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On 2-Variables Konhauser Matrix Polynomials and Their Fractional Integrals

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Abstract: In this paper, we first introduce the 2-variables Konhauser matrix polynomials; then, we investigate some properties of these matrix polynomials such as generating matrix relations, integral representations, and finite sum formulae. Finally, we obtain the fractional integrals of the 2-variables Konhauser matrix polynomials.

Keywords: Konhauser matrix polynomial; generating matrix function; integral representation; fractional integral

MSC: 33C25; 33C45; 33D15; 65N35

1. Introduction

Special functions play a very important role in analysis, physics, and other applications, and solutions of some differential equations or integrals of some elementary functions can be expressed by special functions. In particular, the family of special polynomials is one of the most useful and applicable family of special functions. The Konhauser polynomials which were first introduced by J.D.E. Konhauser [1] include two classes of polynomials $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$, where $Y_n^\alpha(x;k)$ are polynomials in x and $Z_n^\alpha(x;k)$ are polynomials in x^k, $\alpha > -1$ and $k \in \mathbb{Z}^+$. Explicit expressions for the polynomials $Z_n^\alpha(x;k)$ are given by

$$Z_n^\alpha(x;k) = \frac{\Gamma(\alpha+kn+1)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^{kr}}{\Gamma(\alpha+kr+1)}, \tag{1}$$

where $\Gamma(\cdot)$ is the classical Gamma function and for the polynomials $Y_n^\alpha(x;k)$, Carlitz [2] subsequently showed that

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+\alpha+1}{k}\right)_n, \tag{2}$$

where $(a)_n$ is Pochhammer's symbol of a as follows:

$$(a)_n = \begin{cases} a(a+1)(a+2)\dots(a+n-1), & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{3}$$

It is easy to verify that the polynomials $Y_n^{\alpha}(x;k)$ and $Z_n^{\alpha}(x;k)$ are biorthogonal with respect to the weight function $w(x) = x^{\alpha}e^{-x}$ over the interval $(0,\infty)$, which means

$$\int_0^{\infty} x^{\alpha}e^{-x}Y_i^{\alpha}(x;k)Z_j^{\alpha}(x;k)dx = \frac{\Gamma(kj+\alpha+1)}{j!}\delta_{ij}, \tag{4}$$

where $\alpha > -1$, $k \in \mathbb{Z}^+$ and δ_{ij} is the Kronecker delta.

The Laguerre polynomials $\mathcal{L}_n^{\alpha}(x)$ are defined as (see, e.g., [3])

$$\mathcal{L}_n^{\alpha}(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} \sum_{r=0}^{n}(-1)^r \binom{n}{r} \frac{x^r}{\Gamma(\alpha+r+1)}. \tag{5}$$

For $p,q \in \mathbb{N}$, we can define the general hypergeometric functions of p-numerator and q-denominator by

$${}_pF_q\left[\begin{array}{c} \alpha_1,\alpha_2,\ldots,\alpha_p \\ \beta_1,\beta_2,\ldots,\beta_q \end{array} ;x\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{x^n}{n!}, \tag{6}$$

such that $\beta_j \neq 0,-1,-2,\ldots; j = 1,2,\ldots,q$. Then, according to [3], we can rewrite $\mathcal{L}_n^{\alpha}(x)$ as

$$\mathcal{L}_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left[\begin{array}{c} -n \\ \alpha+1 \end{array} ;x\right]. \tag{7}$$

For $k = 1$, we note that the Konhauser polynomials (1) and (2) reduce to the Laguerre Polynomials $\mathcal{L}_n^{\alpha}(x)$ and their special cases; when $k = 2$, the case was encountered earlier by Spencer and Fano [4] in certain calculations involving the penetration of gamma rays through matter and was subsequently discussed in [5].

On the other hand, the matrix theory has become pervasive to almost every area of mathematics, especially in orthogonal polynomials and special functions. The special matrix functions appear in the literature related to statistics [6], Lie theory [7], and in connection with the matrix version of Laguerre, Hermite, and Legendre differential equations and the corresponding polynomial families (see, e.g., [8–10]). In the past few years, the extension of the classical Konhauser polynomials to the Konhauser matrix polynomials of one variable has been a subject of intensive studies [11–14]. Recently, many authors (see, e.g., [15–18]) have proposed the generating relations of Konhauser matrix polynomials of one variable from the Lie algebra method point of view and found some properties of Konhauser matrix polynomials of one variable via the Lie algebra technique; they also obtained operational identities for Laguerre–Konhauser-type matrix polynomials and their applications for the matrix framework.

Some studies have been presented on polynomials in two variables such as 2-variables Shivley's matrix polynomials [19], 2-variables Laguerre matrix polynomials [20], 2-variables Hermite generalized matrix polynomials [21–24], 2-variables Gegenbauer matrix polynomials [25], and the second kind of Chebyshev matrix polynomials of two variables [26].

The purpose of the present work is to introduce and study 2-variables Konhauser matrix polynomials and find the hypergeometric matrix function representations; we try to establish some basic properties of these polynomials which include generating matrix functions, finite sum formulae, and integral representations, and we will also discuss the fractional integrals of the 2-variables Konhauser matrix polynomials.

The rest of this paper is structured as follows. In the next section, we give basic definitions and previous results to be used in the following sections. In Section 3, we introduce the definition of 2-variables Konhauser matrix polynomials for parameter matrices A and B and some generating matrix relations involving 2-variables Konhauser matrix polynomials deriving the integral representations. Finally, we provide some results on the fractional integrals of 2-variables Konhauser matrix polynomials in Section 4.

2. Preliminaries

In this section, we give the brief introduction related to Konhauser matrix polynomials and recall some previously known results.

Let $\mathbb{C}^{N\times N}$ be the vector space of N-square matrices with complex entries; for any matrix $A \in \mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ is the set of all eigenvalues of A,

$$\alpha(A) = \max\{\mathbf{Re}(z) : z \in \sigma(A)\}, \quad \beta(A) = \min\{\mathbf{Re}(z) : z \in \sigma(A)\}. \tag{8}$$

A square matrix $A \in \mathbb{C}^{N\times N}$ is said to be positive stable if and only if $\beta(A) > 0$. Furthermore, the identity matrix and the null matrix or zero matrix in $\mathbb{C}^{N\times N}$ will be symbolized by $\mathbf{I}$ and $\mathbf{0}$, respectively. If $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of the complex variable z, which are defined as an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$, then, from the properties of the matrix functional calculus [27,28], we have

$$\Phi(A)\Psi(A) = \Psi(A)\Phi(A). \tag{9}$$

Furthermore, if $B \in \mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and also if $AB = BA$, then

$$\Phi(A)\Psi(B) = \Psi(B)\Phi(A). \tag{10}$$

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. Then, $\Gamma(A)$ is well defined as

$$\Gamma(A) = \int_0^\infty t^{A-I} e^{-t} dt, \tag{11}$$

where $t^{A-I} = \exp((A-I)\ln t)$. Then, the matrix Pochhammer symbol $(A)_n$ of A is denoted as follows (see, e.g., [29–31]):

$$(A)_n = \begin{cases} A(A+I)...(A+(n-1)I) = \Gamma^{-1}(A)\Gamma(A+nI), & n \geq 1, \\ I, & n = 0, \end{cases} \tag{12}$$

The Laguerre matrix polynomials are defined by Jódar et al. [8]

$$\mathcal{L}_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!} (A+I)_n [(A+I)_k]^{-1} x^k, \tag{13}$$

where $A \in \mathbb{C}^{N\times N}$ is a matrix such that $-k \notin \sigma(A)$, $\forall k \in \mathbb{Z}^+$, $(A+I)_k$ are given by Equation (12) and λ is a complex number with $\mathbf{Re}(\lambda) > 0$.

For $p, q \in \mathbb{N}$, $1 \leq i \leq p$, $1 \leq j \leq q$, if $A_i, B_j \in \mathbb{C}^{N\times N}$ are matrices such that $B_j + kI$ are invertible for all integers $k \geq 0$, the generalized hypergeometric matrix functions are defined as [32]

$${}_pF_q\left[\begin{array}{c} A_1, A_2, \ldots, A_p \\ B_1, B_2, \ldots, B_q \end{array}; x\right] = \sum_{n\geq 0} \frac{(A_1)_n(A_2)_n \ldots (A_p)_n[(B_1)_n]^{-1}[(B_2)_n]^{-1} \ldots ([B_p)_n]^{-1}}{n!} x^n. \tag{14}$$

It follows that, for $\lambda = 1$ in (13), we have

$$\mathcal{L}_n^{A}(x) = \frac{(A+I)_n}{n!}\, {}_1F_1\left[\begin{array}{c} -nI, \\ A+I \end{array}; x\right]. \tag{15}$$

For commuting matrices A_i, B_i, C_i, D_i, E_i and F_i in $\mathbb{C}^{N\times N}$, we define the Kampé de Fériet matrix series as [32]

$$F^{m_1,n_1,l_1}_{m_2,n_2,l_2}\left[\begin{array}{c} A,B,C \\ D,E,F \end{array};x,y\right]= \\ \sum_{m,n\geq 0}\prod_{i=1}^{m_1}(A_i)_{m+n}\prod_{i=1}^{n_1}(B_i)_m\prod_{i=1}^{l_1}(C_i)_n\prod_{i=1}^{m_2}[(D_i)_{m+n}]^{-1}\prod_{i=1}^{n_2}[(E_i)_m]^{-1}\prod_{i=1}^{l_2}[(F_i)_n]^{-1}\frac{x^m y^n}{m!n!}, \tag{16}$$

where A abbreviates the sequence of matrices $A_1,....,A_{m_1}$, etc. and D_i+kI, E_i+kI and F_i+kI are invertible for all integers $k\geq 0$.

If $A\in\mathbb{C}^{N\times N}$ is a matrix satisfying the condition

$$\mathbf{Re}(z)>-1, \quad \forall z\in\sigma(A), \tag{17}$$

and λ is a complex numbers with $\mathbf{Re}(\lambda)>0$, we recall the following explicit expression for the Konhauser matrix polynomials (see, e.g., [11])

$$Z_n^{(A,\lambda)}(x,k)=\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\Gamma^{-1}(A+(kr+1)I)(\lambda x)^{kr}, \tag{18}$$

and

$$Y_n^{(A,\lambda)}(x;k)=\frac{1}{n!}\sum_{r=0}^{n}\frac{(\lambda x)^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{A+(s+1)I}{k}\right)_n, \tag{19}$$

which are biorthogonal with respect to matrix weight function $w(x)=x^A e^{-\lambda x}$ over the interval $(0,\infty)$.

3. 2-Variables Konhauser Matrix Polynomials

In this section, we first introduce the 2-variables Konhauser matrix polynomials with parameter matrices A and B; then, we get the hypergeometric matrix function representations, generating matrix functions, finite summation formulas, and related results for the 2-variables Konhauser matrix polynomials.

Definition 1. *Let $A,B\in\mathbb{C}^{N\times N}$ be matrices satisfying the condition (17). Then, for $k,l\in\mathbb{Z}^+$, the 2-variables Konhauser matrix polynomials $Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)$ are defined as follows:*

$$Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)=\frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)}{(n!)^2} \\ \times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,(\lambda x)^{ks}(\rho y)^{lr}}{r!s!}\Gamma^{-1}(A+(ks+1)I)\Gamma^{-1}(B+(lr+1)I), \tag{20}$$

where λ and ρ are complex numbers with $\mathbf{Re}(\lambda)>0$ and $\mathbf{Re}(\rho)>0$.

Remark 1. *Furthermore, we note the following special cases of the 2-variables Konhauser matrix polynomials $Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)$ as follows:*

i. *Letting $l=1$, $B=\mathbf{0}$ and $y=0$ in (20), we get the Konhauser matrix polynomials defined in (18);*

ii. *Letting $k=l=1$ and $\rho=1$ in (20), we get the 2-variables analogue of Laguerre's matrix polynomials $\mathcal{L}_n^{(A,B,\lambda)}(x,y)$ as follows:*

$$Z_n^{(A,B,\lambda,1)}(x,y,1,1)=\frac{(A+I)_n(B+I)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,(\lambda x)^s(y)^r}{r!s!}\left[(A+I)_s(B+I)_r\right]^{-1}; \tag{21}$$

iii. *Letting $k=l=1$, $B=\mathbf{0}$ and $y=0$ in (20), we obtain the Laguerre's matrix polynomials $\mathcal{L}_n^{(A,\lambda)}(x)$ defined in (13);*

iv. *Letting $A = \alpha \in \mathbb{C}^{1\times 1}$ and $B = \beta \in \mathbb{C}^{1\times 1}$ in (20), we find the scaler 2-variables Konhauser polynomials (see, e.g., [33]);*
v. *Letting $A = \alpha \in \mathbb{C}^{1\times 1}$, and $B = \mathbf{0}$ in (20), we find Konhauser polynomials defined in (1).*

3.1. Hypergeometric Representation

Now, by using (16) and (20), we obtain the hypergeometric matrix function representations

$$Z_n^{(A,B,\lambda,\rho)}(x,y,k,l) = \frac{(A+I)_{kn}(B+I)_{ln}}{(n!)^2} F^1_{k,l}\left[\begin{array}{c} -nI \\ \Delta(k;A+I),\Delta(l;B+I) \end{array} ;(\frac{\lambda x}{k})^k,(\frac{\rho y}{l})^l\right], \tag{22}$$

where $\Delta(k;A)$ abbreviates the array of k parameters such that

$$\Delta(k;A) = (\frac{A}{k})(\frac{A+I}{k})(\frac{A+2I}{k})\ldots(\frac{A+(k-1)I}{k}), \quad k \geq 1, \tag{23}$$

and $F^1_{k,l}$ is defined in (16).

Remark 2. *If $A \in \mathbb{C}^{N\times N}$ is a matrix satisfying the condition (17), letting $B = \mathbf{0}$ and $y = 0$ in (22), we obtain*

$$Z_n^{(A,0,\lambda)}(x,0;k) = \frac{(A+I)_{kn}}{n!}\, {}_1F_k\left[\begin{array}{c} -nI \\ \Delta(k;A+I) \end{array} ;(\frac{\lambda x}{k})^k\right] = Z_n^{(A,\lambda)}(x;k), \tag{24}$$

where $Z_n^{(A,\lambda)}(x;k)$ are Konhauser matrix polynomials in [11] and ${}_1F_k$ is hypergeometric matrix function of 1-numerator and k-denominator defined in (14).

Remark 3. *If $A \in \mathbb{C}^{N\times N}$ is a matrix satisfying the condition (17), let $k = 1$, $B = \mathbf{0}$ and $y = 0$ in (22), then we get*

$$Z_n^{(A,\lambda)}(x;1) = \frac{(A+I)_n}{n!}\, {}_1F_1\left[\begin{array}{c} -nI, \\ A+I \end{array} ;x\right] = \mathcal{L}_n^A(x), \tag{25}$$

where $\mathcal{L}_n^A(x)$ are the Laguerre's matrix polynomials defined in (15).

3.2. Generating Matrix Relations for the 2-Variables of Konhauser Matrix Polynomials

Generating matrix relations always play an important role in the study of polynomials, first, we give some generating matrix relations for the 2-variables of Konhauser matrix polynomials as follows:

Theorem 1. *Letting $A, B \in \mathbb{C}^{N\times N}$ be matrices satisfying the condition (17), we obtain the explicit formulae of matrix generating relations for the 2-variables Konhauser matrix polynomials as follows:*

$$\begin{aligned}&\sum_{n=0}^{\infty} Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}(n!t^n) \\ &= e^t\, {}_0F_k\left[\begin{array}{c} - \\ \Delta(k;A+I) \end{array} ;(\frac{-\lambda x}{k})^k\right] {}_0F_l\left[\begin{array}{c} - \\ \Delta(p;B+I) \end{array} ;(\frac{-\rho y}{l})^l\right],\end{aligned} \tag{26}$$

where ${}_0F_k$ and ${}_0F_l$ are hypergeometric matrix functions of 0-numerator and k,l-denominator as (14), $\Delta(k;A+I)$ and $\Delta(l;B+I)$ are defined as (23), and the short line "$-$" means that the number of parameters is zero.

Proof. From Equation (20), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}(n!t^n) \\
&= \sum_{n=0}^{\infty} n! \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\lambda x)^{ks}(\rho y)^{lr}}{r!s!(n!)^2}[(A+I)_{ks}]^{-1}[(B+I)lr]^{-1}t^n \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{s=0}^{\infty}\frac{(-1)^s(\lambda x)^{ks}}{s!}[(A+I)_{ks}]^{-1}t^s\sum_{r=0}^{\infty}\frac{(-1)^r(\rho y)^{lr}}{r!}[(B+I)_{lr}]^{-1}t^r,
\end{aligned} \tag{27}$$

by using

$$(A)_{km} = k^{km}\, \Big(\frac{A}{k}\Big)_m \Big(\frac{A+I}{k}\Big)_m \cdots \Big(\frac{A+(k-1)I}{k}\Big)_m,$$

we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}(n!t^n) \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{s=0}^{\infty}\frac{(-1)^s(\lambda x)^{ks}}{k^{ks}s!}\Big[\prod_{m=1}^{k}\Big(\frac{A+mI}{k}\Big)_s\Big]^{-1}t^s\sum_{r=0}^{\infty}\frac{(-1)^r(\rho y)^{lr}}{l^{lr}r!}\Big[\prod_{n=1}^{l}\Big(\frac{B+nI}{l}\Big)_r\Big]^{-1}t^r \\
&= e^t\,{}_0F_k\left[\begin{matrix} - \\ \Delta(k;A+I)\end{matrix};\Big(\frac{-\lambda x}{k}\Big)^k\right]{}_0F_l\left[\begin{matrix} - \\ \Delta(l;B+I)\end{matrix};\Big(\frac{-\rho y}{l}\Big)^l\right].
\end{aligned} \tag{28}$$

This completes the proof. □

For a matrix E in $\mathbb{C}^{N\times N}$, we can easily obtain the following generating relations for the 2-variables Konhauser matrix polynomial similar to Theorem 1

$$\begin{aligned}
&\sum_{n=0}^{\infty}(E)_n[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}(n!t^n) \\
&= (1-t)^{-E}F_{k,l}^{1}\left[\begin{matrix} -E \\ \Delta(k;A+I),\Delta(l;B+I)\end{matrix};\frac{t}{t-1}\Big(\frac{\lambda x}{k}\Big)^k,\frac{t}{t-1}\Big(\frac{\rho y}{l}\Big)^l\right],
\end{aligned} \tag{29}$$

where $F_{k,p}^{1}$ are defined in Equation (16), $\Delta(k;A+I)$ and $\Delta(l;B+I)$ are defined as Equation (23).

Corollary 1. *Letting $A,B\in\mathbb{C}^{N\times N}$ be matrices satisfying the condition (17), the following generating matrix relations of the 2-variables Konhauser matrix polynomials hold:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\Gamma^{-1}(A+(nk+1)I)\Gamma^{-1}(B+(nl+1)I)(n!t^n) \\
&= e^t\,\Gamma^{-1}(A+I)\,\Gamma^{-1}(B+I)\,{}_0F_k\left[\begin{matrix} - \\ \Delta(k;A+I)\end{matrix};\Big(\frac{-\lambda x}{k}\Big)^k\right]{}_0F_l\left[\begin{matrix} - \\ \Delta(l;B+I)\end{matrix};\Big(\frac{-\rho y}{l}\Big)^l\right],
\end{aligned} \tag{30}$$

where ${}_0F_k$ and ${}_0F_l$ are hypergeometric matrix functions of 0-numerator and k,l-denominator as (14).

Corollary 2. *Letting A, B, and E be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (17), we give explicit formulae of matrix generating relations for the 2-variables Konhauser matrix polynomials as follows:*

$$\begin{aligned}&\sum_{n=0}^{\infty}(E)_n Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\Gamma^{-1}(A+(nk+1)I)\Gamma^{-1}(B+(nl+1)I)(n!t^n)\\&=(1-t)^{-E}\,\Gamma^{-1}(A+I)\,\Gamma^{-1}(B+I)F^{1}_{k,l}\left[\begin{array}{c}-E\\ \Delta(k;A+I),\Delta(l;B+I)\end{array};\frac{t}{t-1}(\frac{\lambda x}{k})^k,\frac{t}{t-1}(\frac{\rho y}{l})^l\right].\end{aligned}\tag{31}$$

Considering the double series,

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{[(m+n)!]^2}{n!\;m!}Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{k(m+n)}]^{-1}[(B+I)l(m+n)]^{-1}\sigma^m\tau^n\\&=\sum_{n=0}^{\infty}n!Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\tau^n[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}\,{}_1F_0\left[\begin{array}{c}-nI\\ -\end{array};\frac{-\sigma}{\tau}\right]\\&=\sum_{n=0}^{\infty}n!Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{kn}]^{-1}[(B+I)ln]^{-1}(\sigma+\tau)^n.\end{aligned}\tag{32}$$

Now, by making use of Theorem 1, we find

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{[(m+n)!]^2}{n!\;m!}Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)[(A+I)_{k(m+n)}]^{-1}[(B+I)l(m+n)]^{-1}\sigma^m\tau^n\\&=e^{\sigma+\tau}\,{}_0F_k\left[\begin{array}{c}-\\ \Delta(k;A+I)\end{array};(\frac{-\lambda x}{k})^k(\sigma+\tau)\right]{}_0F_l\left[\begin{array}{c}-\\ \Delta(l;B+I)\end{array};(\frac{-\rho y}{l})^l(\sigma+\tau)\right].\end{aligned}\tag{33}$$

Here, Equation (33) may be regarded as a double generating matrix relations for (20).

Remark 4. *For A in $\mathbb{C}^{N\times N}$, letting $k=1$, $B=\mathbf{0}$ and $y=0$ in (33), we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{m+n}{n}[(A+I)_{(m+n)}]^{-1}\mathcal{L}^{A}_{m+n}(x)\;t^n\\&=\sum_{n=m}^{\infty}\frac{(-1)^n m![(A+I)_n]^{-1}x^n}{(n-m)!n!}\,{}_1F_1\left[\begin{array}{c}-(n+1)I,\\ (n-m+1)I\end{array};t\right]\\&=\sum_{n=m}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^n n!t^{n-m}[(A+I)_n]^{-1}(n+1)_j t^j}{m!(n-m)!n!(n-m+1)_j j!}\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^n(n+1)_j(A+I)_n]^{-1}t^j}{(1)_j n!j!},\end{aligned}\tag{34}$$

we find generating matrix relations of the Laguerre's matrix polynomials.

3.3. Some Properties of the 2-Variables Konhauser Matrix Polynomials

For the finite sum property of the 2-variables Konhauser matrix polynomials $Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)$, we get the generating relations together as follows:

$$\begin{aligned}&e^t\,{}_0F_k\left[\begin{array}{c}-\\ \Delta(k;A+I)\end{array};(\frac{-\lambda xw}{k})^k t\right]{}_0F_l\left[\begin{array}{c}-\\ \Delta(l;B+I)\end{array};(\frac{-\rho yw}{l})^l t\right]\\&=e^{(1-w^k)t}e^{w^k t}\,{}_0F_k\left[\begin{array}{c}-\\ \Delta(k;A+I)\end{array};(\frac{-\lambda xw}{k})^k t\right]{}_0F_l\left[\begin{array}{c}-\\ \Delta(l;B+I)\end{array};(\frac{-\rho yw}{l})^l t\right],\end{aligned}\tag{35}$$

and

$$
\begin{aligned}
&\sum_{r=0}^{n} Z_n^{(A,B,\lambda,\rho)}(xw, yw, k, k)[(A+I)_{kn}]^{-1}[(B+I)_{kn}]^{-1}t^n n! \\
&= \Big(\sum_{n=0}^{\infty} \frac{1-w^{kn}t^n}{n!}\Big)\Big(\sum_{n=0}^{\infty} Z_n^{(A,B,\lambda,\rho)}(x,y,k,k)w^{kn}[(A+I)_{kn}]^{-1}[(B+I)_{kn}]^{-1}t^n n!\Big).
\end{aligned} \tag{36}
$$

By comparing the coefficients of t^n on both sides, we have

$$
\begin{aligned}
&Z_n^{(A,B,\lambda,\rho)}(xw, yw, k, k) \\
&= \sum_{r=0}^{n} \frac{r! w^{kr}(1-w^k)^{n-r}}{n!(n-r)!}[(A+I)_{kr}]^{-1}[(B+I)_{kr}]^{-1}(A+I)_{kn}(B+I)_{kn} Z_n^{(A,B,\lambda,\rho)}(x,y,k,k).
\end{aligned} \tag{37}
$$

The integral representations for the 2-variables Konhauser matrix polynomials are derived in the following theorem.

Theorem 2. *Letting $A, B \in \mathbb{C}^{N\times N}$ be matrices satisfying the condition (17), and, if $\left|\frac{t}{\lambda x}\right| < 1, \left|\frac{v}{\rho y}\right| < 1$, then we have the integral representation of the 2-variables Konhauser matrix polynomials $Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)$ as follows:*

$$
\begin{aligned}
Z_n^{(A,B,\lambda,\rho)}(x,y,k,l) &= \frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)}{(n!)^2(2\pi i)^2} \\
&\times \int_{c_1}\int_{c_2} \left(t^k v^l - (\lambda x)^k v^l - (\rho y)^k t^l\right)^n e^{t+v} \ t^{-(A+(kn+1)I)} \ v^{-(B+(ln+1)I)} dt dv,
\end{aligned} \tag{38}
$$

where c_1, c_2 are the paths around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Proof. The right side of the above formulae are deformed into

$$
\begin{aligned}
&\frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)}{(n!)^2} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\lambda x)^{kr}(\rho y)^{lr}}{r!s!} \\
&\times \frac{1}{2\pi i}\int_{c_1} t^{-(A+(ks+1)I)} e^t dt \ \times \frac{1}{2\pi i}\int_{c_2} v^{-(B+(lr+1)I)} e^v dv,
\end{aligned} \tag{39}
$$

and using the integral representation of the reciprocal Gamma function, which are given in [34]

$$
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_c e^t t^{-z} dt, \tag{40}
$$

where c is the path around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis. Thus, from Equation (40), we obtain the following integral matrix functional

$$
\Gamma^{-1}(A+(kn+1)I) = \frac{1}{2\pi i}\int_{c_1} e^t t^{-(A+(kn+1)I)} dt. \tag{41}
$$

By Equation (41), we can transfer (39) to

$$
\begin{aligned}
&\frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)}{(n!)^2} \times \\
&\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\lambda x)^{kr}(\rho y)^{lr}}{r!s!}\Gamma^{-1}(A+(ks+1)I)\Gamma^{-1}(B+(kr+1)I) \\
&= Z_n^{(A,B,\lambda,\rho)}(x,y,k,l).
\end{aligned} \tag{42}
$$

This completes the proof of the theorem. □

4. Fractional Integrals of the 2-Variable Konhauser Matrix Polynomials

In this section, we study the fractional integrals of the Konhauser matrix polynomials of one and two variables. The fractional integrals of Riemann–Liouville operators of order μ and $x > 0$ are given by (see [35,36])

$$(\mathbf{I}_a^\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \quad \mathbf{Re}(\mu) > 0. \tag{43}$$

Recently, the authors (see, e.g., [28]) introduced the fractional integrals with matrix parameters as follows: suppose $A \in \mathbb{C}^{N\times N}$ is a positive stable matrix and $\mu \in \mathbb{C}$ is a complex number satisfying the condition $\mathbf{Re}(\mu) > 0$. Then, the Riemann–Liouville fractional integrals with matrix parameters of order μ are defined by

$$\mathbf{I}^\mu(x^A) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^A dt. \tag{44}$$

Lemma 1. *Supposing that $A \in \mathbb{C}^{N\times N}$ is a positive stable matrix and $\mu \in \mathbb{C}$ is a complex number satisfying the condition $\mathbf{Re}(\mu) > 0$, then the Riemann–Liouville fractional integrals with matrix parameters of order μ are defined and we have (see, e.g., [28])*

$$\mathbf{I}^\mu(x^{A-I}) = \Gamma(A)\Gamma^{-1}(A+\mu I)x^{A+(\mu-1)I}. \tag{45}$$

Theorem 3. *If $A \in \mathbb{C}^{N\times N}$ is a matrix satisfying the condition (17), then the Riemann–Liouville fractional integrals of Konhauser matrix polynomials of one variable are as follows:*

$$\mathbf{I}^\mu\left[(\lambda x)^A Z_n^{(A,\lambda)}(x,k)\right] = \Gamma^{-1}(A+(kn+\mu+1)I)\Gamma(A+(kn+1)I)(\lambda x)^{A+\mu I} Z_n^{(A+\mu I,\lambda)}(x,k), \tag{46}$$

where λ is a complex numbers with $\mathbf{Re}(\lambda) > 0$, and $k \in \mathbb{Z}^+$.

Proof. From Equation (44), we find

$$\begin{aligned} \mathbf{I}^\mu\left[(\lambda x)^A Z_n^{(A,\lambda)}(x,k)\right] &= \int_0^x \frac{\big(\lambda(x-t)\big)^{\mu-1}}{\Gamma(\mu)} t^A Z_n^{(A,\lambda)}(t,k)dt \\ &= \frac{\Gamma(A+(kn+1)I)}{\Gamma(\mu)} \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!}\Gamma^{-1}(A+(kr+1)I) \int_0^x (\lambda x)^{A+krI}\big(\lambda(x-t)\big)^{\mu-1}dt \\ &= \Gamma(A+(kn+1)I) \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!}(\lambda x)^{A+(kr+\mu)I}\,\Gamma^{-1}(A+(kr+\mu+1)I), \end{aligned} \tag{47}$$

and we can write

$$\mathbf{I}^\mu\left[(\lambda x)^A Z_n^{(A,\lambda)}(x,k)\right] = \Gamma^{-1}(A+(kn+\mu+1)I)\Gamma(A+(kn+1)I)(\lambda x)^{A+\mu I} Z_n^{(A+\mu I,\lambda)}(x,k). \tag{48}$$

□

The 2-variables analogue of Riemann–Liouville fractional integrals $\mathbf{I}^{\nu,\mu}$ may be defined as follows

Definition 2. *Letting $A, B \in \mathbb{C}^{N\times N}$ be positive stable matrices, if $\mathbf{Re}(\nu) > 0$ and $\mathbf{Re}(\mu) > 0$, then the 2-variables Riemann–Liouville fractional integrals of orders ν, μ can be defined as follows:*

$$\mathbf{I}^{\nu,\mu}\left[x^A y^B\right] = \frac{1}{\Gamma(\nu)\Gamma(\mu)} \int_0^x \int_0^y (x-u)^{\nu-1}(y-v)^{\mu-1} u^A v^B du dv. \tag{49}$$

Theorem 4. *Letting $A, B \in \mathbb{C}^{N\times N}$ be matrices satisfying the condition (17), $\mathbf{Re}(\lambda) > 0$, $\mathbf{Re}(\rho) > 0$, then, for the Riemann–Liouville fractional integral of a 2-variables Konhauser matrix polynomial, we have the following:*

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] \\ &= \Gamma^{-1}(A+(kn+\nu+1)I)\Gamma^{-1}(B+(ln+\mu+1)I)\Gamma(A+(kn+1)I) \\ &\Gamma(B+(ln+1)I)(\lambda x)^{A+\nu I}(\rho y)^{B+\mu I} Z_n^{(A+\nu I,B+\mu I,\lambda,\rho)}(x,y,k,l), \end{aligned} \tag{50}$$

where λ and ρ are complex numbers and $k, l \in \mathbb{Z}^+$.

Proof. By using Equation (49), we obtain

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] = \frac{1}{\Gamma(\nu)\Gamma(\mu)} \\ &\times \int_0^x \int_0^y \big(\lambda(x-u)\big)^{\nu-1}\big(\rho(y-v)\big)^{\mu-1}(\lambda u)^A(\rho v)^B Z_n^{(A,B,\lambda,\rho)}(u,v,k,l)dudv. \end{aligned} \tag{51}$$

By putting $u = xt$ and $v = yw$, we get

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] = \frac{(\lambda x)^{A+\nu I}(\rho y)^{B+\mu I}}{\Gamma(\nu)\Gamma(\mu)} \\ &\times \int_0^1 \int_0^1 (\lambda t)^A(\rho w)^B\big(\lambda(1-t)\big)^{\nu-1}\big(\rho(1-w)\big)^{\mu-1} Z_n^{(A,B,\lambda,\rho)}(xt,yw,k,l)dtdw, \end{aligned} \tag{52}$$

from definition (20), we have

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] \\ &= \frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)(\lambda x)^{A+\nu I}(\rho y)^{B+\mu I}}{(n!)^2\Gamma(\nu)\Gamma(\mu)} \\ &\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,(\lambda x)^{ks}(\rho y)^{lr}}{r!s!}\Gamma^{-1}(A+(ks+1)I)\Gamma^{-1}(B+(lr+1)I). \\ &\times \int_0^1 (\lambda t)^{A+ksI}\big(\lambda(1-t)\big)^{\nu-1}dt \int_0^1 (\rho w)^{B+lrI}\big(\rho(1-w)\big)^{\mu-1}dw, \end{aligned} \tag{53}$$

and

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] \\ &= \frac{\Gamma(A+(kn+1)I)\Gamma(B+(ln+1)I)(\lambda x)^{A+\nu I}(\rho y)^{B+\mu I}}{(n!)^2\Gamma(\nu)\Gamma(\mu)} \\ &\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,(\lambda x)^{ks}(\rho y)^{lr}}{r!s!}\Gamma^{-1}(A+(ks+\nu+1)I)\Gamma^{-1}(B+(lr+\mu+1)I). \end{aligned} \tag{54}$$

We thus arrive at

$$\begin{aligned} &\mathbf{I}^{\nu,\mu}\left[(\lambda x)^A(\rho y)^B Z_n^{(A,B,\lambda,\rho)}(x,y,k,l)\right] \\ &= \Gamma^{-1}(A+(kn+\nu+1)I)\Gamma^{-1}(B+(ln+\mu+1)I)\Gamma(A+(kn+1)I) \\ &\Gamma(B+(ln+1)I)(\lambda x)^{A+\nu I}(\rho y)^{B+\mu I} Z_n^{(A+\nu I,B+\mu I,\lambda,\rho)}(x,y,k,l). \end{aligned} \tag{55}$$

This completes the proof of Theorem 4. □

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