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# On Null-Continuity of Monotone Measures

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**Abstract:** The null-continuity of monotone measures is a weaker condition than continuity from below and possesses many special properties. This paper further studies this structure characteristic of monotone measures. Some basic properties of null-continuity are shown and the characteristic of null-continuity is described by using convergence of sequence of measurable functions. It is shown that the null-continuity is a necessary condition that the classical Riesz's theorem remains valid for monotone measures. When considered measurable space $(X, \mathcal{A})$ is $S$-compact, the null-continuity condition is also sufficient for Riesz's theorem. By means of the equivalence of null-continuity and property (S) of monotone measures, a version of Egoroff's theorem for monotone measures on $S$-compact spaces is also presented. We also study the Sugeno integral and the Choquet integral by using null-continuity and generalize some previous results. We show that the monotone measures defined by the Sugeno integral (or the Choquet integral) preserve structural characteristic of null-continuity of the original monotone measures.

**Keywords:** fuzzy measure; monotone measure; null-continuity; Sugeno integral; Choquet integral; nonlinear integral

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## 1. Introduction

In generalized measure (or monotone measure) and nonlinear integral theory, the various continuities of set functions, such as continuity from below and from above, autocontinuity, order continuity, strongly order continuity and weak asymptotic null-additivity, etc., were investigated (e.g., see [1–7]). These structural characteristics of set functions played important roles in generalizing many well-known theorems in classical measure and integral theory to non-additive measure (i.e., monotone measure) and nonlinear integral theory [4,5,7–9]). In [6] (see also [1]), the concept of null-continuity of non-additive measures was proposed, and some relations among the several different continuity of non-additive measures were discussed.

In this paper, we further investigate the null-continuity of monotone measures and present some of its new characteristics. In the following section, we give some preliminaries and recall several different continuities of monotone measures and show the relations between the null-continuity and other continuities. We also give a set of sufficient conditions of null-continuity. In Section 3, a deeper discussion concerning null-continuity is presented, as follows: (1) We describe the characteristic of null-continuity of monotone measures by using the convergence of sequence of measurable functions. (2) We show that null-continuity is a necessary condition in which the classical Riesz's theorem ([10]) remains valid for monotone measures. (3) We show that null-continuity is a necessary and sufficient condition in which Riesz's theorem holds for monotone measures on $S$-compact spaces. By means of the equivalence of null-continuity and property (S) of monotone measures, we present a version of Egoroff's theorem for monotone measures on $S$-compact spaces. (4) The several results concerning the convergence in measure in classical measure theory are generalized to monotone measure spaces under the condition of null-continuity. In Section 4, we use null-continuity to generalize some previous results for the Sugeno integral and the Choquet integral. For the Sugeno integral and the Choquet integral, we show that the monotone measures defined by integral preserve structural characteristic of null-continuity of the original monotone measures. In Section 5, we briefly discuss other two important nonlinear integrals, the pan-integral [7,11], and the concave integral [12,13].

## 2. Preliminaries

Let $X$ be a nonempty set and $\mathcal{A}$ a $\sigma$-algebra of subsets of $X$ and $(X, \mathcal{A})$ denote measurable space. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{A}$.

### 2.1. Monotone Measures

**Definition 1** ([5,7])**.** *A monotone measure on $(X, \mathcal{A})$ is an extended real valued set function $\mu : \mathcal{A} \to [0, +\infty]$ satisfying*
(1) $\mu(\emptyset) = 0$ *(vanishing at $\emptyset$);*
(2) $\mu(A) \leq \mu(B)$ *whenever $A \subset B$ and $A, B \in \mathcal{A}$ (monotonicity).*

When $\mu$ is a monotone measure on $(X, \mathcal{A})$, the triple $(X, \mathcal{A}, \mu)$ is called monotone measure space.

In many literature works, a monotone measure is also known as "fuzzy measure", "non-additive measure", "capacity", or "non-additive probability", etc. (see [3,9,14–16]).

Let $\mathcal{M}$ denote the set of all monotone measures defined on $(X, \mathcal{A})$.

We recall some structural characteristics of monotone measures, as follows:

A monotone measure $\mu \in \mathcal{M}$ is said to be *finite*, if $\mu(X) < \infty$; *continuous from below* [17], if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *continuous from above* [17], if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \searrow A$ and there exists $n_0$ with $\mu(A_{n_0}) < +\infty$; *continuous*, if $\mu$ is continuous both from below and above; *order continuous* [18], if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow \emptyset$; *strongly order continuous* [4], if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow A$ and $\mu(A) = 0$; have *pseudometric generating property* (briefly the $(p.g.p.)$ [19], if for any sequences $\{A_n\}$ and $\{B_n\}$, $\mu(A_n) \vee \mu(B_n) \to 0$ $(n \to \infty)$ implies $\mu(A_n \cup B_n) \to 0$ $(n \to \infty)$; *weakly asymptotic null-additive* [3], if for any decreasing sequences $\{A_n\}$ and $\{B_n\}$, $\mu(A_n) \vee \mu(B_n) \to 0$ $(n \to \infty)$ implies $\mu(A_n \cup B_n) \to 0$ $(n \to \infty)$; *null-additive* [7], if $\mu(E \cup F) = \mu(E)$ whenever $E, F \in \mathcal{A}$ and $\mu(F) = 0$; *weakly null-additive* [7], if $\mu(E \cup F) = 0$ whenever $\mu(E) = \mu(F) = 0$; *converse null-additive* [7], if for any $A, B \in \mathcal{A}, A \subset B$ and $\mu(A) = \mu(B) < \infty$ imply $\mu(A - B) = 0$.

### 2.2. Null-Additivity of Monotone Measures

**Definition 2** ([1])**.** *A monotone measure $\mu$ defined on $\mathcal{A}$ is called null-continuous, if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}_{n\in N} \subset \mathcal{A}$ such that $\mu(A_n) = 0, n = 1, 2, \cdots$.*

The following example shows that not all monotone measures are null-continuous:

**Example 1.** *Let $X = [0, 1)$ and $\mathcal{A} = 2^X$. The weakest capacity $\mu : \mathcal{A} \to [0, 1]$ is defined by*

$$\mu(A) = \begin{cases} 0 & \text{if } A \neq X, \\ 1 & \text{if } A = X. \end{cases}$$

*Then, $\mu$ is not null-continuous. In fact, take $A_n = [0, 1 - \frac{1}{n+1}), n = 1, 2, \ldots$, then $A_n \nearrow X$ and $\mu(A_n) = 0, n = 1, 2, \ldots$, but $\mu(X) = 1$. Note that $\mu$ is not weakly null-additive, and hence it is not null-additive either.*

Obviously, the continuity from below of $\mu$ implies null-continuity. The following example shows that null-continuity of monotone measures is weaker than continuity from below (see also [1]).

**Example 2.** *Let $X = \mathbb{N}$ and $\mathcal{A} = 2^{\mathbb{N}}$. Define the monotone measure $\mu : \mathcal{A} \to [0,1]$ as*

$$\mu(E) = \begin{cases} 1 & \text{if } |E^c| < \infty, \\ \sum_{i \in E} \frac{1}{2^i} & \text{if } |E^c| = \infty, \\ 0 & \text{if } E = \emptyset, \end{cases}$$

*where $E^c$ stands for the complement of $E$ and $|E^c|$ stands for the number of element of $E^c$. Obviously, for any $A \in \mathcal{A}$, $\mu(A) = 0$ iff $A = \emptyset$, thus $\mu$ is null-continuous, null-additive and weakly null-additive. However, it is not continuous from below. In fact, we take $A_n = \{2, 3, \ldots, n\}, n = 2, 3, \ldots$, then $A_n \nearrow X \setminus \{1\}$ and $\mu(A_n) = \sum_{i=2}^{n} \frac{1}{2^i}$. Thus, $\mu(A_n) \to \frac{1}{2} (n \to \infty)$, but $\mu(X \setminus \{1\}) = 1$.*

**Definition 3** ([20])**.** *A monotone measure $\mu$ is called countably weakly null-additive, if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ whenever $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ and $\mu(A_n) = 0, n = 1, 2, \cdots$.*

**Proposition 1.** *$\mu$ is countably weakly null-additive if and only if $\mu$ is both weakly null-additive and null-continuous.*

The weak null-additivity and null-continuity are independent each other, as the following examples show:

**Example 3.** *Let $X = \mathbb{N}$ and $\mathcal{A} = 2^{\mathbb{N}}$. Define the monotone measure $\mu : \mathcal{A} \to [0,1]$ by*

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty, \\ \sum_{i \in A} \frac{1}{2^i} & \text{if } |A| = \infty, \end{cases}$$

*where $|A|$ stands for the number of elements of $A$. Obviously, $\mu$ is weakly null-additive. It is easy to see that $\mu$ is not null-continuous. In fact, we take $A_n = \{1, 2, \ldots, n\}, n = 1, 2, 3, \ldots$, then $A_n \nearrow X$ and $\mu(A_n) = 0$. However, $\mu(X) = 1$. Note that $\mu$ is not strongly order continuous (in fact, we take $A = \{1\}$ and $A_n = \{1\} \cup \{n+1, n+2, \ldots\}, n = 1, 2, \ldots$, then $A_n \searrow A$ and $\mu(A) = 0$. However, $\mu(A_n) > \frac{1}{2}, n = 1, 2, \ldots$, so $\mu(A_n) \nrightarrow 0$).*

**Example 4.** *Let $X = \{a, b\}$ and $\mathcal{A} = 2^X$. Put*

$$\mu(E) = \begin{cases} 0 & \text{if } E \neq X, \\ 1 & \text{if } E = X. \end{cases}$$

*Then, obviously $\mu$ is not a weakly null-additive monotone measure, but it is strongly order continuous and null-continuous.*

We show some relations among the above introduced structural characteristics.

**Proposition 2** ([1])**.** *If $\mu$ is strongly order continuous and weakly null-additive, then it is null-continuous.*

**Proposition 3.** *If $\mu$ is strongly order continuous and weakly null-additive, then it is countably weakly null-additive.*

By using Propositions 2 and 3, we get a set of sufficient conditions that a monotone measure is null-continuous, as follows:

**Proposition 4.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then, each of the following conditions (i)–(v) implies null-continuity of $\mu$:*
*(i) $\mu$ is countably weakly null-additive;*
*(ii) $\mu$ is order continuous and null-additive;*
*(iii) $\mu$ is order continuous and sub-additive;*
*(iv) $\mu$ is order continuous and weakly asymptotic null-additive;*
*(v) $\mu$ is $\sigma$-subadditive, i.e., $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$.*

**Remark 1.** *In [5], Pap introduced the concept of $\sigma$-null-additivity for a general set function (See Definition 2.7 in [5]). When $\mu$ is a monotone measure on $\mathcal{A}$, the $\sigma$-null-additivity of $\mu$ implies countable weak null-additivity. A countably weakly null-additive monotone measure may not be $\sigma$-null-additive. From Proposition 2.3 in [5], we can deduce the following result: if $\mu$ is a monotone measure on $\mathcal{A}$, then $\mu$ is $\sigma$-null-additive if and only if $\mu$ is null-additive and countably weakly null-additive. Therefore, the $\sigma$-null-additivity of monotone measure implies null-continuity.*

## 3. Null-Continuity and Convergence of a Sequence of Measurable Functions

Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $\mathcal{F}$ be the class of all finite real-valued $\mathcal{A}$-measurable functions on $(X, \mathcal{A})$ and let $\mathcal{F}^+$ denote the class of all finite nonnegative real-valued $\mathcal{A}$-measurable functions on $(X, \mathcal{A})$.

Let $f, f_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$. We say that $\{f_n\}$ *converges almost everywhere to $f$ on $X$* , and denote it by $f_n \xrightarrow{a.e.} f[\mu]$, if there is a subset $E \in \mathcal{A}$ such that $\mu(E) = 0$ and $f_n \to f$ on $X \setminus E$; $\{f_n\}$ *converges almost uniformly to $f$ on $X$*, and denote it by $f_n \xrightarrow{a.u.} f[\mu]$, if, for any $\epsilon > 0$, there is a subset $E_\epsilon \in \mathcal{A}$ such that $\mu(X \setminus E_\epsilon) < \epsilon$ and $f_n$ converges to $f$ uniformly on $E_\epsilon$; $\{f_n\}$ *converge to $f$ in measure $\mu$* on $X$, in symbols $f_n \xrightarrow{\mu} f$, if, for any $\sigma > 0$, $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \sigma\}) = 0$.

Let $A \in \mathcal{A}$. We say *a proposition $P$ with respect to points in $A$ is almost everywhere true on $A$*, denoted by "$P$ *a.e.*$[\mu]$ on $A$", if there exists $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $P$ is true on $A - N$. When $X = A$, "$P$ *a.e.*$[\mu]$ on $A$" is denoted by "$P$ *a.e.*$[\mu]$". For example, for $f, g \in \mathcal{F}$, $f = g$ *a.e.*$[\mu]$, means that $\mu(\{x \in X \mid f \neq g\}) = 0$.

In [1], the relations among the null-continuity and other different continuity were discussed. Now, we use the convergence of sequence of measurable functions to describe the null-continuity.

**Proposition 5.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then, the following statements are equivalent:*
*(1) $\mu$ is null-continuous;*
*(2) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n = 0$ a.e.$[\mu]$, $n = 1, 2, \ldots$, if $f_n \nearrow f$, then $f = 0$ a.e.$[\mu]$;*
*(3) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n \leq C$ a.e.$[\mu]$, $n = 1, 2, \ldots$, if $f_n \nearrow f$, then $f \leq C$ a.e.$[\mu]$, where $C$ is a constant.*

**Proof.** (1) $\Rightarrow$ (2). Suppose that $\mu$ is null-continuous. For $f \in \mathcal{F}^+$ and $\{f_n\}_n \subset \mathcal{F}^+$, denote $A = \{x \mid f(x) > 0\}$ and $A_n = \{x \mid f_n(x) > 0\}$ $(n = 1, 2, \ldots)$. If $f_n = 0$ *a.e.*$[\mu]$, $n = 1, 2, \ldots$, and $f_n \nearrow f$, then $A_1 \subset A_2 \subset \ldots$ and $\mu(A_n) = 0$, $n = 1, 2, \ldots$. Noting that $A = \bigcup_{n=1}^{\infty} A_n$ and $\mu$ is null-continuous, we have $\mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n) = 0$, i.e., $f = 0$ *a.e.*$[\mu]$.

(2) $\Rightarrow$ (1). For every increasing sequence $\{A_n\}_{n\in N} \subset \mathcal{A}$ such that $\mu(A_n) = 0, n = 1, 2, \cdots$, it is easy to see that $\chi_{A_n} \nearrow \chi_A$ (where $A = \bigcup_{n=1}^{\infty} A_n$, $\chi_{A_n}$ and $\chi_A$ are the characteristic functions of $A_n$ and $A$, respectively) and $\chi_{A_n} = 0$ *a.e.*$[\mu]$. It follows from condition (2) that $\chi_A = 0$ *a.e.*$[\mu]$, i.e., $\mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n) = 0$. Therefore, $\mu$ is null-continuous.

(1) $\Leftrightarrow$ (3). The proof is similar. $\square$

The Riesz theorem is one of the most important convergence theorems in classical measure theory. It states that, to each sequence of measurable functions which converges in measure, there is a subsequence converging almost everywhere, i.e., for any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, if $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $f_{n_i} \xrightarrow{a.e.} f[\mu]$ (see [10]).

This important theorem was generalized to monotone measure spaces. In [8], the concept of property $(S)$ of monotone measures was introduced, and it was shown that the conclusion of the classical Riesz theorem holds for a monotone measure $\mu$ if and only if $\mu$ has property $(S)$ ([8], also see [5,7,21]). We recall these results.

**Definition 4** ([8])**.** *A monotone measure $\mu$ is called to have property $(S)$, if for any $\{A_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty}\mu(A_n) = 0$, there exists a subsequence $\{A_{n_i}\}_{i\in\mathbb{N}}$ of $\{A_n\}_{n\in\mathbb{N}}$ such that $\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_{n_i}\right) = 0$.*

The following result is a version of Riesz's theorem for monotone measures [8]:

**Theorem 1.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then, the following statements are equivalent:*
*(1) $\mu$ has property $(S)$.*
*(2) For any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, if $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $f_{n_i} \xrightarrow{a.e.} f[\mu]$.*

The following implication between the property (S) and null-continuity is shown in [6]:

**Proposition 6.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If $\mu$ has property (S), then $\mu$ is null-continuous.*

Thus, we obtain a necessary condition that Riesz's theorem holds for monotone measures:

**Proposition 7.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If the classical Riesz's theorem remains valid for the monotone measure $\mu$, then $\mu$ is null-continuous.*

Note that null-continuity may not imply property (S) (see [6]).

In the following, we concentrate on the discussion of convergence of measurable functions on $S$-compact spaces.

A measurable space $(X, \mathcal{A})$ is said to be $S$-compact, if, for any sequence of sets in $\mathcal{A}$, there exists some convergent subsequence, i.e., $\forall\, \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ , $\exists\, \{A_{n_i}\}_{i\in\mathbb{N}} \subset \{A_n\}_{n\in\mathbb{N}}$ such that

$$\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} A_{n_i} = \bigcup_{s=1}^{\infty}\bigcap_{i=s}^{\infty} A_{n_i}. \tag{1}$$

Observe that, if $X$ is countable, then $(X, \mathcal{A})$ is $S$-compact space ([7]). The converse may not be true (see [7]).

When $(X, \mathcal{A})$ is an $S$-compact space, the converse of Proposition 6 is true:

**Proposition 8.** *Let $(X, \mathcal{A})$ be an $S$-compact space and $\mu \in \mathcal{M}$. Then, $\mu$ is null-continuous if and only if $\mu$ has property (S).*

**Proof.** If $\mu$ has property $(S)$, it follows from Proposition 6 that $\mu$ is null-continuous. Conversely, assume $\mu$ is null-continuous. For any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\lim_{n\to\infty}\mu(A_n) = 0$, since $(X, \mathcal{A})$ is $S$-compact, so there exists a subsequence $\{A_{n_i}\}_{i\in\mathbb{N}}$ of $\{A_n\}_{n\in\mathbb{N}}$ such that

$$\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} A_{n_i} = \bigcup_{s=1}^{\infty}\bigcap_{i=s}^{\infty} A_{n_i}. \tag{2}$$

For any fixed $s = 1, 2, \cdots$, since $\lim_{i\to\infty}\mu(A_{n_i}) = 0$ and

$$0 \leq \mu\left(\bigcap_{i=s}^{\infty} A_{n_i}\right) \leq \mu(A_{n_i}) \quad \forall i \geq s,$$

it follows that $\mu\left(\bigcap_{i=s}^{\infty} A_{n_i}\right) = 0$. By the null-continuous and Equation (2), then $\mu\left(\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} A_{n_i}\right) = \mu\left(\bigcup_{s=1}^{\infty}\bigcap_{i=s}^{\infty} A_{n_i}\right) = 0$. Therefore, $\mu$ has property (S). $\square$

The following is a direct consequence of Propositions 2 and 8:

**Corollary 1.** *Let $(X, \mathcal{A})$ be an $S$-compact space and $\mu \in \mathcal{M}$. If $\mu$ is strongly order continuous and weakly null-additive, then $\mu$ has property (S).*

Combining Theorem 1 and Proposition 6, we obtain a version of Riesz's theorem for monotone measure on $S$-compact spaces.

**Theorem 2.** (Riesz's theorem) *Let $(X, \mathcal{A})$ be an $S$-compact space and $\mu \in \mathcal{M}$. Then, the following statements are equivalent:*
*(1) $\mu$ is null-continuous.*
*(2) For any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, if $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $f_{n_i} \xrightarrow{a.e.} f[\mu]$.*

From Propositions 2, 4, and Theorem 2, we obtain the following corollaries.

**Corollary 2.** *Let $(X, \mathcal{A})$ be an $S$-compact space and let $\mu \in \mathcal{M}$ be strongly order continuous and weakly null-additive. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $f_{n_i} \xrightarrow{a.e.} f[\mu]$.*

**Corollary 3.** *Let $(X, \mathcal{A})$ be an $S$-compact space and $\mu \in \mathcal{M}$. Then, each of the conditions (i)–(v) in Proposition 4 is a sufficient condition that the conclusion of the classical Riesz theorem holds for monotone measure $\mu$.*

Egoroff's theorem is one of the most important convergence theorems in classical measure theory. It states that almost everywhere convergence implies almost uniform convergence on a measurable set of finite measure ([17,22]). In general, the Egoroff theorem is not valid for monotone measure without additional conditions. This well-known theorem was generalized to monotone measure spaces ([23,24]).

By using the equivalence between property $(S)$ and null-continuity on $S$-compact spaces, we present a version of Egoroff's theorem for monotone measures on $S$-compact spaces, as follows:

**Theorem 3** (Egoroff's theorem)**.** *Let $(X, \mathcal{A})$ be an $S$-compact space and let $\mu \in \mathcal{M}$ be weakly null-additive and strongly order continuous. Then, for any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,*

$$f_n \xrightarrow{a.e.} f[\mu] \implies f_n \xrightarrow{a.u.} f[\mu]. \tag{3}$$

**Proof.** From Propositions 2 and 8, then $\mu$ has property $(S)$. By Egoroff's theorem for monotone measures (see Theorem 1 in [23]), we get the conclusion (3). $\square$

Observe that weak null-additivity and strong order continuity are independent of each other, as shown in Examples 3 and 4.

By using the fact that null-additivity implies property (S) on $S$-compact spaces (Proposition 8), we can obtain the following results (see also [25]).

**Theorem 4.** *Let $(X, \mathcal{A})$ be an S-compact space and $\mu \in \mathcal{M}$ be strongly order continuous. The following statements are equivalent:*
*(1) $\mu$ is weakly null-additive;*
*(2) $\mu$ is weakly asymptotic null-additive;*
*(3) $\mu$ has (p.g.p.);*
*(4) For any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, if $f_n \xrightarrow{a.e.} f[\mu]$ and $g_n \xrightarrow{a.e.} g[\mu]$, then $f_n + g_n \xrightarrow{a.e.} f + g[\mu]$;*
*(5) For any $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, then $f_n + g_n \xrightarrow{\mu} f + g$.*

**Proof.** It is obvious that $(3) \Rightarrow (2) \Rightarrow (1)$. Now, we prove $(1) \Rightarrow (3)$. Suppose that $\mu$ is weakly null-additive. Since $\mu$ is strongly order continuous, from Proposition 2, then $\mu$ is null-continuous and hence it follows from Proposition 8 that $\mu$ has property (S). We assume that $\mu$ has not (p.g.p.). Then, there exist $\epsilon_0 > 0$ and two sequences $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ such that

$$\mu(E_n) \vee \mu(F_n) \to 0 \quad \text{while} \quad \mu(E_n \cup F_n) \geq \epsilon_0, \quad \forall\, n \geq 1.$$

By using property (S), there exist subsequences $\{E_{n_k}\}_{k\in\mathbb{N}}$ and $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\mu\left(\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} E_{n_k}\right) = 0 \quad \text{and} \quad \mu\left(\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} F_{n_k}\right) = 0.$$

By the weak null-additivity of $\mu$, we have

$$\mu\left(\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} (E_{n_k} \cup F_{n_k})\right) = \mu\left((\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} E_{n_k}) \cup (\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} F_{n_k})\right) = 0.$$

Therefore, from the strong order continuity of $\mu$, we have

$$\begin{aligned}
\limsup_{s\to\infty} \mu(E_{n_s} \cup F_{n_s}) &\leq \lim_{s\to\infty} \mu\left(\bigcup_{k=s}^{\infty} (E_{n_k} \cup F_{n_k})\right) \\
&= \mu\left(\bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} (E_{n_k} \cup F_{n_k})\right) \\
&= 0.
\end{aligned}$$

This is in contradiction with the fact that

$$\mu(E_{n_s} \cup F_{n_s}) \geq \epsilon_0, \quad \forall s \geq 1.$$

This shows that $(1) \Rightarrow (3)$. Therefore, $(1) \Leftrightarrow (2) \Leftrightarrow (3)$.
For other equivalences, see [25]. □

The following is a generalization of result in classical measure theory (see [22]).

**Theorem 5.** *Let $(X, \mathcal{A})$ be an S-compact space and $\mu \in \mathcal{M}$ be weakly null-additive and strongly order continuous. If $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then $f = g$ a.e.$[\mu]$*

**Proof.** For any given $\epsilon > 0$, we have

$$\left\{x \in X \middle| |f(x) - g(x)| \geq \epsilon\right\} \subset \left\{x \in X \middle| |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \bigcup \left\{x \in X \middle| |f_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}.$$

Since $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, we have

$$\mu\Big(\{x \in X||f_n(x) - f(x)| \geq \frac{\epsilon}{2}\}\Big) \bigvee \mu\Big(\{x \in X||f_n(x) - g(x)| \geq \frac{\epsilon}{2}\}\Big) \longrightarrow 0 \ (n \to \infty).$$

From Theorem 4, we know that $\mu$ has $(p.g.p.)$, and therefore

$$\mu\Big(\{x \in X||f_n(x) - f(x)| \geq \frac{\epsilon}{2}\} \cup \{x \in X||f_n(x) - g(x)| \geq \frac{\epsilon}{2}\}\Big) \longrightarrow 0 \ (n \to \infty).$$

Hence, it is clear that

$$\mu\Big(\Big\{x \in X||f(x) - g(x)| \geq \epsilon\Big\}\Big) = 0.$$

Denote $A_k = \{x \in X||f(x) - g(x)| \geq \frac{1}{k}\}, k = 1, 2, \ldots$, then $\{A_k\}_{k \in \mathbb{N}}$ is an increasing sequence of measurable subsets in $\mathcal{A}$ and $\mu(A_k) = 0, k = 1, 2, \ldots$. Noting that, from Proposition 2, $\mu$ is null-continuous, then

$$\mu\Big(\Big\{x \in X|f(x) \neq g(x)\Big\}\Big) = \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = 0,$$

i.e., $f = g \ a.e.[\mu]$. $\square$

Similarly, we have the following:

**Theorem 6.** *Let $(X, \mathcal{A})$ be an S-compact space and $\mu \in \mathcal{M}$ be weakly null-additive and strongly order continuous. If $f_n \xrightarrow{\mu} f$ and $g \in \mathcal{F}$ such that $f = g \ a.e.[\mu]$, then $f_n \xrightarrow{\mu} g$.*

## 4. Null-Continuity and Integrals

We recall two kinds of basic nonlinear integrals based on monotone measures, namely, the Sugeno integral (also called fuzzy integral) [15] and the Choquet integral [14].

Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and let $\mu \in \mathcal{M}$ be fixed, and $f \in \mathcal{F}^+$.

The Sugeno integral (or fuzzy integral) of $f$ on $X$ with respect to $\mu$, is defined by

$$\int^{Su} f \, d\mu = \sup_{0 \leq \alpha < +\infty} \Big[\min\Big\{\alpha, \mu(\{x : f(x) \geq \alpha\})\Big\}\Big]. \tag{4}$$

The Choquet integral of $f$ on $X$ with respect to $\mu$, is defined by

$$\int^{Cho} f \, d\mu = \int_0^{\infty} \mu(\{x : f(x) \geq t\}) \, dt, \tag{5}$$

where the right-hand side integral is the improper Riemann integral.

For $A \in \mathcal{A}$, define $\int_A^{Su} f d\mu = \int^{Su} f\chi_A d\mu$ and $\int_A^{Cho} f d\mu = \int^{Cho} f\chi_A d\mu$.

Note that when $\mu$ is a $\sigma$-additive measure, the Choquet integral coincides with the Lebesgue integral. Thus, it is a generalization of the Lebesgue integral. The Sugeno integral, which is based on monotone measure and relates the logic addition " $\vee$ "and logic multiplication " $\wedge$ ", is a special kind of nonlinear integral. It is not linear even for a probability measure, so the Sugeno integral is not a generalization of the Lebesgue integral.

We recall some basic properties of the Sugeno integral and the Choquet integral, as follows:

**Proposition 9.** *For any $A, B \in \mathcal{A}$, $f, g \in \mathcal{F}^+$, we have*
*(i) if $\mu(A) = 0$ then $\int_A^{(*)} f d\mu = 0$;*
*(ii) if $f \leq g$ on $A$, then $\int_A^{(*)} f d\mu \leq \int_A^{(*)} g d\mu$;*
*(iii) if $A \subset B$, then $\int_A^{(*)} f d\mu \leq \int_B^{(*)} f d\mu$,*
*where the integral $\int^{(*)}$ in the above formulas stands for the Sugeno integral or the Choquet integral.*

**Proposition 10** ([7])**.** *Let $A \in \mathcal{A}$, $a \geq 0$, $f \in \mathcal{F}^+$. Then,*
*(i)* $\int_A^{Su}(a \wedge f)d\mu = a \wedge \int^{Su} f d\mu$; $\int_A^{Su} a d\mu = a \wedge \mu(A)$.
*(ii)* $\int_A^{Cho} a \cdot f d\mu = a \cdot \int_A^{Cho} f d\mu$, *in particular,* $\int_A^{Cho} a d\mu = a \cdot \mu(A)$.

**Proposition 11.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$. If $f = 0$ a.e.$[\mu]$ on $A$, then $\int_A^{Su} f d\mu = 0$ and $\int_A^{Cho} f d\mu = 0$.*

**Theorem 7.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $\mu$ be null-continuous. For any $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$, if $\int_A^{Su} f d\mu = 0$, then $f = 0$ a.e.$[\mu]$ on $A$.*

**Proof.** We use a proof by contradiction. Assume the conclusion is not true, i.e.,

$$\mu(\{x \in X \mid f(x) > 0\} \cap A) > 0.$$

Denote $A_k = \{x \in X \mid f(x) \geq \frac{1}{k}\} \cap A$, $k = 1, 2, \ldots$, then

$$\{x \in X \mid f(x) > 0\} \cap A = \bigcup_{k=1}^{\infty} \{x \in X \mid f(x) \geq \frac{1}{k}\} \cap A.$$

Note that $\{A_k\}_{k \in N}$ is an increasing sequence of measurable subsets in $\mathcal{A}$ and $\mu$ is null-continuous, so there exists $k_0$ such that $\mu(\{x \in X \mid f(x) \geq \frac{1}{k_0}\} \cap A) = c_0 > 0$ (otherwise, by the null-continuity of $\mu$, $\mu(\{x \in X \mid f(x) > 0\} \cap A) = 0$). Consequently, we have

$$\begin{aligned}
\int_A^{Su} f\, d\mu &= \sup_{0 \leq \alpha < +\infty} \left[\min\left\{\alpha, \mu(\{x : f(x) \geq \alpha\}) \cap A\right\}\right] \\
&\geq \min\left\{\frac{1}{k_0}, \mu(\{x \in X \mid f(x) \geq \frac{1}{k_0}\} \cap A)\right\} \\
&\geq \min\left\{\frac{1}{k_0}, c_0\right\} > 0.
\end{aligned}$$

This contradicts $\int_A^{Su} f d\mu = 0$. □

**Theorem 8.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $\mu$ be null-continuous. For any $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$, if $\int_A^{Cho} f d\mu = 0$, then $f = 0$ a.e.$[\mu]$ on $A$.*

**Proof.** Denote $A_k = \{x \in X \mid f(x) \geq \frac{1}{k}\} \cap A$, $k = 1, 2, \ldots$. From Propositions 9 and 10, for any $k = 1, 2, \ldots$ we have

$$\int_A^{Cho} f d\mu \geq \int_{A_k}^{Cho} f d\mu \geq \int_{A_k}^{Cho} \frac{1}{k} d\mu = \frac{1}{k} \cdot \mu(A_k).$$

If $\int_A^{Cho} f d\mu = 0$, then for any $k = 1, 2, \ldots$, $\mu(A_k) = 0$. Noting that

$$\{x \in X \mid f(x) > 0\} \cap A = \bigcup_{k=1}^{\infty} A_k$$

and $A_1 \subset A_2 \subset \ldots$, by the null-continuity of $\mu$, we have $\mu(\{x \in X \mid f(x) > 0\} \cap A) = 0$, i.e., $f = 0$ *a.e.*$[\mu]$ on $A$. □

As special results of Theorems 7 and 8, we have the following:

**Corollary 4** ([7])**.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If $\mu$ is continuous from below, then the conclusions of Theorem 7 and 8 hold.*

**Corollary 5.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If $\mu$ satisfies one of the following conditions:*
*(i) $\mu$ is countably weakly null-additive;*
*(ii) $\mu$ is order continuous and null-additive;*
*(iii) $\mu$ is order continuous and sub-additive;*
*(iv) $\mu$ is order continuous and weakly asymptotic null-additive;*
*(v) $\mu$ is $\sigma$-subadditive,*
*then the conclusions of Theorems 7 and 8 hold.*

**Proposition 12.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then, the following statements are equivalent:*
*(1) $\mu$ is null-continuous;*
*(2) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n = 0$ a.e.$[\mu], n = 1, 2, \ldots$, if $f_n \nearrow f$, then*

$$\int^{Su} f d\mu = 0;$$

*(3) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n = 0$ a.e.$[\mu], n = 1, 2, \ldots$, if $f_n \nearrow f$, then*

$$\int^{Cho} f d\mu = 0.$$

**Proof.** We only prove (1) $\Leftrightarrow$ (2), for (1) $\Leftrightarrow$ (3) the proof is similar. If $\mu$ is null-continuous, then, from Proposition 5, we have $f = 0$ *a.e.*$[\mu]$. Therefore, it follows from Proposition 11 that $\int^{Su} f d\mu = 0$. Conversely, if condition (2) holds, then it follows from Theorem 7 that $\int^{Su} f d\mu = 0$ implies $f = 0$ *a.e.*$[\mu]$. Using Proposition 5 again, then $\mu$ is null-continuous. □

In the following, we discuss the null-continuity of monotone measures defined by integral.

Given $\mu \in \mathcal{M}$ and $f \in \mathcal{F}^+$, then the Choquet integral (resp. the Sugeno integral) of $f$ with respect to $\mu$ determines a new monotone measure $\lambda_f^{(Cho)}$ (resp. $\lambda_f^{(Su)}$), as follows:

$$\lambda_f^{(Cho)}(A) = \int_A^{Cho} f d\mu, \quad \forall A \in \mathcal{A}$$

$$\left(\text{resp.} \quad \lambda_f^{(Su)}(A) = \int_A^{Su} f d\mu, \quad \forall A \in \mathcal{A}\right).$$

Such the new monotone measures $\lambda_f^{(Cho)}$ and $\lambda_f^{(Su)}$ are absolutely continuous with respect to the monotone measure $\mu$, respectively, i.e., $\lambda_f^{(Cho)} \ll_I \mu$ and $\lambda_f^{(Su)} \ll_I \mu$ (see [7,17]).

In the following, we show that the new monotone measures $\lambda_f^{(Cho)}$ and $\lambda_f^{(Su)}$ preserve the structural characteristic of null-continuity of the original monotone measure $\mu$.

**Theorem 9.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If $\mu$ is null-continuous, then so are both $\lambda_f^{(Cho)}$ and $\lambda_f^{(Su)}$.*

**Proof.** We only prove the case of $\lambda_f^{(Cho)}$; for $\lambda_f^{(Su)}$, the proof is similar.

Let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ be an increasing sequence with $\lambda_f^{(Cho)}(A_n) = 0, n = 1, 2, \cdots$. Then, $\int_{A_n}^{Cho} f d\mu = 0, n = 1, 2, \cdots$ and, from Theorem 8, we have $f = 0$ *a.e.*$[\mu]$ on $A_n, n = 1, 2, \ldots$, i.e., $\mu(\{x \in X \mid f(x) > 0\} \cap A_n) = 0, n = 1, 2, \ldots$. Since

$$\{x \in X \mid f(x) > 0\} \cap A_n \nearrow \bigcup_{n=1}^{\infty} \{x \in X \mid f(x) > 0\} \cap A_n$$

and $\mu$ is null-continuous, then we have

$$\mu\Big(\bigcup_{n=1}^{\infty}\{x \in X \mid f(x) > 0\} \cap A_n\Big) = 0$$

and hence $f = 0$ $a.e.[\mu]$ on $\bigcup_{n=1}^{\infty} A_n$. Therefore, $\int_{\bigcup_{n=1}^{\infty} A_n}^{Cho} f d\mu = 0$, i.e., $\lambda_f^{(Cho)}(\bigcup_{n=1}^{\infty} A_n) = 0$. This shows that $\lambda_f^{(Cho)}$ is null-continuous. □

## 5. Pan-Integral and Concave Integral

In the above discussions, we only discussed the Sugeno integral and the Choquet integral. There are two other important nonlinear integrals, the pan-integral [7,11], and the concave integral [12,13] (see also [26,27]), as follows:

Given $\mu \in \mathcal{M}$ and $f \in \mathcal{F}^+$.

The *pan-integral* of $f$ on $X$ with respect to $\mu$ (and the standard addition + and multiplication $\cdot$) is defined by

$$\begin{aligned}\int^{pan} f d\mu \;=\; \sup\Big\{ & \sum_{i=1}^{n} \lambda_i \mu(E_i) : \sum_{i=1}^{n} \lambda_i \chi_{E_i} \le f, \\ & \{E_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a finite measurable partition of } X, \lambda_i \ge 0, n \in \mathbb{N}\Big\},\end{aligned}$$

where $\chi_{E_i}$ are the characteristic functions of $E_i$, respectively.

The *concave integral* of $f$ on $X$ is defined by

$$\begin{aligned}\int^{cav} f d\mu \;=\; \sup\Big\{ & \sum_{i=1}^{n} \lambda_i \mu(E_i) : \sum_{i=1}^{n} \lambda_i \chi_{E_i} \le f, \\ & \{E_i\}_{i=1}^{n} \subset \mathcal{A}, \lambda_i \ge 0, n \in \mathbb{N}\Big\},\end{aligned}$$

where $\chi_{E_i}$ are the characteristic functions of $E_i$, respectively.

Note that, when $\mu$ is a $\sigma$-additive measure, the Choquet integral, the pan-integral, and the concave integral coincide with the Lebesgue integral. Thus, these three integrals are all generalizations of the Lebesgue integral. However, they are significantly different from each other. Observing that the Choquet integral is based on chains of measurable subsets of $X$, the pan-integral is related to finite measurable partitions of $X$ and the concave integral to any finite set systems of measurable subsets of $X$.

In general, for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$, we have

$$\int^{Cho} f d\mu \le \int^{cav} f d\mu \quad \text{and} \quad \int^{pan} f d\mu \le \int^{cav} f d\mu,$$

but the converse inequalities may not be true, and $\int^{Ch} f d\mu$ and $\int^{pan} f d\mu$ are incomparable (see [28–30]).

For the pan-integral and the concave integral, the conclusions of Theorem 7 (or Theorem 8) and Proposition 12 remain valid.

**Theorem 10.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $\mu$ be null-continuous. For any $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$, if $\int_A^{pan} f d\mu = 0$ (or $\int_A^{cav} f d\mu = 0$), then $f = 0$ $a.e.[\mu]$ on $A$.*

**Proof.** It is similar to the proof of Theorem 8. □

**Proposition 13.** *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then, the following statements are equivalent:*
*(1) $\mu$ is null-continuous;*
*(2) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n = 0$ a.e.$[\mu]$, $n = 1, 2, \ldots$, if $f_n \nearrow f$, then*

$$\int^{pan} f d\mu = 0;$$

*(3) For any $f \in \mathcal{F}^+$, $\{f_n\}_n \subset \mathcal{F}^+$ with $f_n = 0$ a.e.$[\mu]$, $n = 1, 2, \ldots$, if $f_n \nearrow f$, then*

$$\int^{cav} f d\mu = 0.$$

**Proof.** It is similar to the proof of Proposition 12. □

## 6. Conclusions

In this paper, we have discussed a kind of special continuity of monotone measures—null-continuity. The main results are Propositions 5, 7, and 8, Theorem 2 (Riesz's theorem), Theorem 3 (Egoroff's theorem), and Theorems 5 and 7–9. As we have seen, under the condition of null-continuity, many results of classical measure and integral theory are generalized to monotone measure spaces, and it is shown that null-continuity is a necessary condition that the classical Riesz's theorem remains valid for monotone measures. Thus, the null-continuity is a noteworthy structure characteristic of monotone measures. On the other hand, our several results on the convergence of measurable functions have been obtained on $S$-compact spaces. Observe that countable measurable spaces are $S$-compact ([7]) and countable measurable spaces with finite monotone measures are often encountered in practice [31], so our results seem to be useful.

We point out that, in our discussions, we only consider the Sugeno integral, the Choquet integral, pan-integral, and concave integral. In fact, several results we obtained in Section 4 can be easily extended to the class of generalized Sugeno integrals (e.g., see [32,33]), such as, seminormed fuzzy integrals under the semicopula. For instance, Theorem 7 remains valid under the condition that semicopula has no zero divisors, and the convergence result of Proposition 12 is also true for generalized Sugeno integrals. In addition, Theorem 9 can be improved in this direction.

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