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The Existence of Solutions to Nonlinear Matrix Equations via Fixed Points of Multivalued F-Contractions

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Abstract: In this paper, we set up an adequate condition for the presence of a solution of the nonlinear matrix equation. To do so, we prove the existence of fixed points for multi-valued modified F-contractions in the context of complete metric spaces, which generalize, refine, and extend several existing results in the literature. An example is accompanies the obtained results to show that derived results are a proper generalization.

Keywords: F-contraction; fixed points; matrix equation; multi-valued mapping

1. Introduction and Preliminaries

It has always been an attractive problem to find an adequate method to solve matrix equations because the existence of solutions of matrix equations arises in a number of applications such as in stochastic filtering, system theory, dynamic programming, control theory, statistics, ladder networks, and many other fields. In 2003, Ran and Reurings [1] obtained a sufficient condition for the presence of positive definite solution of two classes of matrix equations

$$V = \mathcal{Q} \pm \sum_{j=1}^{f} \mathscr{C}_j^* \mathcal{F}(V) \mathscr{C}_j, \tag{1}$$

where $\mathcal{F}$ is a order-preserving (or order-reversing) mapping on $H(s)$, $\mathcal{Q} \in \mathcal{P}(s)$ and $\mathscr{C}_j$ is an $s \times s$ complex matrix. Since then, many fixed point theorems have been presented by several authors to find solutions for different classes of matrix equations (see [2,3]). In [4], Berzig proved the existence and uniqueness of solution of the matrix equations of the form

$$V = \mathcal{Q} + \sum_{i=1}^{f} \mathscr{C}_i^* V \mathscr{C}_i - \sum_{i=1}^{k} \mathscr{D}_i^* V \mathscr{D}_i. \tag{2}$$

In the present paper, our goal is to find a sufficient condition to determine a solution for nonlinear matrix equations of the form

$$V = \mathcal{Q} + \sum_{i=1}^{f} \mathscr{C}_i \varrho(V) \mathscr{C}_i^* - \sum_{j=1}^{k} \mathscr{D}_j \varrho(V) \mathscr{D}_j^*, \tag{3}$$

where $\mathcal{Q}$ is a positive definite matrix, $\mathscr{C}_i$, $\mathscr{D}_j$ are arbitrary $s \times s$ matrices for all $i = 1, ..., f$, $j = 1, ..., k$, and ϱ is a self mapping on the set of all $s \times s$ Hermitian matrices, which maps the set of all $s \times s$ Hermitian positive definite matrices onto itself. To do this, we prove the existence of fixed points for multi-valued modified F-contractions in the frame of complete metric spaces. Henceforth, for a metric space (U, m), define

$$
\begin{aligned}
2^U &= \{\mathscr{A} \subseteq U \,:\, \mathscr{A} \neq \emptyset\}, \\
C(U) &= \left\{\mathscr{A} \subseteq 2^U \,:\, \mathscr{A} \text{ is closed}\right\}, \\
CB(U) &= \left\{\mathscr{A} \subseteq 2^U \,:\, \mathscr{A} \text{ is closed and bounded}\right\}, \\
K(U) &= \left\{\mathscr{A} \subseteq 2^U \,:\, \mathscr{A} \text{ is compact}\right\}.
\end{aligned}
$$

Note that $K(U) \subseteq CB(U) \subseteq C(U) \subseteq 2^U$. Let

$$
H(\mathscr{B}, \mathscr{A}) = \max\left\{\sup_{\mu \in \mathscr{B}} M(\mu, \mathscr{A}), \sup_{\omega \in \mathscr{A}} M(\omega, \mathscr{B})\right\},
$$

where $M(\mu, \mathscr{B}) = \inf\{m(\mu, \omega) : \omega \in \mathscr{B}\}$ and $\mathscr{A}, \mathscr{B} \in C(U)$. Symbolize

$$
\mathfrak{F} = \{F : \mathbb{R}^+ \to \mathbb{R} : F \text{ satisfies } (F1), (F2) \text{ and } (F3)\}
$$

and

$$
\mathfrak{F}_* = \{F \in \mathfrak{F} : F \text{ satisfies } (F4)\},
$$

where

($F1$) F is strictly increasing;
($F2$) for all, sequence $\{r_q\} \subseteq R^+$, $\lim_{q \to \infty} r_q = 0$ if and only if $\lim_{q \to \infty} F(r_q) = -\infty$;
($F3$) there exist $0 < k < 1$ such that $\lim_{n \to 0^+} t^k F(t) = 0$; and
($F4$) $F(\inf \mathscr{A}) = \inf F(\mathscr{A})$ for all $\mathscr{A} \subset (0, \infty)$ with $\inf \mathscr{A} > 0$.

Feng and Liu gave an important and interesting generalization of Nadler's fixed point theorem [5] as:

Theorem 1. *[6] Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$. If there exists $a, c \in (0, 1)$ such that $c < a$ and for any $\mu \in U$, there is $\omega \in I_a^\mu$ satisfying*

$$
M(\omega, \mathcal{G}\omega) \leq c\, m(\mu, \omega), \tag{4}
$$

where $I_a^\mu = \{\omega \in \mathcal{G}\mu : a\, m(\mu, \omega) \leq M(\mu, \mathcal{G}\mu)\}$. Then, $\mathcal{G}$ has a fixed point, provided that the map $\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous.

Altun et al. [7] defined multi-valued F-contractions and found some fixed point results. Further, Minak et al. [8] extended Theorem 1 and claimed that their obtained results are factual or proper generalizations of Feng and Liu's theorem (Theorem 1). However, Nguyen et al. [9] showed that their claim is not true by giving an example (see Example 1.1 in [9]) and gave refinements of Minak et al.'s theorems [9] by replacing "for any $\mu \in U$ there is $\omega \in \mathcal{G}\mu$" by "for any $\mu \in U$ there is $\omega \in U$" and extending functions F to $[0, \infty)$ by putting $F(0) = -\infty$. Very recently, Nashine and Kadelburg [10] proved the following result as generalization of Theorem 1.

Theorem 2. *Let (U, m) be a complete metric space, $\mathcal{G} : U \to C(U)$ and $F \in \mathfrak{F}_*$ [10]. If there exist two functions $\xi : (0, \infty) \to (0, \infty)$ and $\pi : (0, \infty) \to (\xi, \infty)$ such that*

$$\pi(t) > \xi(t), \quad \lim_{t \to s^+} \inf \pi(t) > \lim_{t \to s^+} \inf \xi(t) \qquad \text{for} \quad s \geq 0 \tag{5}$$

and, for any $\mu \in U$ with $M(\mu, \mathcal{G}\mu) > 0$, there exists $\omega \in \mathbb{F}^{\mu}_{\xi}$ satisfying

$$\pi(M_{\mu,\omega}) + F(M(\omega, \mathcal{G}\omega)) \leq F(m(\mu, \omega)), \tag{6}$$

where $\mathbb{F}^{\mu}_{\xi} = \{\omega \in \mathcal{G}\mu : F(m(\mu, \omega)) \leq F(\max\{m(\mu, \mathcal{G}\mu), m(\omega, \mathcal{G}\omega)\}) + \xi(M_{\mu,\omega})\}$ and $M_{\mu,\omega}$ is defined in Equation (8), then $\mathcal{G}$ has a fixed point, provided that the map $\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous.

However, the following example shows that Theorem 2 is not proper generalization of Theorem 1.

Example 1. *Let $U = [0, 1] \subset \mathbb{R}$ with the usual metric m. Then, (U, m) is complete metric space. Consider $\xi : (0, \infty) \to (0, \infty)$ and $\pi : (0, \infty) \to (\xi, \infty)$ are two functions satisfying Equation (5). Define $\mathcal{G} : U \to C(U)$ by*

$$\mathcal{G}\mu = \begin{cases} U & \text{if} \quad \mu = 0, \\ \{0\} & \text{if} \quad \mu \neq 0. \end{cases}$$

Then, $M(\mu, \mathcal{G}\mu) = \mu$ for all $x \in V$. In this example, Theorem 2 cannot be applied. Indeed, for $\mu \neq 0$, we have $M(\mu, \mathcal{G}\mu) > 0$. Therefore, if $\omega \in \mathcal{G}\mu$, then $\omega = 0$ and $M(\omega, \mathcal{G}\omega) = 0$, thus Equation (6) is not satisfied for any F.

Motivated by Nguyen et al. [9], we overcome the error mentioned in Example 1.1 of [9] by another way. We define contractions involving F-functions and prove fixed point results for these type of contractions. In our results, the domain of the function F is not extended from $(0, \infty)$ to $[0, \infty)$. Our results generalize (see [11–31]), refine, and extend the results of [6,8,32,33].

2. Fixed Point Results

Let $\mathcal{G} : U \to 2^U$ be the multi-valued map, $F \in \mathfrak{F}$, $\xi : (0, \infty) \to (\xi, \infty)$, $\xi > 0$ and $\mu \in U$ with $M(\mu, \mathcal{G}\mu) > 0$. Define the set

$$\mathbb{F}^{\mu}_{\xi} = \{\omega \in \mathcal{G}\mu : M(\omega, \mathcal{G}\omega) > 0 \text{ and } F(m(\mu, \omega)) \leq F(M_{\mu,\omega}) + \xi(m(\mu, \omega))\}, \tag{7}$$

where

$$M_{\mu,\omega} = \max\left\{m(\mu, \omega), M(\mu, \mathcal{G}\mu), M(\omega, \mathcal{G}\omega), \frac{M(\omega, \mathcal{G}\mu) + M(\mu, \mathcal{G}\omega)}{2}\right\}. \tag{8}$$

By considering $\xi : (0, \infty) \to (\xi, \infty), \xi > 0$ a constant function, that is, $\xi(t) = \xi$+constant$= b$, Equation (7) becomes

$$\mathbb{F}^{\mu}_{b} = \{\omega \in \mathcal{G}\mu : M(\omega, \mathcal{G}\omega) > 0 \text{ and } F(m(\mu, \omega)) \leq F(M(\mu, \mathcal{G}\mu)) + b\}. \tag{9}$$

Definition 1. *Let $\mathcal{G} : U \to 2^U$ be a multi-valued mapping on a metric space (U, m); then, $\mathcal{G}$ is said to be modified-F-contraction on U, if there exists $\pi : (0, \infty) \to (0, \infty)$, $\xi : (0, \infty) \to (\xi, \infty), \xi > 0$ and a function $F \in \mathfrak{F}$ such that, for all $\mu \in U$ with $M(\mu, \mathcal{G}\mu) > 0$, there exists $\omega \in \mathbb{F}^{\mu}_{\xi}$ satisfying*

$$\pi(m(\mu, \omega)) + F(M(\omega, \mathcal{G}\omega)) \leq F(M_{\mu,\omega}), \tag{10}$$

where $M_{\mu,\omega}$ is defined in Equation (8).

Now, we prove our main results.

Theorem 3. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to K(U)$ be a multi-valued mapping satisfying the following assertions:*

1. *$\mathcal{G}$ is modified-F-contraction for $F \in \mathfrak{F}$;*
2. *$\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous mapping; and*
3. *$\pi : (0, \infty) \to (0, \infty)$ and $\xi : (0, \infty) \to (\xi, \infty), \xi > 0$ satisfy*

$$\pi(t) > \xi(t) \tag{11}$$

and

$$\lim_{t \to s^+} \inf \pi(t) > \lim_{t \to s^+} \inf \xi(t) \quad \text{for all} \quad s \geq 0. \tag{12}$$

Then, $\mathcal{G}$ has a fixed point in U.

Proof. Assume that $\mathcal{G}$ has no fixed point in U. Let $\mu_0 \in U$. Then, $M(\mu_0, \mathcal{G}\mu_0) > 0$, otherwise μ_0 is the fixed point of $\mathcal{G}$. Since $\mathcal{G}\mu \in K(U)$ for every μ, there exists $\mu_1 \in \mathcal{G}\mu_0$ such that $m(\mu_0, \mu_1) = M(\mu_0, \mathcal{G}\mu_0)$. It also follows that

$$F(m(\mu_0, \mu_1)) \leq F(M_{\mu_0,\mu_1}) + \xi(m(\mu_0, \mu_1)) \tag{13}$$

$M(\mu_1, \mathcal{G}\mu_1) > 0$, otherwise μ_1 is the fixed point of $\mathcal{G}$. Thus, $\mu_1 \in \mathbb{F}_{\xi}^{\mu_0}$ and $\mu_1 \neq \mu_0$, therefore, from Equation (10), we have

$$\pi(m(\mu_0, \mu_1)) + F(M(\mu_1, \mathcal{G}\mu_1)) \leq F(M_{\mu_0,\mu_1}), \tag{14}$$

where

$$\begin{aligned} M_{\mu_0,\mu_1} = \max \{ & m(\mu_0, \mu_1), M(\mu_0, \mathcal{G}\mu_0), M(\mu_1, \mathcal{G}\mu_1), \\ & \left. \frac{M(\mu_1, \mathcal{G}\mu_0) + M(\mu_0, \mathcal{G}\mu_1)}{2} \right\}. \end{aligned} \tag{15}$$

Since $\mathcal{G}\mu_0$ and $\mathcal{G}\mu_1$ are compact, Equation (15) gives

$$M_{\mu_0,\mu_1} = \max \left\{ m(\mu_0, \mu_1), m(\mu_1, \mu_2), \frac{m(\mu_0, \mu_2)}{2} \right\}. \tag{16}$$

Since

$$\begin{aligned} \frac{m(\mu_0, \mu_2)}{2} &\leq \frac{m(\mu_0, \mu_1) + m(\mu_1, \mu_2)}{2} \\ &\leq \max\{m(\mu_0, \mu_1), m(\mu_1, \mu_2)\}, \end{aligned}$$

it follows that

$$M_{\mu_0,\mu_1} \leq \max\{m(\mu_0, \mu_1), m(\mu_1, \mu_2)\}. \tag{17}$$

Suppose that $m(\mu_0, \mu_1) < m(\mu_1, \mu_2)$; then, Equation (14) implies that

$$\pi(m(\mu_0, \mu_1)) + F(M(\mu_1, \mathcal{G}\mu_1)) \leq F(m(\mu_1, \mu_2)); \tag{18}$$

consequently,

$$\pi(m(\mu_0,\mu_1)) + F(m(\mu_1,\mu_2)) \leq (m(\mu_1,\mu_2)), \tag{19}$$

or $F(m(\mu_1,\mu_2)) \leq F(m(\mu_1,\mu_2)) - \pi(m(\mu_0,\mu_1))$, which is a contradiction. Hence, $M_{\mu_0,\mu_1} \leq m(\mu_0,\mu_1)$; therefore, by using $(F1)$, Equations (13) and (14) imply that

$$F(m(\mu_0,\mu_1)) \leq F(m(\mu_0,\mu_1)) + \xi(m(\mu_0,\mu_1)) \tag{20}$$

and

$$\pi(m(\mu_0,\mu_1)) + F(m(\mu_1,\mu_2) \leq F(m(\mu_0,\mu_1)). \tag{21}$$

On continuing recursively, we get a sequence $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}_{n\in\mathbb{N}}$ in U, where $\mu_{q+1} \in \mathbb{F}_{\xi}^{\mu_q}$, $\mu_{q+1} \notin \mathcal{G}\mu_{q+1}$, $M_{\mu_q,\mu_{q+1}} \leq m(\mu_q,\mu_{q+1})$ and

$$\pi(m(\mu_q,\mu_{q+1})) + F(M(\mu_{q+1},\mathcal{G}\mu_{q+1})) \leq F(m(\mu_q,\mu_{q+1})). \tag{22}$$

Since $\mu_{q+1} \in \mathbb{F}_{\xi}^{\mu_q}$ and $\mathcal{G}\mu_q$ and $\mathcal{G}\mu_{q+1}$ are compact, we have

$$\pi(m(\mu_q,\mu_{q+1})) + F(m(\mu_{q+1},\mu_{q+2})) \leq F(m(\mu_q,\mu_{q+1})) \tag{23}$$

and

$$F(m(\mu_q,\mu_{q+1})) \leq F(m(\mu_q,\mu_{q+1})) + \xi(m(\mu_q,\mu_{q+1})). \tag{24}$$

Combining Equations (23) and (24) gives

$$\begin{aligned} F(m(\mu_{q+1},\mu_{n+2})) \leq{}& F(m(\mu_q,\mu_{q+1})) + \xi(m(\mu_q,\mu_{q+1})) \\ &- \pi(m(\mu_q,\mu_{q+1})) \end{aligned} \tag{25}$$

Set $m(\mu_q,\mu_{q+1}) = L_q$. From Equation (25), we get

$$\begin{aligned} F(L_{q+1}) &\leq F(L_q) + \xi(L_q) - \pi(L_q) \\ &\leq F(L_{q-1}) + \xi(L_q) + \xi(L_{q-1}) - \pi(L_q) - \pi(L_{q-1}) \\ &\vdots \\ &\leq F(L_0) + \xi(L_q) + \xi(L_{q-1}) + \cdots + \xi(L_0) \\ &- \pi(L_q) - \pi(L_{q-1}) - \cdots - \pi(L_0). \end{aligned} \tag{26}$$

Let $\pi(L_{p_n}) = \min\{\pi(L_0), \pi(L_1), \cdots, \pi(L_q)\}$ and $\xi(L_{q_n}) = \max\{\xi(L_0), \xi(L_1), \cdots, \xi(L_q)\}$ for all $n \in \mathbb{N}$. From Equation (26), we get

$$F(L_{q+1}) \leq F(L_0) + q(\xi(L_{q_n}) - \pi(L_{p_n})). \tag{27}$$

From Equation (22), we also get

$$\begin{aligned} F(M(\mu_{q+1},\mathcal{G}\mu_{q+1})) \leq{}& F(M(\mu_0,\mathcal{G}\mu_0)) + \\ & n(\xi(L_{q_n}) - \pi(L_{p_n})). \end{aligned} \tag{28}$$

Equations (12) and (28) imply $\lim_{q\to\infty} F(L_q) = -\infty$, thus, by $(F2)$, $\lim_{q\to\infty} L_q = 0$. Now, we prove that $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}$ is a Cauchy sequence. From $(F3)$, there exists $0 < r < 1$ such that

$$\lim_{q\to\infty} (L_q)^r F(L_q) = 0. \tag{29}$$

By Equation (27), we get for all $n \in \mathbb{N}$

$$(L_q)^r F(L_q) - (L_q)^r F(L_0) \leq (L_q)^r q(\xi(L_{q_n}) - \pi(L_{p_n})) \leq 0. \tag{30}$$

Letting $q \to \infty$ in Equation (30), we obtain

$$\lim_{q\to\infty} q(L_q)^r = 0 \tag{31}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $q(L_q)^r \leq 1$ or $L_q \leq \frac{1}{q^{1/r}}$, for all $q > q_1$. Next, for $m > q \geq q_1$, we have

$$m(\mu_q, \mu_m) \leq \sum_{i=q}^{f-1} m(\mu_i, \mu_{i+1}) \leq \sum_{l=q}^{f-1} \frac{1}{l^{1/k}},$$

since $0 < k < 1$, $\sum_{l=q}^{f-1} \frac{1}{l^{1/k}}$ converges. Therefore, $m(\mu_q, \mu_f) \to 0$ as $f, q \to \infty$. Thus, $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}$ is a Cauchy sequence. Since U is complete, there exists $\mu^* \in U$ such that $\mu_q \to \mu^*$ as $q \to \infty$. From Equation (28) and $(F2)$, we have

$$\lim_{q\to\infty} M(\mu_q, \mathcal{G}\mu_q) = 0.$$

From the hypothesis in Equation (2), we obtain

$$0 \leq M(\mu, \mathcal{G}\mu) \leq \lim_{q\to\infty} \inf M(\mu_q, \mathcal{G}\mu_q) = 0,$$

which is a contradiction. Thus, $\mathcal{G}$ has a fixed point. □

In the following theorem, we take $C(U)$ instead of $K(U)$; thus, we need to take $F \in \mathfrak{F}_*$.

Theorem 4. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a multi-valued mapping such that $\mathcal{G}$ is modified-F-contraction for $F \in \mathfrak{F}_*$ and satisfying the assertions in Equations (2) and (3) of Theorem 3. Then, $\mathcal{G}$ has a fixed point in U.*

Proof. Assume that $\mathcal{G}$ has no fixed point in U. Let $\mu_0 \in U$, then $M(\mu_0, \mathcal{G}\mu_0) > 0$, otherwise μ_0 is the fixed point of $\mathcal{G}$. Since $\mathcal{G}\mu \in C(U)$ for every $\mu \in U$ and $F \in \mathfrak{F}_*$, there exist $\mu_1 \in \mathcal{G}\mu_0$ such that

$$\begin{aligned}
F(m(\mu_0, \mu_1)) &\leq \inf\{F(m(\mu_0, \mu_1)) : \mu_1 \in \mathcal{G}\mu_0\} + \xi(m(\mu_0, \mu_1)) \\
&= F(\inf\{m(\mu_0, \mu_1) : \mu_1 \in \mathcal{G}\mu_0\}) + \xi(m(\mu_0, \mu_1)) \\
&= F(M(\mu_0, \mathcal{G}\mu_0)) + \xi(m(\mu_0, \mu_1)) \\
&\leq F(M_{\mu_0,\mu_1}) + \xi(m(\mu_0, \mu_1))
\end{aligned}$$

and $M(\mu_1, \mathcal{G}\mu_1) > 0$, otherwise μ_1 is the fixed point of $\mathcal{G}$. Thus, from Equation (10), we have

$$\pi(m(\mu_0, \mu_1)) + F(M(\mu_1, \mathcal{G}\mu_1)) \leq F(M_{\mu_0,\mu_1}), \tag{32}$$

where

$$\begin{aligned} M_{\mu_0,\mu_1} &= \max\Big\{m(\mu_0,\mu_1), M(\mu_0,\mathcal{G}\mu_0), M(\mu_1,\mathcal{G}\mu_1),\\ &\quad \frac{M(\mu_1,\mathcal{G}\mu_0)+M(\mu_0,\mathcal{G}\mu_1)}{2}\Big\}\\ &\le \max\Big\{m(\mu_0,\mu_1), m(\mu_0,\mu_1), m(\mu_1,\mu_2),\\ &\quad \frac{m(\mu_1,\mu_1)+m(\mu_0,\mu_2)}{2}\Big\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{m(\mu_0,\mu_2)}{2} &\le \frac{m(\mu_0,\mu_1)+m(\mu_1,\mu_2)}{2}\\ &\le \max\{m(\mu_0,\mu_1), m(\mu_1,\mu_2)\}, \end{aligned}$$

it follows that

$$M_{\mu_0,\mu_1} \le \max\{m(\mu_0,\mu_1), m(\mu_1,\mu_2)\}. \tag{33}$$

Due to $(F4)$, we obtain

$$F(M(\mu_1,\mathcal{G}\mu_1)) = \inf_{\omega\in\mathcal{G}\mu} F(m(\mu_1,\omega)) \tag{34}$$

Suppose that $m(\mu_0,\mu_1) < m(\mu_1,\mu_2)$; then, Equations (32) and (34) imply that

$$\inf_{\omega\in\mathcal{G}\mu} F(m(\mu_1,\omega)) \le F(m(\mu_1,\mu_2)) - \pi(m(\mu_0,\mu_1)). \tag{35}$$

Then, by Equation (35), there exists $\mu_2 \in \mathcal{G}\mu_1$ such that

$$F(m(\mu_1,\mu_2)) \le F(m(\mu_1,\mu_2)) - \pi(m(\mu_0,\mu_1)), \tag{36}$$

which is a contradiction. Hence, $M_{\mu_0,\mu_1} \le m(\mu_0,\mu_1)$. Therefore, from Equations (32) and (34), we obtain

$$F(m(\mu_1,\mu_2)) \le F(m(\mu_0,\mu_1)) - \pi(m(\mu_0,\mu_1)), \tag{37}$$

The rest of the proof can be completed as in the proof of Theorem 3. $\square$

By defining $\xi : (0,\infty) \to (\xi,\infty), \xi > 0$ as $\xi(t) = \xi +$ constant $= b > 0$ for all $t \in [0,\infty)$, we get

Corollary 1. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to K(U)$ be a multi-valued mapping. If there exists $b > 0$ and a function $\pi : (0,\infty) \to (b,\infty)$ such that*

$$\lim_{t\to s^+} \inf \pi(t) > b \qquad \textit{for} \quad s \ge 0$$

and for $F \in \mathfrak{F}$, $\mu \in U$ with $M(\mu,\mathcal{G}\mu) > 0$, there exists $\omega \in \mathbb{F}_b^{\mu}$ satisfying

$$\pi(m(\mu,\omega)) + F(M(\omega,\mathcal{G}\omega)) \le F(M_{\mu,\omega}).$$

Then, $\mathcal{G}$ has a fixed point in U, provided that $\mu \to M(\mu,\mathcal{G}\mu)$ is a lower semi-continuous mapping.

Corollary 2. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a multi-valued mapping satisfying all the assertions of Corollary 1 for $F \in \mathfrak{F}_*$. Then, $\mathcal{G}$ has a fixed point in U.*

Corollary 3. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to K(U)$ be a multi-valued mapping. If there exists $b > 0$ and a function $\pi : (0, \infty) \to (b, \infty)$ such that*

$$\lim_{t \to s^+} \inf \pi(t) > b \qquad \textit{for} \quad s \geq 0$$

and for $F \in \mathfrak{F}$, $x \in U$ with $M(\mu, \mathcal{G}\mu) > 0$, there exists $y \in \mathbb{F}_b^x$ satisfying at least one of the following:

(F_1) $\pi(m(\mu, \omega)) + F(M(\omega, \mathcal{G}\omega)) \leq F(m(\mu, \omega))$;

(F_2) $\pi(m(\mu, \omega)) + F(M(\omega, \mathcal{G}\omega)) \leq F(\max\{M(\mu, \mathcal{G}\mu), M(\omega, \mathcal{G}\omega)\})$; *and*

(F_3) $\pi(m(\mu, \omega)) + F(M(\omega, \mathcal{G}\omega)) \leq F\left(\frac{1}{2}[M(\mu, \mathcal{G}\omega) + M(\omega, \mathcal{G}\mu)]\right)$.

Then, $\mathcal{G}$ has a fixed point in U provided that the map $\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous.

Corollary 4. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a multi-valued mapping satisfying all the assumptions of Corollary 3 for $F \in \mathfrak{F}_*$. Then, $\mathcal{G}$ has a fixed point in U.*

Corollary 5. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a multi-valued mapping. If there exists a function $\varphi : [0, \infty) \to (0, 1)$ and a non-decreasing function $b : [0, \infty) \to [b, 1), b > 0$ such that*

$$\varphi(t) < b(t) \quad \textit{and} \quad \lim_{t \to s^+} \sup \varphi(t) < \lim_{t \to s^+} \sup b(t)$$

for all $t \in [0, \infty)$ and for any $\mu \in U$ there is $\omega \in \mathcal{G}\mu$ satisfying the following two conditions:

$$b(m(\mu, \omega))m(\mu, \omega) \leq M_{\mu,\omega}$$

and

$$M(\omega, T\omega) \leq \varphi(m(\mu, \omega))M_{\mu,\omega},$$

where $M_{\mu,\omega}$ is defined in Equation (8). Then, $\mathcal{G}$ has a fixed point in U provided that $\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous.

Proof. Define $F : [0, \infty) \to \mathbb{R}$, $\pi : (0, \infty) \to (0, \infty)$ and $\xi : (0, \infty) \to (\xi, \infty), \xi > 0$ by $F(r) = \ln r$ for $r \in (0, 1)$, $\pi(t) = -\ln \varphi(t)$ and $\xi(t) = -\ln b(t)$ for $t \in (0, \infty)$. Then, all conditions of Theorem 4 hold true and thus $\mathcal{G}$ has a fixed point in U. $\square$

Corollary 6. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to K(U)$ be a multi-valued mapping satisfying all the assertions of Corollary 5 for $F \in \mathfrak{F}$. Then, $\mathcal{G}$ has a fixed point in U.*

Corollary 7. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a multi-valued mapping. If there exists constants $b, c \in (0, 1)$ such that $c < b$ and for any $\mu \in U$ there is $\omega \in Tx$ satisfying the following conditions:*

$$bd(\mu, \omega) \leq M_{\mu,\omega}$$

and

$$M(\omega, \mathcal{G}\omega) \leq cM_{\mu,\omega},$$

where $M_{\mu,\omega}$ is defined in Equation (8). Then, $\mathcal{G}$ has a fixed point in U provided that $\mu \to M(\mu, \mathcal{G}\mu)$ is lower semi-continuous.

Proof. Define $\varphi : [0,\infty) \to (0,1)$ and $b : [0,\infty) \to [b,1)$ by $\varphi(t) = c$ and $b(t) = b$, respectively, for all $t \in [0,\infty)$, where $b, c \in (0,1)$. Then, all conditions of Corollary 5 are satisfied and hence $\mathcal{G}$ has a fixed point. □

Remark 1. *Corollary 5 generalizes Theorem 6 of [33] and Corollary 7 generalizes the Theorem 1.*

Example 2. *Let* $U = \{0,1,2,3,...\}$ *with*

$$m(\mu,\omega) = \begin{cases} 0 & \text{if } \mu = \omega, \\ \mu + \omega & \text{if } \mu \neq \omega, \end{cases}$$

then (U, m) *is complete metric space. Define* $\mathcal{G} : U \to C(U)$, $F : \mathbb{R}^+ \to \mathbb{R}$, $\pi : (0,\infty) \to (0,\infty)$ *and* $\xi : (0,\infty) \to (\xi,\infty)$ *by*

$$\mathcal{G}\mu = \begin{cases} \{0,4\} & \text{if } \mu \in \{0,3\}, \\ \{2,3\} & \text{if } \mu \in \{2\}, \\ \{1,2,...,\mu-1\} & \text{if } \mu \in \{4,5,...\} \cup \{1\} \end{cases}$$

$F(t) = \ln(t)$, $\pi(t) = \frac{1}{t} + 0.12$, *and* $\xi(t) = \frac{1}{t} + 0.05$ *for all* $t > 0$. *Then,*

$$M(\mu,\mathcal{G}\mu) = \begin{cases} 0 & \text{if } \mu \in \{0,1,2\} \\ 3 & \text{if } \mu = 3 \\ \mu+1 & \text{if } \mu \in \{4,5,...\}, \end{cases}$$

$\pi(t) > \xi(t)$ *and* $\lim_{t\to s^+} \inf \pi(t) > \lim_{t\to s^+} \inf \xi(t)$ *for all* $s \geq 0$. *Now, let* $M(\mu,\mathcal{G}\mu) > 0$; *then, there exists two cases:*

Case 1. *When* $\mu = 3$, $\mathcal{G}\mu = \{0,4\}$. *Thus, for* $\omega = 4 \in \mathcal{G}\mu$ *such that* $M(4,\mathcal{G}4) = 5 > 0$, *we have*

$$F(m(\mu,\omega)) - F(M_{\mu,\omega}) = F(7) - F(7) = 0 < \xi(m(\mu,\omega)).$$

In addition,

$$\begin{aligned} F(M(\omega,\mathcal{G}\omega)) - F(M_{\mu,\omega}) &= F(5) - F(7) \\ &= \ln\left(\frac{5}{7}\right) \\ &= -0.336 \\ &< -\left(\frac{1}{7} + 0.12\right) \\ &= -\pi(m(\mu,\omega)). \end{aligned}$$

Case 2. *When* $\mu \in \{4,5,...\}$,
$\mathcal{G}\mu = \{1,2,...,\mu-1\}$. *Thus, for* $\omega = 3 \in \mathcal{G}\mu$ *such that* $M(3,\mathcal{G}3) = 3 > 0$, *we have*

$$\begin{aligned} F(m(\mu,\omega)) - F(M_{\mu,\omega}) &= F(\mu+1) - F(\mu+1) \\ &= 0 < \xi(m(\mu,\omega)). \end{aligned}$$

In addition,

$$\begin{aligned} F(M(\omega,\mathcal{G}\omega)) - F(M_{\mu,\omega}) &= F(3) - F(\mu+1) \\ &= \ln\left(\frac{3}{\mu+1}\right) \\ &\le -\left(\frac{1}{\mu+3} + 0.12\right) \\ &= -\pi(m(\mu,\omega)). \end{aligned}$$

Hence, $\mathcal{G}$ is modified-F-contraction.
Next, let $\lim_{q\to\infty} m(\mu_q,\mu) = 0$. Then,

$$\lim_{q\to\infty} \inf M(\mu_q, \mathcal{G}\mu_q) = M(\mu, \mathcal{G}\mu).$$

Hence, $\mathcal{G}$ is a lower semi-continuous mapping. Thus, all conditions of Theorem 4 hold and 0, 1, and 2 are fixed points of $\mathcal{G}$.

Remark 2. *In Example 2, Theorem 2 cannot be applied. Indeed, for $\mu = 3$, $\mathcal{G}\mu = \{0,4\}$. Thus, for $\omega = 4 \in \mathcal{G}\mu$ such that $M(4,\mathcal{G}4) = 5 > 0$, we have*

$$\begin{aligned} &F(m(\mu,\omega)) - F\left(\max\{M(\mu,\mathcal{G}\mu), M(\omega,\mathcal{G}\omega)\}\right) \\ =&F(7) - F(5) \\ =&\ln\left(\frac{7}{5}\right) \\ <&\xi(m(\mu,\omega)). \end{aligned}$$

Then,

$$\begin{aligned} &F(M(\omega,\mathcal{G}\omega)) - F(m(\mu,\omega)) \\ =&F(5) - F(7) \\ =&-\ln\left(\frac{7}{5}\right) \\ \ge&-\xi(m(\mu,\omega)) \\ \ge&-\pi(M_{\mu,\omega}). \end{aligned}$$

Hence, Equation (6) does not hold.

Definition 2. *Let $\mathcal{G} : U \to 2^U$ be a multi-valued mapping on a metric space (U, m), $F_1 : (0,\infty) \to \mathbb{R}$ be a nondecreasing function, and $F_2 : (0,\infty) \to \mathbb{R}$ satisfy (F2) and (F3). Then, $\mathcal{G}$ is said to be F_1-F_2-contraction on U, if there exists $\pi : (0,\infty) \to (0,\infty)$ and $\xi : (0,\infty) \to (\xi,\infty)$, $\xi > 0$ such that, for all $x \in U$ with $M(\mu,\mathcal{G}\mu) > 0$, there exists $\omega \in \mathcal{G}\mu$ with $M(\omega,\mathcal{G}\omega) > 0$ satisfying*

$$\pi(m(\mu,\omega)) + F_1(M(\omega,\mathcal{G}\omega)) \le F_1(M_{\mu,\omega}) \tag{38}$$

and

$$F_2(m(\mu,\omega)) \le F_1(M(\mu,\mathcal{G}\mu)) + \xi(m(\mu,\omega)), \tag{39}$$

where $M_1(\mu,\omega)$ is defined in Equation (8).

Theorem 5. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to K(U)$ be a F_1-F_2-contraction satisfying the hypotheses in Equations (2) and (3) of Theorem 3. Then, $\mathcal{G}$ has a fixed point in U.*

Proof. Assume that $\mathcal{G}$ has no fixed point in U. Let $\mu_0 \in U$; then, $M(\mu_0, \mathcal{G}\mu_0) > 0$, otherwise μ_0 is the fixed point of $\mathcal{G}$. Since $\mathcal{G}\mu \in K(U)$ for every μ, there exists $\mu_1 \in \mathcal{G}\mu_0$ such that $m(\mu_0, \mu_1) = M(\mu_0, \mathcal{G}\mu_0)$ with $M(\mu_1, \mathcal{G}\mu_1) > 0$, otherwise μ_1 is the fixed point of $\mathcal{G}$. Thus, from Equations (38) and (39), we have

$$\pi(m(\mu_0, \mu_1)) + F_1(M(\mu_1, \mathcal{G}\mu_1)) \leq F_1(M_{\mu_0,\mu_1}), \tag{40}$$

and

$$F_2(m(\mu_0, \mu_1)) \leq F_1(M(\mu_0, \mathcal{G}\mu_0)) + \xi(m(\mu_0, \mu_1)) \tag{41}$$

where

$$\begin{aligned} M_{\mu_0,\mu_1} = \max \{ & m(\mu_0, \mu_1), M(\mu_0, \mathcal{G}\mu_0), M(\mu_1, \mathcal{G}\mu_1), \\ & \frac{M(\mu_1, \mathcal{G}\mu_0) + M(\mu_0, \mathcal{G}\mu_1)}{2} \bigg\}. \end{aligned} \tag{42}$$

Since $\mathcal{G}\mu_0$ and $\mathcal{G}\mu_1$ are compact, Equation (42) gives

$$\begin{aligned} M_{\mu_0,\mu_1} &= \max \{ m(\mu_0, \mu_1), m(\mu_0, \mu_1), m(\mu_1, \mu_2), \\ & \quad \frac{m(\mu_1, \mu_1) + m(\mu_0, \mu_2)}{2} \bigg\} \\ &= \max \left\{ m(\mu_0, \mu_1), m(\mu_1, \mu_2), \frac{m(\mu_0, \mu_2)}{2} \right\}. \end{aligned} \tag{43}$$

Since

$$\begin{aligned} \frac{m(\mu_0, \mu_2)}{2} &\leq \frac{m(\mu_0, \mu_1) + m(\mu_1, \mu_2)}{2} \\ &\leq \max\{m(\mu_0, \mu_1), m(\mu_1, \mu_2)\}, \end{aligned}$$

it follows that

$$M_{\mu_0,\mu_1} \leq \max\{m(\mu_0, \mu_1), m(\mu_1, \mu_2)\}. \tag{44}$$

Suppose that $m(\mu_0, \mu_1) < m(\mu_1, \mu_2)$; then, Equation (40) implies that

$$\pi(m(\mu_0, \mu_1)) + F_1(M(\mu_1, \mathcal{G}\mu_1)) \leq F_1(m(\mu_1, \mu_2)); \tag{45}$$

consequently,

$$\pi(m(\mu_0, \mu_1)) + F_1(m(\mu_1, \mu_2)) \leq F_1(m(\mu_1, \mu_2)), \tag{46}$$

or $F(m(\mu_1, \mu_2)) \leq F(m(\mu_1, \mu_2)) - \pi(m(\mu_0, \mu_1))$, which is a contradiction. Hence, $M_{\mu_0,\mu_1} \leq m(\mu_0, \mu_1)$. Since F_1 is nondecreasing, from Equation (40), we get

$$\pi(m(\mu_0, \mu_1)) + F_1(m(\mu_1, \mu_2) \leq F_1(m(\mu_0, \mu_1)). \tag{47}$$

On continuing recursively, we get a sequence $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}_{n\in\mathbb{N}}$ in U, where $M(\mu_q, \mathcal{G}\mu_q) > 0$, $\mu_{q+1} \notin \mathcal{G}\mu_{q+1}$ and $M_{\mu_q,\mu_{q+1}} \leq m(\mu_q, \mu_{q+1})$ satisfying

$$\pi(m(\mu_q, \mu_{q+1})) + F_1(M(\mu_{q+1}, \mathcal{G}\mu_{q+1})) \leq F_1(m(\mu_q, \mu_{q+1})). \tag{48}$$

and

$$F_2(m(\mu_q,\mu_{q+1})) \leq F_1(m(\mu_q,\mu_{q+1})) + \xi(m(\mu_q,\mu_{q+1})). \tag{49}$$

Since $\mu_{q+1} \in \mathbb{F}_{1,2}^{\xi,\mu_q}$ and $\mathcal{G}\mu_q$ and $\mathcal{G}\mu_{q+1}$ are compact, we have

$$\pi(m(\mu_q,\mu_{q+1})) + F_1(m(\mu_{q+1},\mu_{q+2})) \leq F_1(m(\mu_q,\mu_{q+1})) \tag{50}$$

and

$$F_2(m(\mu_q,\mu_{q+1})) \leq F_1(m(\mu_q,\mu_{q+1})) + \xi(m(\mu_q,\mu_{q+1})). \tag{51}$$

From Equation (50) and by monotonicity of F_1, we obtain that $\{m(\mu_q,\mu_{q+1})\}$ is nondecreasing sequence. Hence, there exists $r \geq 0$ such that $m(\mu_q,\mu_{q+1}) \to r$ as $q \to \infty$. Assume that $r > 0$; then, combining Equations (50) and (51) gives

$$\begin{aligned} F_2(m(\mu_{q+1},\mu_{q+2})) \leq F_1(m(\mu_q,\mu_{q+1})) + \xi(m(\mu_q,\mu_{q+1})) \\ - \pi(m(\mu_q,\mu_{q+1})) \end{aligned} \tag{52}$$

From Equation (52), we get

$$\begin{aligned} F_2(L_{q+1}) &\leq F_1(L_q) + \xi(L_q) - \pi(L_q) \\ &\leq F_1(L_{q-1}) + \xi(L_q) + \xi(L_{q-1}) - \pi(L_q) - \pi(L_{q-1}) \\ &\vdots \\ &\leq F_1(L_0) + \xi(L_q) + \xi(L_{q-1}) + \cdots + \xi(L_0) \\ &- \pi(L_q) - \pi(L_{q-1}) - \cdots - \pi(L_0). \end{aligned} \tag{53}$$

Let $\pi(L_{p_n}) = \min\{\pi(L_0), \pi(L_1), \cdots, \pi(L_q)\}$ and $\xi(L_{q_n}) = \max\{\xi(L_0), \xi(L_1), \cdots, \xi(L_q)\}$ for all $n \in \mathbb{N}$. From Equation (53), we get

$$F_2(L_{q+1}) \leq F_1(L_0) + q(\xi(L_{q_n}) - \pi(L_{p_n})). \tag{54}$$

From Equation (48), we also get

$$F_2(M(\mu_{q+1},\mathcal{G}\mu_{q+1})) \leq F_1(M(\mu_0,\mathcal{G}\mu_0)) + q(\xi(L_{q_n}) - \pi(L_{p_n})). \tag{55}$$

Equations (12) and (55) imply $\lim_{q\to\infty} F_2(L_q) = -\infty$; thus, by $(F2)$, $\lim_{q\to\infty} L_q = 0$. Now, we prove that $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}$ is a Cauchy sequence. Since F_2 satisfies $(F3)$, there exists $0 < r < 1$ such that

$$\lim_{q\to\infty} (L_q)^r F_2(L_q) = 0. \tag{56}$$

By Equation (54), we get for all $q \in \mathbb{N}$

$$\begin{aligned} (L_q)^r F_2(L_q) - (L_q)^r F_1(L_0) \leq (L_q)^r q(\xi(L_{q_n}) - \pi(L_{p_n})) \\ \leq 0. \end{aligned} \tag{57}$$

Letting $q \to \infty$ in Equation (57), we obtain

$$\lim_{q\to\infty} n(L_q)^r = 0 \tag{58}$$

This implies that there exists $q_1 \in \mathbb{N}$ such that $q(L_q)^r \leq 1$, or, $L_q \leq \frac{1}{q^{1/r}}$, for all $n > q_1$. Next, for $m > q \geq q_1$, we have

$$\mathit{m}(\mu_q, \mu_m) \leq \sum_{i=q}^{m-1} \mathit{m}(x_i, x_{i+1}) \leq \sum_{i=q}^{m-1} \frac{1}{i^{1/k}},$$

since $0 < k < 1$, $\sum_{i=q}^{m-1} \frac{1}{i^{1/k}}$ converges. Therefore, $\mathit{m}(\mu_q, \mu_m) \to 0$ as $m, q \to \infty$. Thus, $\{\mu_q : \mu_q \in \mathcal{G}\mu_{q-1}\}$ is a Cauchy sequence. Since U is complete, there exists $\mu^* \in U$ such that $\mu_q \to \mu^*$ as $q \to \infty$. Since F_2 satisfies $(F2)$, from Equation (55), we have

$$\lim_{q\to\infty} M(\mu_q, \mathcal{G}\mu_q) = 0.$$

From the hypothesis in Equation (2), we obtain

$$0 \leq M(\mu, \mathcal{G}\mu) \leq \lim_{q\to\infty} \inf M(\mu_q, \mathcal{G}\mu_q) = 0,$$

which is a contradiction. Thus, $\mathcal{G}$ has a fixed point. □

Theorem 6. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$ be a F_1-F_2-contraction satisfying the hypotheses in Equations (2) and (3) of Theorem 3. Assume that F_1 satisfies $(F4)$, then $\mathcal{G}$ has a fixed point in U.*

Proof. Assume that $\mathcal{G}$ has no fixed point in U. Let $\mu_0 \in U$, then $M(\mu_0, \mathcal{G}\mu_0) > 0$, otherwise μ_0 is the fixed point of $\mathcal{G}$. Since $\mathcal{G}$ is F_1-F_2-contraction there exists $\mu_1 \in Tx_0$ with $M(\mu_1, \mathcal{G}\mu_1) > 0$, otherwise μ_1 is the fixed point of $\mathcal{G}$, satisfying

$$\pi(\mathit{m}(\mu_0, \mu_1)) + F_1(M(\mu_1, \mathcal{G}\mu_1)) \leq F_1(M_{\mu_0,\mu_1}), \tag{59}$$

and

$$F_2(\mathit{m}(\mu_0, \mu_1)) \leq F_1(M(\mu_0, \mathcal{G}\mu_0)) + \xi(\mathit{m}(\mu_0, \mu_1)) \tag{60}$$

where

$$\begin{aligned} M_{\mu_0,\mu_1} &= \max\Big\{\mathit{m}(\mu_0, \mu_1), M(\mu_0, \mathcal{G}\mu_0), M(\mu_1, \mathcal{G}\mu_1), \\ &\qquad \frac{M(\mu_1, \mathcal{G}\mu_0) + M(\mu_0, \mathcal{G}\mu_1)}{2}\Big\} \\ &\leq \max\Big\{\mathit{m}(\mu_0, \mu_1), \mathit{m}(\mu_0, \mu_1), \mathit{m}(\mu_1, \mu_2), \\ &\qquad \frac{\mathit{m}(\mu_1, \mu_1) + \mathit{m}(\mu_0, \mu_2)}{2}\Big\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{\mathit{m}(\mu_0, \mu_2)}{2} &\leq \frac{\mathit{m}(\mu_0, \mu_1) + \mathit{m}(\mu_1, \mu_2)}{2} \\ &\leq \max\{\mathit{m}(\mu_0, \mu_1), \mathit{m}(\mu_1, \mu_2)\}, \end{aligned}$$

it follows that

$$M_{\mu_0,\mu_1} \leq \max\{\mathit{m}(\mu_0, \mu_1), \mathit{m}(\mu_1, \mu_2)\}. \tag{61}$$

Since F_1 satisfies $(F4)$, we obtain

$$F_1(M(\mu_1, \mathcal{G}\mu_1)) = \inf_{\omega\in\mathcal{G}\mu} F_1(\mathit{m}(\mu_1, \omega)) \tag{62}$$

Suppose that $m(\mu_0, \mu_1) < m(\mu_1, \mu_2)$; then, Equations (59) and (62) imply that

$$\inf_{\omega \in \mathcal{G}\mu} F_1(m(\mu_1, \omega)) \leq F_1(m(\mu_1, \mu_2)) - \pi(m(\mu_0, \mu_1)). \tag{63}$$

Then, by Equation (63), there exists $\mu_2 \in \mathcal{G}\mu_1$ such that

$$F_1(m(\mu_1, \mu_2)) \leq F_1(m(\mu_1, \mu_2)) - \pi(m(\mu_0, \mu_1)), \tag{64}$$

which is a contradiction. Hence, $M_{\mu_0,\mu_1} \leq m(\mu_0, \mu_1)$. Therefore, from Equations (59) and (62), we obtain

$$F_1(m(\mu_1, \mu_2)) \leq F_1(m(\mu_0, \mu_1)) - \pi(m(\mu_0, \mu_1)), \tag{65}$$

The rest of the proof follows as the proof of Theorem 5. □

Theorem 7. *Let (U, m) be a complete metric space and $\mathcal{G} : U \to C(U)$. Assume that $F_1 : (0, \infty) \to \mathbb{R}$ is a nondecreasing function, $F_2 : (0, \infty) \to \mathbb{R}$ satisfies $(F2)$ and $(F3)$, and there exists $\pi : (0, \infty) \to (0, \infty)$ and $\xi : (0, \infty) \to (\xi, \infty)$, $\xi > 0$ such that, for all $\mu \in U$ with $M(\mu, \mathcal{G}\mu) > 0$, we have $\omega \in \mathcal{G}\mu$ with $M(\omega, \mathcal{G}\omega) > 0$ satisfying*

$$\pi(m(\mu, \omega)) + F_1(M(\omega, \mathcal{G}\omega)) \leq F_1(M(\mu, \mathcal{G}\mu)) \tag{66}$$

and

$$F_2(m(\mu, \omega)) \leq F_1(M(\mu, \mathcal{G}\mu)) + \xi(m(\mu, \omega)). \tag{67}$$

If the hypotheses in Equations (2) and (3) of Theorem 3 hold, then $\mathcal{G}$ has a fixed point in U.

Proof. Assume that $\mathcal{G}$ has no fixed point in U. Let $\mu_0 \in U$, we can construct a sequence $\{\mu_q\}$ in U satisfying

$$\pi(m(\mu_q, \mu_{q+1})) + F_1(M(\mu_{q+1}, \mathcal{G}\mu_{q+1})) \leq F_1(M(\mu_q, \mathcal{G}\mu_q)), \tag{68}$$

and

$$F_2(m(\mu_q, \mu_{q+1})) \leq F_1(M(\mu_q, \mathcal{G}\mu_q)) + \xi(m(\mu_q, \mu_{q+1})), \tag{69}$$

$\mu_{q+1} \in \mathcal{G}\mu_q$ and $M(\mu_q, \mathcal{G}\mu_q) > 0$. The rest of the proof follows as the proof of Theorem 5. □

3. Notations and Setting of the Problem

Consider the following nonlinear matrix equation

$$V = \mathcal{Q} + \sum_{i=1}^{f} \mathscr{C}_i \varrho(V) \mathscr{C}_i^* - \sum_{j=1}^{k} \mathscr{D}_j \varrho(V) \mathscr{D}_j^*, \tag{70}$$

where $\mathcal{Q}$ is a positive definite matrix, $\mathscr{C}_i$, $\mathscr{D}_j$ are arbitrary $s \times s$ matrices for all $i = 1, ..., f$, $j = 1, ..., k$ and ϱ is a self mapping on the set of all $s \times s$ Hermitian matrices, which maps set of all $s \times s$ Hermitian positive definite matrices into itself. Designate

$$H(s) = \{V : V \text{ is } s \times s \text{ Hermitian matrix}\},$$

which is a complete metric space in respect of the Ky Fan norm $||.||_1$, defined by

$$||\mathscr{C}||_1 = \sum_{l=1}^{n} s_l(\mathscr{C}),$$

where $s_l(\mathscr{C})$, $l = 1, ..., n$, are the singular values of $\mathscr{C}$. In addition,

$$||\mathscr{C}||_1 = tr((\mathscr{C}^*\mathscr{C})^{1/2}),$$

which is $tr(\mathscr{C})$ for (Hermitian) nonnegative matrices and

$$\mathcal{P}(s) = \{V \in H(s) : V \text{ is positive definite}\}.$$

Define $\mathcal{G} : H(s) \to H(s)$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{G}(V) = \mathcal{Q} + \sum_{i=1}^{f} \mathscr{C}_i \varrho(V) \mathscr{C}_i^* - \sum_{j=1}^{k} \mathscr{D}_j \varrho(V) \mathscr{D}_j^* \tag{71}$$

and $F(r) = \ln r$, respectively. Then, $F \in \mathfrak{F}$. For a function $\xi : (0, \infty) \to (\xi, \infty)$, $\xi > 0$ and $V \in U$ with $M(V, \mathcal{G}V) = \inf\{m(V, Y) : Y \in \mathcal{G}V\} > 0$, define the set

$$\mathbb{F}_\xi^V = \{Y \in \mathcal{G}V : M(Y, \mathcal{G}Y) > 0 \text{ and } F(m(V, Y)) \leq F(M_{V,Y}) + \xi(m(V, Y))\}, \tag{72}$$

where

$$M_{V,Y} = \max\left\{m(V, Y), M(V, \mathcal{G}V), M(Y, \mathcal{G}Y), \frac{M(Y, \mathcal{G}V) + M(V, \mathcal{G}Y)}{2}\right\}. \tag{73}$$

Note that a fixed point of $\mathcal{G}$ is a solution of Equation (70).

4. Existence of Solution to Nonlinear Matrix Equations

In this section, we prove the existence of the positive definite solution to the nonlinear matrix equation in Equation (70) by using the fixed point results in Section 2.

Theorem 8. *Let $\varrho : H(s) \to H(s)$, which maps $\mathcal{P}(s)$ into $\mathcal{P}(s)$ and $\mathcal{Q} \in \mathcal{P}(s)$. Assume the following*

(1) *there exists a positive number N for which $\sum_{i=1}^{f} \mathscr{C}_i \mathscr{C}_i^* + \sum_{j=1}^{k} \mathscr{D}_j \mathscr{D}_j^* < NI_s$; and*

(2) *for all $V, Y \in \mathcal{P}(s)$, $\|\varrho(Y) - \varrho(V)\|_1 \leq N^{-1}(M_{V,Y}) exp\left(-\left(\frac{2\|Y-V\|+1}{2}\right)\right)$.*

Then, Equation (70) has a solution in $\mathcal{P}(s)$.

Proof. Let $V \in H(s)$. For the functions $\pi : (0, \infty) \to (0, \infty)$ and $\xi : (0, \infty) \to (\xi, \infty)$ defined by $\pi(t) = t + \frac{1}{2}$ and $\xi(t) = \frac{1}{t} + \frac{1}{4}$, there exist $Y \in \mathbb{F}_\xi^V$ such that $Y = \mathcal{G}V$. Thus, for $M(V, \mathcal{G}V) > 0$, Equation (11) holds true, $M(Y, TY) > 0$, and

$$\begin{aligned}
F(m(V, Y)) &= F(m(V, \mathcal{G}V)) \\
&= \ln(||\mathcal{G}V - V||_1) \\
&< \ln(||\mathcal{G}V - V||_1) + \frac{1}{||\mathcal{G}V - V||_1} + \frac{1}{||\mathcal{G}V - V||_1} \\
&= F(M(V, \mathcal{G}V)) + \xi(m(V, Y)).
\end{aligned}$$

Now,

$$\begin{aligned}
M(Y,\mathcal{G}Y) &= m(\mathcal{G}V,\mathcal{G}Y) \\
&= ||\mathcal{G}Y-\mathcal{G}V||_1 \\
&= tr(\mathcal{G}Y-\mathcal{G}V) \\
&= tr\left(\sum_{i=1}^{f}(\mathscr{C}_i\varrho(V)\mathscr{C}_i^*-\mathscr{C}_i\varrho(Y)\mathscr{C}_i^*)+\sum_{j=1}^{k}(\mathscr{D}_j\varrho(V)\mathscr{D}_j^*-\mathscr{D}_j\varrho(Y)\mathscr{D}_j^*)\right) \\
&\leq \sum_{i=1}^{f}\|\mathscr{C}_i\mathscr{C}_i^*\|\,\|\varrho(V)-\varrho(Y)\|+\sum_{j=1}^{k}\left\|\mathscr{D}_j\mathscr{D}_j^*\right\|\,\|\varrho(Y)-\varrho(V)\| \\
&= \left[\sum_{i=1}^{f}\|\mathscr{C}_i\mathscr{C}_i^*\|+\sum_{j=1}^{k}\left\|\mathscr{D}_j\mathscr{D}_j^*\right\|\right]\|\varrho(Y)-\varrho(V)\| \\
&\leq \frac{\sum_{i=1}^{f}|\mathscr{C}_i\mathscr{C}_i^*|+\sum_{j=1}^{k}\left|\mathscr{D}_j\mathscr{D}_j^*\right|}{N}(M_{Y,V})\exp\left(-\left(\frac{2\|Y-V\|+1}{2}\right)\right) \\
&< (M_{Y,V})\exp\left(-\left(\frac{2\|Y-V\|+1}{2}\right)\right),
\end{aligned}$$

and, thus,

$$\begin{aligned}
\ln(||\mathcal{G}Y-Y||_1)=\ln(||\mathcal{G}Y-\mathcal{G}V||_1) &< \ln\left((M_{Y,V})e^{-\left(\frac{2\|Y-V\|+1}{2}\right)}\right) \\
&= \ln(M_{V,Y})-\left\{\frac{2\|Y-V\|+1}{2}\right\}.
\end{aligned}$$

This implies that

$$||Y-V||_1+\frac{1}{2}+\ln(||\mathcal{G}Y-Y||_1)<\ln(M_{V,Y}).$$

Consequently,

$$\pi(m(V,Y))+F(m(Y,\mathcal{G}Y))<F(M_{V,Y}).$$

Thus, by using Theorem 3, we conclude that $\mathcal{G}$ has a fixed point and hence Equation (70) has a solution in $\mathcal{P}(s)$. □

Corollary 8. *Consider the matrix equation in Equation (70) with unitary matrices $\mathscr{C}_i$, $\mathscr{D}_j$ for all $i=1,2,...,f$, $j=1,2,...,k$. Assume that there exists a positive number N for which $m+k<N$ and Hypothesis (2) of Theorem 8 holds for all $V,Y\in\mathcal{P}(s)$. Then, Equation (70) has a solution in $\mathcal{P}(s)$.*

Corollary 9. *Consider the matrix equation*

$$V=\mathcal{Q}+\mathscr{C}V\mathscr{C}^*-\mathscr{D}V\mathscr{D}^*. \qquad (74)$$

Assume that there exists a positive number N for which $\mathscr{C}_i\mathscr{C}_i^*+\mathscr{D}_j\mathscr{D}_j^*<NI_s$, and Hypothesis (2) of Theorem 8 holds for all $V,Y\in\mathcal{P}(s)$. Then, Equation (74) has a solution in $\mathcal{P}(s)$.

5. Conclusions

The motivation of the presented work is to get a new approach to the existence of the solution to nonlinear matrix equations via fixed point results for newly introduced multi-valued mappings, named as modified-F-contractions. It is also proved that our obtained results generalize and extend many existing results in the literature and nontrivial examples are provided to verify it. Here, we overcome the error mentioned in Example 1.1 of [9] by adopting a way other than that of Nguyen et al. In addition, we show that the main result of Nashine and Kadelburg [10] (see Theorem 2) is not a proper generalization of Feng and Liu's theorem by giving an example.

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