following assertions hold.

- (1) A w -FF domain is an FF domain.
- (2) D is a w -FF domain if and only if every nonzero w -flat ideal of D is w -invertible.
- (3) A t -locally FF domain with finite t -character is a w -FF domain.
- (4) A Mori domain is a w -FF domain.

Proof. (1) Recall that a nonzero flat ideal is a t -ideal. Thus the assertion follows directly from [12] (Proposition 1), because a flat ideal of a w -FF domain is of w -finite type.

(2) This is an immediate consequence of Lemma 2 and the fact that a nonzero ideal I is w -flat if and only if I_w is w -flat.

(3) Let I be a w -flat w -ideal of a t -locally FF domain D , and let $0 \neq x \in I$. Since D has finite t -character, there exist only finitely many maximal t -ideals of D containing x , say $M_1, \dots, M_n$. Fix an index $i \in \{1, \dots, n\}$. Then ID_{M_i} is a flat ideal of D_{M_i} . Since D_{M_i} is an FF domain, $ID_{M_i} = (x_{i1}, \dots, x_{im_i})D_{M_i}$ for some $x_{i1}, \dots, x_{im_i} \in I$. Let J be the ideal of D generated by x and $\{x_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m_i\}$. Then $ID_{M_i} = JD_{M_i}$. If N is a maximal t -ideal of D which is distinct from $M_1, \dots, M_n$, then $ID_N = D_N = JD_N$. Therefore $ID_M = JD_M$ for all maximal t -ideals M of D , and hence $I = J_w$ [7] (Theorem 4.3) (or [14] (Corollary 2.11)). Thus D is a w -FF domain.

(4) Let I be a w -flat w -ideal of D . Then I is a t -ideal by Lemma 1. Since D is a Mori domain, I is of t -finite type. Let M be a maximal w -ideal of D . Then ID_M is a flat ideal of a Mori domain D_M [15] (§2, Théorème 2); so ID_M is invertible [12] (Corollary 4). Therefore ID_M is principal. Hence I is w -invertible [8] (Corollary 2.7), which implies that I is of w -finite type. Thus D is a w -FF domain. $\square$

Recall that D is a *Prüfer domain* (respectively, *Prüfer v -multiplication domain* (PvMD)) if every nonzero finitely generated ideal of D is invertible (respectively, t -invertible). It is well known that every ideal of D is flat if and only if D is a Prüfer domain [16] (Theorem 4.2). We give the PvMD version of this result.

Proposition 2. *The following statements are equivalent.*

- (1) D is a PvMD.
- (2) Every w -ideal of D is w -flat.
- (3) Every nonzero finitely generated ideal of D is w -flat.
- (4) Every w -finite type w -ideal of D is w -flat.

Proof. (1) $\Rightarrow$ (2) Let I be a w -ideal of D . Then $I = \bigcup \{J_w \mid J \text{ is a nonzero finitely generated subideal of } I\}$. Let M be any maximal t -ideal of D . Then we have

$$\begin{aligned} ID_M &= \left(\bigcup \{J_w \mid J \text{ is a nonzero finitely generated subideal of } I\} \right) D_M \\ &= \bigcup \{(J_w)D_M \mid J \text{ is a nonzero finitely generated subideal of } I\}. \end{aligned}$$

Since D is a PvMD, J_w is w -invertible; so J_w is w -flat by Lemma 2. Therefore $(J_w)D_M$ is flat, and hence ID_M is flat [17] (Proposition 10.3). Thus I is w -flat.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) These implications follow because a nonzero ideal I is w -flat if and only if I_w is w -flat.

(4) $\Rightarrow$ (1) This is an immediate consequence of Lemma 2. $\square$

It is well known that D is a PvMD if and only if $D[X]$ is a PvMD [8] (Theorem 3.7). Thus by Theorem 2, we obtain the following result.

Corollary 1. *Every w -ideal of D is w -flat if and only if every w -ideal of $D[X]$ is w -flat.*

Recall that D is a *Dedekind domain* if every nonzero ideal of D is invertible. We give new characterizations of Krull domains and Dedekind domains via the (w) -FF property.

Corollary 2. *The following assertions hold.*

- (1) D is a Krull domain if and only if D is both a PvMD and a w -FF domain.
- (2) D is a Dedekind domain if and only if D is both a Prüfer domain and an FF domain.
- (3) A valuation domain V is an FF domain if and only if V is a rank-one discrete valuation domain.

Proof. (1) The necessary condition follows because every nonzero ideal of a Krull domain is t -invertible. For the converse, assume that D is both a PvMD and a w -FF domain. Let I be a nonzero ideal of D .

Then by Proposition 2, I_w is w -flat and so I_w is of w -finite type. Therefore by Lemma 2, I_w , and hence, I , are w -invertible. Thus D is a Krull domain.

(2) This follows directly from (1) because D is a Dedekind domain (respectively, Prüfer domain) if and only if D is a Krull domain (respectively, PvMD) and each nonzero ideal of D is a w -ideal.

(3) Note that a valuation domain is a quasi-local Prüfer domain and that a quasi-local domain is a Dedekind domain if and only if it is a rank-one discrete valuation domain. Thus the result is an immediate consequence of (2). $\square$

Let $D[X]$ denote the polynomial ring over D . For an $f \in D[X]$, the *content* of f , denoted by $c_D(f)$, is the ideal of D generated by the coefficients of f . Let $N_v(D) := \{f \in D[X] \mid c_D(f)_v = D\}$. (If there is no confusion, we simply denote $c_D(f)$ and $N_v(D)$ by $c(f)$ and N_v , respectively.) Then N_v is a (saturated) multiplicative subset of $D[X]$, and the quotient ring $D[X]_{N_v}$ is called the *t -Nagata ring* of D . For the sake of convenience, we sometimes use the notation $D\{X\}$ instead of $D[X]_{N_v}$.

Lemma 3. *The following assertions hold.*

- (1) *If A is a flat ideal of $D[X]$, then A/XA is a flat ideal of D .*
- (2) *If I is a w -flat ideal of $D[X]$, then I/XI is a w -flat ideal of D .*

Proof. (1) Assume that A is a flat ideal of $D[X]$. Since $\bigcap_{n=0}^{\infty} X^n D[X] = (0)$, we can find an integer $k \geq 0$ such that $A \subseteq X^k D[X]$ but $A \not\subseteq X^{k+1} D[X]$; therefore, $A = X^k B$ for some ideal B of $D[X]$ with $B \not\subseteq X D[X]$. Hence we may assume that $A \not\subseteq X D[X]$ by replacing A with B , because $A \cong B$. Note that for any integer $n \geq 1$, $A \cap X^n D[X] = X^n A$ [18] (Lemma 1.4). Therefore the natural homomorphism $A/X^n A \rightarrow D[X]/X^n D[X]$ is a monomorphism for any integer $n \geq 1$. In particular, $A/XA \rightarrow D[X]/XD[X]$ is a monomorphism. Now the flatness follows from the facts that $A/XA \otimes_D M \cong A \otimes_{D[X]} (D[X]/XD[X]) \otimes_D M$ and $(D[X]/XD[X]) \otimes_D M \cong M$ for any D -module M .

(2) Note that if I is a w -flat ideal of $D[X]$, then $I \cap X^n D[X] = X^n I$ for all integers $n \geq 1$ [4] (Lemma 2.8), and a D -module M is a w -flat D -module if and only if $M[X]$ is a w -flat $D[X]$ -module [4] (Theorem 1.7). Thus the result follows by applying the same argument as in the proof of (1). $\square$

Next, we study the w -FF property of polynomial rings and t -Nagata rings.

Theorem 1. *The following statements are equivalent.*

- (1) *D is a w -FF domain.*
- (2) *$D[X]$ is a w -FF domain.*
- (3) *$D[X]_{N_v}$ is a w -FF domain.*
- (4) *$D\{X\}$ is an FF domain.*

Proof. (1) $\Rightarrow$ (2) Let I be a w -flat w -ideal of $D[X]$. Then by Lemma 3(2), I/XI is a w -flat ideal of D . Since D is a w -FF domain, $(I/XI)_w$ is a w -finite type ideal of D . Let $f_1, \dots, f_m \in I$ such that $I = (XI + (f_1, \dots, f_m))_w$, and let M be a maximal w -ideal of D . Then $ID_M[X] = XID_M[X] + (f_1, \dots, f_m)D_M[X]$ [7] (Theorem 4.3) (or [14] (Proposition 2.10)); so $ID_M[X] = X^n ID_M[X] + (f_1, \dots, f_m)D_M[X]$ for all integers $n \geq 1$. Let g be a nonzero element of $ID_M[X]$ with degree l . Then for any integer $s > l$, there exists an element $h \in ID_M[X]$ and $h_1, \dots, h_m \in D_M[X]$ such that

$g = X^s h + h_1 f_1 + \cdots + h_m f_m$. Hence $c_{D_M}(g) \subseteq (c_D(f_1) + \cdots + c_D(f_m))D_M$; so $c_{D_M}(ID_M[X]) = (c_D(f_1) + \cdots + c_D(f_m))D_M$. Therefore we have

$$\begin{aligned} c_D(I)_w &= \bigcap_{M \in w\text{-Max}(D)} c_D(I)D_M \\ &= \bigcap_{M \in w\text{-Max}(D)} c_{D_M}(ID_M[X]) \\ &= \bigcap_{M \in w\text{-Max}(D)} (c_D(f_1) + \cdots + c_D(f_m))D_M \\ &= (c_D(f_1) + \cdots + c_D(f_m))_w. \end{aligned}$$

Thus I is of w -finite type [4] (Theorem 2.6).

(2) $\Rightarrow$ (3) Let M be a maximal t -ideal of $D[X]$ and let $ID[X]_{N_v}$ be a w -flat w -ideal of $D[X]_{N_v}$, where I is an ideal of $D[X]$. If $M \cap D \neq (0)$, then $M \cap D$ is a maximal t -ideal of D and $M = (M \cap D)[X]$ [19] (Proposition 1.1). Since $(M \cap D)[X]_{N_v}$ is a maximal t -ideal of $D[X]_{N_v}$ and $(ID[X]_{N_v})_{MD[X]_{N_v}} = ID[X]_M$, $ID[X]_M$ is a flat ideal. If $M \cap D = (0)$, then $D[X]_M$ is a local PID; so $ID[X]_M$ is principal. Therefore $ID[X]_M$ is flat. Hence I is w -flat in $D[X]$. Since $D[X]$ is a w -FF domain, I_w is a w -finite type w -ideal of $D[X]$, and so $ID[X]_{N_v}$ is of w -finite type. Thus $D[X]_{N_v}$ is a w -FF domain.

(3) $\Rightarrow$ (4) Proposition 1(1).

(4) $\Rightarrow$ (1) Let I be a w -flat w -ideal of D . Then $ID[X]_{N_v}$ is a flat ideal of an FF domain $D[X]_{N_v}$ [4] (Theorem 1.7); so $ID[X]_{N_v}$ is invertible. Hence I is w -invertible [8] (Corollary 2.5), and thus D is a w -FF domain. $\square$

Let S be a (not necessarily saturated) multiplicative subset of D , and for set $D + XD_S[X] := \{f \in D_S[X] \mid \text{the constant term of } f \text{ belongs to } D\}$. Then $D[X] \subseteq D + XD_S[X] \subseteq D_S[X]$, and $D + XD_S[X]$ is both the symmetric algebra $S_D(D_S)$ of D_S considered as a D -module and the direct limit of $D[\frac{X}{s}]$, where $s \in S$. This kind of ring is usually called the *composite polynomial ring* and was first introduced by Costa, Mott and Zafrullah in [20]. For more on this construction, the readers can refer to [21–27].

Corollary 3. Let S be a (not necessarily saturated) multiplicative subset of D . Then the following assertions hold.

- (1) $D + XD_S[X]$ is a w -FF domain if and only if D is a w -FF domain and S consists of units of D .
- (2) (cf. [25] (Corollary 1.7)) $D + XD_S[X]$ is a Krull domain if and only if D is a Krull domain and S consists of units of D .

Proof. (1) $(\Rightarrow)$ Assume that $D + XD_S[X]$ is a w -FF domain, and fix an $s \in S$. Then $\left\{ \frac{X}{s^n}(D + XD_S[X]) \right\}_{n \geq 1}$ is an ascending chain of principal ideals of $D + XD_S[X]$, so $\bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X])$ is flat [17] (Proposition 10.3). Therefore $\bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X])$ is a w -flat w -ideal of $D + XD_S[X]$. Since $D + XD_S[X]$ is a w -FF domain, $\bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X])$ is of w -finite type, so $\bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X]) = (f_1, \dots, f_k)_w$ for some $f_1, \dots, f_k \in \bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X])$. Hence we can find a suitable integer $m \geq 1$ such that $(f_1, \dots, f_k)_w \subseteq \frac{X}{s^m}(D + XD_S[X])$. Therefore $\bigcup_{n \geq 1} \frac{X}{s^n}(D + XD_S[X]) = \frac{X}{s^m}(D + XD_S[X])$, which implies that s is a unit in D . Thus S consists of units of D and D is a w -FF domain by Theorem 1.

$(\Leftarrow)$ This implication was shown in Theorem 1.

(2) Recall that D is a Krull domain if and only if $D[X]$ is a Krull domain [28] (Theorem 43.11). Thus the equivalence follows directly from (1) and Corollary 2(1). $\square$

Let M denote a nonzero maximal ideal of an integral domain T , $K := T/M$ be the residue field, $\varphi : T \rightarrow K$ be the natural projection and D be a proper subring of K . Assume that $T = K + M$. Then $R = D + M$, which is the integral domain arising from the following pullback (R, D, T, K) of canonical homomorphisms:

$$\begin{array}{ccc} R = \varphi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M. \end{array} \quad (\square)$$

We next study the w -FF property of R when T is t -local. (Recall that an integral domain is t -local if it is quasi-local whose maximal ideal is a t -ideal.) To do this, we need a simple lemma whose proof is word for word that of [2] (Lemma 3.5).

Lemma 4. *Given a pullback diagram $(\square)$, if R is a w -FF domain, then D is a field.*

Proof. Let d be a nonzero element of D and m be a nonzero element of M . Then $\frac{m}{d^n} \in M$ for all positive integers n ; so $\frac{m}{d^n} \in R$. Since $\{\frac{m}{d^n}R\}_{n \geq 1}$ forms an ascending chain of principal ideals, $\bigcup_{n \geq 1} \frac{m}{d^n}R$ is a w -flat w -ideal of R . Since R is a w -FF domain, a simple modification of the proof of Corollary 3(1) shows that $\bigcup_{n \geq 1} \frac{m}{d^n}R = \frac{m}{d^k}R$ for some integer $k \geq 1$. Hence $\frac{m}{d^{k+1}} \in \frac{m}{d^k}R$, which indicates that d is a unit in D . Thus D is a field. $\square$

Let $R_1 \subseteq R_2$ be an extension of integral domains. Recall that $R_1 \subseteq R_2$ is a t -linked extension (or R_2 is t -linked over R_1) if $I^{-1} = R_1$ for a nonzero finitely generated ideal I of R_1 implies $(IR_2)^{-1} = R_2$.

Theorem 2. *Consider a pullback diagram $(\square)$. If T is a w -FF domain and M is a t -ideal of T , then R is a w -FF domain if and only if D is a field.*

Proof. $(\Rightarrow)$ This was shown in Lemma 4.

$(\Leftarrow)$ Assume that D is a field. In order to avoid the trivial case, we assume that K properly contains D . Let I be a w -flat w -ideal of R . Note that T is t -linked over R [29] (Proposition 3.1), so $I \otimes_R T \cong IT$ is a w -flat ideal of T [4] (Lemma 1.5). Since T is a w -FF domain, there exists a finitely generated ideal $J \subseteq I$ of R such that $(IT)_w = (JT)_w$. Note that M is a maximal t -ideal of R [30] (Proposition 2.1), because D is a field. Hence IR_M is flat in R_M . If IR_M is not principal, then $IR_M = IMR_M$ [1] (Lemma 2.1); so $IT_M = IMT_M$. Note that M is a maximal t -ideal of T ; so we obtain

$$JT_M = ((JT)_w)T_M = ((IT)_w)T_M = IT_M,$$

which shows that $JT_M = JMT_M$. By Nakayama's lemma, $JT = (0)$, a contradiction. Hence $IR_M = aR_M$ for some $a \in I$. Let $F = J + aR$. Then $IR_M = FR_M$ and $(IT)_w = (FT)_w$. Let N be a maximal t -ideal of R with $N \neq M$. Then there exists the unique prime ideal Q of T with $Q \cap R = N$ and $R_N = T_Q$ [31] (page 335). Note that Q is a maximal t -ideal of T [29] (Lemma 3.3). Therefore we have

$$IR_N = IT_Q = (IT)_wT_Q = (FT)_wT_Q = FT_Q = FR_N.$$

Hence we obtain

$$I = \bigcap_{P \in t\text{-Max}(R)} IR_P = \bigcap_{P \in t\text{-Max}(R)} FR_P = F_w.$$

Thus R is a w -FF domain. $\square$

Corollary 4. *With the notation as in $(\square)$, if T is t -local, then R is a w -FF domain if and only if D is a field and T is a $(w\text{-})$ FF domain.*

Proof. By Lemma 4 and Theorem 2, it suffices to show that if R is a w -FF domain, then T is a w -FF domain. Since T is t -local, T is a w -FF domain if and only if T is an FF domain. Thus the result is an immediate consequence of Proposition 1(1) and [2] (Corollary 3.7). $\square$

It is well known that D is a UFD if and only if every t -ideal of D is principal [13] (Section 1). Additionally, it was shown that every nonzero flat ideal is a t -ideal, and hence every flat ideal of a UFD is principal. We will say that D is an *FP domain* (respectively, *w-FP domain*) if every flat ideal (respectively, w -flat w -ideal) of D is principal. If every nonzero ideal is a w -ideal, then the notion of FP domains coincides with that of w -FP domains. Additionally, it is clear that an FP domain (respectively, w -FP domain) is an FF domain (respectively, w -FF domain).

Proposition 3. *The following assertions hold.*

- (1) *A w -FP domain is an FP domain.*
- (2) *Every invertible ideal of an FP domain is principal.*
- (3) *Every w -invertible w -ideal of a w -FP domain is principal.*

Proof. (1) This follows directly from the fact that any nonzero flat ideal is a w -flat w -ideal.

For Propositions (2) and (3): these results come easily from the fact that an invertible (respectively, w -invertible) ideal is flat (respectively, w -flat). $\square$

Corollary 5. *Let $V = K + M$ be a valuation domain and set $R = D + M$, where K is a field, D is a subring of K and M is the maximal ideal of V . Then the following conditions are equivalent.*

- (1) *R is a w -FF domain.*
- (2) *R is an FF domain.*
- (3) *R is a w -FP domain.*
- (4) *R is an FP domain.*
- (5) *V is a rank-one discrete valuation domain and D is a field.*

Proof. (1) $\Rightarrow$ (2) This was already shown in Proposition 1(1).

(2) $\Rightarrow$ (3) Assume that R is an FF domain. Then D is a field [2] (Corollary 3.8); so R is t -local. Thus R is a w -FP domain.

(3) $\Rightarrow$ (4) Proposition 3(1).

(4) $\Rightarrow$ (5) Since R is an FP domain, D is a field [2] (Corollary 3.8); so R is t -local. Hence R is a (w) -FF domain. Since V is t -local, V is a rank-one discrete valuation domain by Corollaries 2(3) and 4.

(5) $\Rightarrow$ (1) Note that V is a pullback as in $(\square)$ and is t -local. Thus the implication comes directly from Corollaries 2(3) and 4. $\square$

We give new characterizations of UFDs and PIDs in terms of the (w) -FP property.

Proposition 4. *The following statements hold.*

- (1) *D is a UFD if and only if D is both a PvMD and a w -FP domain.*
- (2) *D is a PID if and only if D is both a Prüfer domain and an FP domain.*

Proof. (1) $(\Rightarrow)$ This implication is an immediate consequence of Lemma 1 and Corollary 2(1).

$(\Leftarrow)$ Let I be a nonzero ideal of D . Since D is a PvMD, I_w is w -flat by Proposition 2; so I_w is principal because D is a w -FP domain. Thus D is a UFD.

(2) Note that D is a PID (respectively, Prüfer domain) if and only if D is a UFD (respectively, PvMD) and each nonzero ideal of D is a w -ideal; and that if every nonzero ideal of D is a w -ideal, then D is a w -FP domain if and only if D is an FP domain. Thus the equivalence follows from (1). $\square$

We next show that the power series ring over an FP domain is an FP domain.

Theorem 3. *If D is an FP domain, then the power series ring $D[[X]]$ is an FP domain.*

Proof. We adapt the proof of [1] (Theorem 4.1). Let I be a nonzero flat ideal of $D[[X]]$. Since $\bigcap_{n=0}^{\infty} X^n D[[X]] = (0)$, there exists a nonnegative integer m such that $I \subseteq X^m D[[X]]$ but $I \not\subseteq X^{m+1} D[[X]]$; so $I = X^m J$ for some ideal J of $D[[X]]$ with $J \not\subseteq X D[[X]]$. Hence we may assume that $I \not\subseteq X D[[X]]$ by replacing I with J , because $I \cong J$. Let I_0 be the ideal of D generated by constant terms of elements of I . Then $I_0 \cong I/XI$ and I_0 is flat. Since D is an FP domain, I_0 is principal. Let $f \in I$ such that $I = XI + fD[[X]]$. Then $I = X^n I + fD[[X]]$ for all positive integers n . Thus $I = fD[[X]]$ [32] (Proposition 12, §2, Chapter III). $\square$

For the sake of the reader's better understanding, we give a diagram of some integral domains related to w -FF domains.

Figure 1. Integral domains related to w -FF domains.

The next examples show that any of the reverses in “UFD $\Rightarrow$ FP domain $\Rightarrow$ FF domain,” in “ w -FP domain $\Rightarrow$ FP domain,” in “ w -FP domain $\Rightarrow$ w -FF domain” and “Mori domain $\Rightarrow$ w -FF domain” do not generally hold. We also give an example of a t -locally FF domain which is not a w -FF domain. (This shows that the hypothesis “finite t -character” in Proposition 1(3) is essential.). Furthermore, we construct an example of a PvMD that is not a w -FF domain. Finally, we give an example of a w -FP domain D such that $D[[X]]$ is not a w -FP domain.

- Example 1.** (1) Let D be a Dedekind domain (respectively, Krull domain) which is not a PID (respectively, UFD). Then by Corollary 2 and Proposition 4, D is an FF domain (respectively, w -FF domain) that is not an FP domain (respectively, w -FP domain).
- (2) Let D be a t -almost Dedekind domain which is not a Krull domain. (Recall that D is a t -almost Dedekind domain if D_M is a discrete valuation domain for each maximal t -ideal M of D .) Then D is both a PvMD and a t -locally (w) -FF domain (cf. Corollary 2(3)). Note that a t -almost Dedekind domain is a Krull domain if and only if it has finite t -character; and by Corollary 2(1), a t -almost Dedekind domain is a w -FF domain if and only if it is a Krull domain. Thus D is not a w -FF domain.
- (3) This example is due to [33] (Section 4). Let $D = L[X, Y, Z]$, where L is a perfect field of characteristic 2 and X, Y, Z satisfying $Z^2 - X^3 - Y^7 = 0$. Then $D[[X]]$ is not a UFD. Since D is a UFD, D is an FP domain by Propositions 3(1) and 4(1). Thus $D[[X]]$ is an FP domain by Theorem 3. Note that D is a Krull domain; so $D[[X]]$ is a Krull domain [28] (Corollary 44.11); so $D[[X]]$ is a PvMD. Thus by Proposition 4(1), $D[[X]]$ is not a w -FP domain.
- (4) Let $\mathbb{R}$ (respectively, $\mathbb{C}$) be the field of real (respectively, complex) numbers. Then $\mathbb{C}[[X]]$ is a t -local w -FF domain; so by Corollary 4, $\mathbb{R} + X\mathbb{C}[[X]]$ is also a w -FF domain.
- (5) Let $D = L[X, Y, Z]/(XY - Z^2)$, where L is a field and X, Y, Z are indeterminates over L . Let x, y, z denote the images of X, Y, Z , respectively, and let $M = (x, y, z)$. Then D_M is a two-dimensional integrally closed Noetherian domain that is not a UFD [34] (Example 7). Let I be a flat ideal of D_M . Then I is finitely generated, and hence I is invertible. Therefore I is principal because D_M is local. Thus D_M is an

- FP domain. However, since D_M is not a UFD, there exists a w -invertible ideal J of D_M such that J_w is not principal. Therefore J_w is w -flat by Lemma 2 but not principal. Thus D_M is not a w -FP domain.
- (6) Let D be a Mori domain such that $D[X]$ is not a Mori domain. (The existence of such a domain D was shown in [35] (Proposition 8.3).) Then D is a w -FF domain; so by Theorem 1, $D[X]$ is a w -FF domain.
- (7) Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Q}[X]$ is a w -FF domain and $X\mathbb{Q}[X]$ is a (maximal) t -ideal of $\mathbb{Q}[X]$; so $\mathbb{Z} + X\mathbb{Q}[X]$ is not a w -FF domain by Theorem 2 (or Corollary 3(1)). However, $\mathbb{Z} + X\mathbb{Q}[X]$ is a PvMD [20] (Theorem 4.43) (or [21] (Corollary 3.8)).

We end this paper with the following two questions.

Question 1. (1) Is an FF domain generally a w -FF domain?

(2) Can one characterize integral domains in which each t -ideal is w -flat? (Note that each t -ideal of D is flat if and only if D is a generalized GCD-domain [12] (Proposition 10). (Recall that D is a generalized GCD-domain if for every nonzero finitely generated ideal I of D , I_t is invertible.))

3. Conclusions

In this paper, we introduced the concept of a w -FF domain; that is, an integral domain in which every nonzero w -flat ideal is w -invertible. As in Figure 1, the class of w -FF domains contains Noetherian domains and Mori domains. Additionally, w -FF domains are good examples of FF domains and w -LPI domains. We extend the w -FF property to the polynomial ring, the t -Nagata ring, the composite polynomial ring and the $D + M$ construction. Unfortunately, as in Question 1(1), we could not find an example which shows that an FF domain need not be a w -FF domain. In ensuing work, we are going to find such an example.

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