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A New Approach in the Study of Oscillation Criteria of Even-Order Neutral Differential Equations

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Abstract: Based on the comparison with first-order delay equations, we establish a new oscillation criterion for a class of even-order neutral differential equations. Our new criterion improves a number of existing ones. An illustrative example is provided.

Keywords: even-order differential equations; neutral delay; oscillation

1. Introduction

In the last decade, many studies have been carried out on the oscillatory behavior of various types of functional differential equations, see [1–24] and the references cited therein. As a result of numerous applications in technology and natural science, the issue of oscillation of nonlinear neutral delay differential equation has caught the attention of many researchers, see [1,3–5,8,12,17,19,22–24]. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines, see [11].

In this paper, we are concerned with improving the oscillation criteria for the even-order neutral differential equation of the form

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\,x^{\alpha}(\sigma(t)) = 0, \tag{1}$$

where $t \geq t_0$, $n \geq 4$ is an even natural number and $z(t) := x(t) + p(t)\,x(\tau(t))$. In this work, we assume that α is a quotient of odd positive integers, $r \in C[t_0,\infty)$, $r(t) > 0$, $r'(t) \geq 0$, $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s = \infty$, $p, q \in C[t_0,\infty)$, $q(t) > 0$, $0 \leq p(t) < p_0 < \infty$, $q(t)$ is not identically zero for large t, $\tau \in C^1[t_0,\infty)$, $\sigma \in C[t_0,\infty)$, $\tau'(t) > 0$, $\tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of (1) we mean a function $x \in C^3[t_y,\infty)$, $t_y \geq t_0$, which has the property $r(t)\left(z^{(n-1)}(t)\right)^{\alpha} \in C^1[t_y,\infty)$, and satisfies (1) on $[t_y,\infty)$. We consider only those solutions x of (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$, for all $T \geq t_y$. A solution x of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

A neutral delay differential equation is a differential equation in which the highest-order derivative of the unknown function appears both with and without delay.

In the following, we briefly review some important oscillation criteria obtained for higher-order neutral equations which can be seen as a motivation for this paper.

In 1998, based on establishing comparison theorems that compare the nth-order equation with only one first-order delay differential equations, Zafer [23] proved that the even-order differential equation

$$z^{(n)}(t) + q(t)\, x(\sigma(t)) = 0 \tag{2}$$

is oscillatory if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)\, 2^{(n-1)(n-2)}}{\mathrm{e}}, \tag{3}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)\, 2^{(n-1)(n-2)}, \;\; \sigma'(t) \geq 0.$$

where $Q(t) := \sigma^{n-1}(t)\,(1 - p(\sigma(t)))\, q(t)$. In a similar approach, Zhang and Yan [24] proved that (2) is oscillatory if either

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}}, \tag{4}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\,\mathrm{d}s > (n-1)!, \;\; \sigma(t) \geq 0.$$

It's easy to note that $(n-1)! < (n-1)\, 2^{(n-1)(n-2)}$ for $n > 3$, and hence results in [24] improved results of Zafer in [23].

For nonlinear equation, Xing et al. [22] proved that (1) is oscillatory if

$$\left(\sigma^{-1}(t)\right)' \geq \sigma_0 > 0, \; \tau'(t) \geq \tau_0 > 0, \; \tau^{-1}(\sigma(t)) < t$$

and

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{\widehat{q}(s)}{r(s)} \left(s^{n-1}\right)^{\alpha} \mathrm{d}s > \left(\frac{1}{\sigma_0} + \frac{p_0^{\alpha}}{\sigma_0 \tau_0}\right) \frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{5}$$

where $\widehat{q}(t) := \min\left\{q\left(\sigma^{-1}(t)\right), q\left(\sigma^{-1}(\tau(t))\right)\right\}$.

If we apply the previous results to the equation

$$\left(x(t) + \frac{7}{8} x\left(\frac{1}{\mathrm{e}} t\right)\right)^{(4)} + \frac{q_0}{t^4} x\left(\frac{1}{\mathrm{e}^2} t\right) = 0, \; t \geq 1, \tag{6}$$

then we get that (6) is oscillatory if

The condition	(3)	(4)	(5)
The criterion	$q_0 > 113{,}981.3$	$q_0 > 3561.9$	$q_0 > 3008.5$

Hence, Xing et al. [22] improved the results in [23,24].

By establishing a new comparison theorem that compare the higher-order Equation (1) with a couple of first-order delay differential equations, we improve the results in [22–24]. An example is presented to illustrate our main results.

In order to discuss our main results, we need the following lemmas:

Lemma 1 ([13]). *If the function x satisfies $x^{(i)}(t) > 0$, $i = 0, 1, ..., n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n / n!} \geq \frac{x'(t)}{t^{n-1} / (n-1)!}.$$

Lemma 2 ([2] Lemma 2.2.3). *Let $x \in C^n([t_0,\infty),(0,\infty))$. Assume that $x^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists a $t_1 \geq t_0$ such that $x^{(n-1)}(t)\,x^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} x(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(n-1)!}t^{n-1}\left|x^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

Lemma 3 ([3] Lemmas 1 and 2). *Assume that $u, v \geq 0$ and β is a positive real number. Then*

$$(u+v)^\beta \leq 2^{\beta-1}\left(u^\beta + v^\beta\right), \text{ for } \beta \geq 1$$

and

$$(u+v)^\beta \leq u^\beta + v^\beta, \text{ for } \beta \leq 1.$$

2. Main Results

Here, we define the next notation:

$$P_k(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{k-1}}{(\tau^{-1}(t))^{k-1}\,p(\tau^{-1}(\tau^{-1}(t)))}\right), \quad \text{for } k = 2, n,$$

$$R_0(t) = \left(\frac{1}{r(t)}\int_t^\infty q(s)\,P_2^\alpha(\sigma(s))\,\mathrm{d}s\right)^{1/\alpha}$$

and

$$R_m(t) = \int_t^\infty R_{m-1}(s)\,\mathrm{d}s, \quad m = 1, 2, ..., n-3.$$

Lemma 4 ([20] Lemma 1.2). *Assume that x is an eventually positive solution of (1). Then, there exist two possible cases:*

$(\mathbf{I}_1)$ $\quad z(t) > 0,\ z'(t) > 0, z''(t) > 0,\ z^{(n-1)}(t) > 0,\ z^{(n)}(t) < 0,$

$(\mathbf{I}_2)$ $\quad z(t) > 0, z^{(j)}(t) > 0, z^{(j+1)}(t) < 0$ *for all odd integer* $j \in \{1, 3, ..., n-3\}, z^{(n-1)}(t) > 0, z^{(n)}(t) < 0,$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Theorem 1. *Let*

$$\frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{n-1}}{(\tau^{-1}(t))^{n-1}\,p(\tau^{-1}(\tau^{-1}(t)))} \leq 1. \tag{7}$$

Assume that there exist positive functions η, $\zeta \in C^1([t_0,\infty),\mathbb{R})$ satisfying

$$\eta(t) \leq \sigma(t),\ \eta(t) < \tau(t),\ \zeta(t) \leq \sigma(t),\ \zeta(t) < \tau(t),\ \zeta'(t) \geq 0 \text{ and } \lim_{t\to\infty}\eta(t) = \lim_{t\to\infty}\zeta(t) = \infty. \tag{8}$$

If there exists a $\mu \in (0,1)$ such that the differential equations

$$\psi'(t) + \left(\frac{\mu\left(\tau^{-1}(\eta(t))\right)^{n-1}}{(n-1)!r^{1/\alpha}(\tau^{-1}(\eta(t)))}\right)^\alpha q(t)\,P_n^\alpha(\sigma(t))\,\psi\left(\tau^{-1}(\eta(t))\right) = 0 \tag{9}$$

and

$$\phi'(t) + \tau^{-1}(\zeta(t))\,R_{n-3}(t)\,\phi\left(\tau^{-1}(\zeta(t))\right) = 0 \tag{10}$$

are oscillatory, then Equation (1) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we can assume that x is eventually positive. It follows from Lemma 4 that there exist two possible cases $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$.

Assume that Case $(\mathbf{I}_1)$ holds. From the definition of $z(t)$, we see that

$$x(t)=\frac{1}{p\left(\tau^{-1}(t)\right)}\left(z\left(\tau^{-1}(t)\right)-x\left(\tau^{-1}(t)\right)\right).$$

By repeating the same process, we find that

$$\begin{aligned} x(t) &= \frac{z\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)}-\frac{1}{p\left(\tau^{-1}(t)\right)}\left(\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}-\frac{x\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) \\ &\geq \frac{z\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)}-\frac{1}{p\left(\tau^{-1}(t)\right)}\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}. \end{aligned} \tag{11}$$

Using Lemma 1, we get $z(t)\geq\frac{1}{(n-1)}tz'(t)$ and hence the function $t^{1-n}z(t)$ is nonincreasing, which with the fact that $\tau(t)\leq t$ gives

$$\left(\tau^{-1}(t)\right)^{n-1}z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)\leq\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{n-1}z\left(\tau^{-1}(t)\right). \tag{12}$$

Combining Equations (11) and (12), we conclude that

$$\begin{aligned} x(t) &\geq \frac{1}{p\left(\tau^{-1}(t)\right)}\left(1-\frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{n-1}}{\left(\tau^{-1}(t)\right)^{n-1}p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right)z\left(\tau^{-1}(t)\right) \\ &= P_n(t)\,z\left(\tau^{-1}(t)\right). \end{aligned} \tag{13}$$

From Equations (1) and (13), we obtain

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'+q(t)\,P_n^{\alpha}(\sigma(t))\,z^{\alpha}\left(\tau^{-1}(\sigma(t))\right)\leq 0.$$

Since $\eta(t)\leq\sigma(t)$ and $z'(t)>0$, we get

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'\leq -q(t)\,P_n^{\alpha}(\sigma(t))\,z^{\alpha}\left(\tau^{-1}(\eta(t))\right). \tag{14}$$

Now, by using Lemma 2, we have

$$z(t)\geq\frac{\mu}{(n-1)!}t^{n-1}z^{(n-1)}(t). \tag{15}$$

for some $\mu\in(0,1)$. It follows from (14) and (15) that, for all $\mu\in(0,1)$,

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'+\left(\frac{\mu\left(\tau^{-1}(\eta(t))\right)^{n-1}}{(n-1)!}\right)^{\alpha}q(t)\,P_n^{\alpha}(\sigma(t))\left(z^{(n-1)}\left(\tau^{-1}(\eta(t))\right)\right)^{\alpha}\leq 0.$$

Thus, if we set $\psi(t)=r(t)\left(z^{(n-1)}(t)\right)^{\alpha}$, then we see that ψ is a positive solution of the first-order delay differential inequality

$$\psi'(t)+\left(\frac{\mu\left(\tau^{-1}(\eta(t))\right)^{n-1}}{(n-1)!\,r^{1/\alpha}\left(\tau^{-1}(\eta(t))\right)}\right)^{\alpha}q(t)\,P_n^{\alpha}(\sigma(t))\,\psi\left(\tau^{-1}(\eta(t))\right)\leq 0.$$

It is well known (see [21] (Theorem 1)) that the corresponding Equation (9) also has a positive solution, which is a contradiction.

Assume that Case $(\mathbf{I}_2)$ holds. Using Lemma 1, we get that

$$z(t) \geq tz'(t) \tag{16}$$

and thus the function $t^{-1}z(t)$ is nonincreasing, eventually. Since $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, we obtain

$$\tau^{-1}(t)\, z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \tau^{-1}\left(\tau^{-1}(t)\right) z\left(\tau^{-1}(t)\right). \tag{17}$$

Combining (11) and (17), we find

$$\begin{aligned} x(t) &\geq \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{(\tau^{-1}(\tau^{-1}(t)))}{(\tau^{-1}(t))\, p(\tau^{-1}(\tau^{-1}(t)))}\right) z\left(\tau^{-1}(t)\right) \\ &= P_2(t)\, z\left(\tau^{-1}(t)\right), \end{aligned}$$

which with (1) yields

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\, P_2^{\alpha}(\sigma(t))\, z^{\alpha}\left(\tau^{-1}(\sigma(t))\right) \leq 0.$$

Since $\zeta(t) \leq \sigma(t)$ and $z'(t) > 0$, we have that

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' \leq -q(t)\, P_2^{\alpha}(\sigma(t))\, z^{\alpha}\left(\tau^{-1}(\zeta(t))\right). \tag{18}$$

Integrating the (18) from t to ∞, we obtain

$$z^{(n-1)}(t) \geq R_0(t)\, z\left(\tau^{-1}(\zeta(t))\right).$$

Integrating this inequality from t to ∞ a total of $n-3$ times, we obtain

$$z''(t) + R_{n-3}(t)\, z\left(\tau^{-1}(\zeta(t))\right) \leq 0. \tag{19}$$

Thus, if we set $\phi(t) := z'(t)$ and using (16), then we conclude that ϕ is a positive solution of

$$\phi'(t) + \tau^{-1}(\zeta(t))\, R_{n-3}(t)\, \phi\left(\tau^{-1}(\zeta(t))\right) \leq 0. \tag{20}$$

It is well known (see [21] (Theorem 1)) that the corresponding Equation (10) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 1. *Assume that (7) holds and there exist positive functions η, ζ such that (8) holds. If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} \left(\frac{\left(\tau^{-1}(\eta(s))\right)^{n-1}}{r^{1/\alpha}\left(\tau^{-1}(\eta(s))\right)}\right)^{\alpha} q(s)\, P_n^{\alpha}(\sigma(s))\, \mathrm{d}s > \frac{((n-1)!)^{\alpha}}{\mathrm{e}} \tag{21}$$

and

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\zeta(t))}^{t} \tau^{-1}(\zeta(s))\, R_{n-3}(s)\, \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{22}$$

then (1) is oscillatory.

Proof. It is well-known (see, e.g., [14] (Theorem 2)) that Condition (21) and (22) imply oscillation of (9) and (10), respectively. □

Example 1. *Consider the equation*

$$\left(x\left(t\right)+p_0x\left(\delta t\right)\right)^{(n)}+\frac{q_0}{t^n}x\left(\lambda t\right)=0, \tag{23}$$

where $t\geq 1$, $q_0>0$, $\delta\in\left(p_0^{-1/(n-1)},1\right)$ *and* $\lambda\in(0,\delta)$. *We note that* $r(t)=1$, $p(t)=p_0$, $\tau(t)=\delta t$, $\sigma(t)=\lambda t$ *and* $q(t)=q_0/t^n$. *Thus, if we choose* $\eta(t)=\zeta(t)=\lambda t$, *then it's easy to see that (7) and (8) are satisfied. Moreover, we have*

$$\begin{aligned} P_k\left(t\right) &= \frac{1}{p_0}\left(1-\frac{\delta^{1-k}}{p_0}\right), \text{ for } k=2,n, \\ R_0\left(t\right) &= \frac{q_0}{p_0}\left(1-\frac{1}{\delta p_0}\right)\frac{t^{1-n}}{(n-1)}, \end{aligned}$$

and

$$R_{n-3}\left(t\right)=\frac{1}{(n-3)!}\frac{q_0}{p_0}\left(1-\frac{1}{\delta p_0}\right)\frac{1}{(n-2)\left(n-1\right)t^2}.$$

Hence, Condition (21) and (22) become

$$q_0\frac{1}{p_0}\left(\frac{\lambda}{\delta}\right)^{n-1}\left(1-\frac{\delta^{1-n}}{p_0}\right)\ln\frac{\delta}{\lambda}>\frac{(n-1)!}{\mathrm{e}} \tag{24}$$

and

$$q_0\frac{1}{p_0}\frac{\lambda}{\delta}\left(1-\frac{1}{\delta p_0}\right)\ln\frac{\delta}{\lambda}>\frac{(n-1)!}{\mathrm{e}}, \tag{25}$$

respectively. It's easy to see that (24) implies (25).
Therefore, by Corollary 1, we conclude that (23) is oscillatory if (24) holds.

Remark 1. *For Equation (23), in particular case that* $n=4$, $p_0=16$, $\delta=1/2$ *and* $\lambda=1/3$, *Condition (24) yields* $q_0>587.93$. *Whereas, the criterion obtained from the results of [22] is* $q_0>4850.4$. *Hence, our results improve the results in [22].*

3. Conclusions

In this paper, our method is based on presenting a new comparison theorem that compare the higher-order Equation (1) with a couple of first-order equations. There are numerous results concerning the oscillation criteria of first order Equations (9) and (10) (see, e.g., [14,25–27]), which include various forms of criteria as Hille/Nehari, Philos, etc. This allows us to obtain also various criteria for the oscillation of (1). Further, we can try to obtain oscillation criteria of (1) if $z(t):=x(t)-p(t)x(\tau(t))$ in the future work.

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