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Local Convergence for Multi-Step High Order Solvers under Weak Conditions

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Abstract: Our aim in this article is to suggest an extended local convergence study for a class of multi-step solvers for nonlinear equations valued in a Banach space. In comparison to previous studies, where they adopt hypotheses up to 7th Fŕechet-derivative, we restrict the hypotheses to only first-order derivative of considered operators and Lipschitz constants. Hence, we enlarge the suitability region of these solvers along with computable radii of convergence. In the end of this study, we choose a variety of numerical problems which illustrate that our works are applicable but not earlier to solve nonlinear problems.

Keywords: local convergence; multi-step iterative solver; Lipschitz constant; order of convergence; Banach space

MSC: 65G99; 65H10; 47J25; 47J05; 65D10; 65D99

1. Introduction

Finding the approximate solution μ of

$$F(x) = 0, \tag{1}$$

is one of the top priorities in the field of Numerical analysis. We assume that $F : \mathbb{A} \subset \mathbb{E}_1 \to \mathbb{E}_2$ is a Fréchet-differentiable operator, $\mathbb{E}_1, \mathbb{E}_2$ are Banach spaces and $\mathbb{A}$ is a convex subset of $\mathbb{E}_1$. The $\ell B(\mathbb{E}_1,\ \mathbb{E}_2)$ is known as the set of bounded linear operators.

The problem of finding an approximate unique solution μ is very important, since many problems can be written as Equation (1) in References [1–8]. However, it is not always possible to access the solution μ in an explicit form. Hence, most of the solvers are iterative in nature. The analysis of solvers involves local convergence that stands on the knowledge around μ. It also ensures the convergence of iteration procedures. One of the most significant tasks in the analysis of iterative procedures is to yield the convergence region. Hence, it is essential to suggest the radius of convergence.

We redefine the iterative solver suggested in Reference [7], for all $\sigma = 0,\ 1\ 2,\ \ldots$ as

$$\begin{aligned} y_\sigma &= x_\sigma - F'(x_\sigma)^{-1}F(x_\sigma),\\ z_\sigma &= \phi_1(x_\sigma,\ y_\sigma)\\ z_\sigma^{(1)} &= z_\sigma - \phi(x_\sigma,\ y_\sigma)F(z_\sigma),\\ &\ \vdots\\ z_\sigma^{(m-1)} &= z_\sigma^{(m-2)} - \phi(x_\sigma,\ y_\sigma)F(z_\sigma^{m-2}),\\ x_{\sigma+1} &= z_\sigma^{(m-1)} - \phi(x_\sigma,\ y_\sigma)F(z_\sigma^{m-1}), \end{aligned} \tag{2}$$

where $x_0 \in \mathbb{A}$ is a starting guess, $z_\sigma = \phi_1(x_\sigma,\ y_\sigma)$ is a λ-order iteration function solver (for $\lambda \geq 1$) and

$$\phi(s,\ \zeta) = \frac{1}{3}\left\{4[3F'(\zeta) - F'(s)]^{-1} - F'(s)^{-1}\right\}. \tag{3}$$

F' stands for the first-order Fŕechet-derivative of F. The study of these methods is important for various reasons already stated in Reference [7]. For brevity we refer the reader to Reference [7] and the references therein. On top of those reasons, we also mention that method (2) generalizes the existing widely used Newton's type methods such as Newton's, Traub's and other methods. So, it is important to study these methods under the same set of convergence criteria. Keeping the linear operator frozen is also a very cheap and efficient way of increasing the order of convergence. The convergence order of (2) was given in Reference [7] but using hypotheses up to the 7th-order derivative of function F. Only the 1st-order derivative emerges in scheme (2). Such conditions hamper the suitability of solver (2). Consider function F with $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{R}$ on $\mathbb{A} = [-\frac{1}{2}, \frac{3}{2}]$ by

$$\Theta(\kappa) = \begin{cases} \kappa^3 \ln \kappa^2 + \kappa^5 - \kappa^4, & \kappa \neq 0\\ 0, & \kappa = 0 \end{cases}.$$

Using this definition, we get

$$\Theta'(\kappa) = 3\kappa^2 \ln \kappa^2 + 5\kappa^4 - 4\kappa^3 + 2\kappa^2,$$

$$\Theta''(\kappa) = 6\kappa \ln \kappa^2 + 20\kappa^3 - 12\kappa^2 + 10\kappa$$

and

$$\Theta'''(\kappa) = 6 \ln \kappa^2 + 60\kappa^2 - 24\kappa + 22.$$

It is clear from the above that the 3rd-order derivative of $F(x)$ is unbounded in $\mathbb{A}$. We have plenty of research articles on iterative solvers [1–26]. The local convergence analysis of these solvers traditionally requires the usage of Taylor expansions and the operator involved must be sufficiently many times differentiable in a neighborhood of the solution μ. This way, the convergence order is established but derivatives of an order higher than one do not appear in these solvers, as we saw previously with the motivational example restricting the applicability of solvers. Another problem is that this approach does not provide error estimates on $\|x_n - \mu\|$ that can be used to predetermine the number of steps required to attain a prescribed error tolerance. The uniqueness of the solution μ also cannot be established in any set containing it. Moreover, the starting guess is a shot in the dark. Therefore, it is important to find a technique other than the preceding. This is what we offer in this article. Furthermore, (COC) and (ACOC) [27] are used to compute the convergence order (to be explained in Remark 1 (d)).

These formulas do not require higher than one derivative, and in the case of ACOC, knowledge of μ is not needed. It is worth noting that the iterates are obtained by using (2), which involves the first derivative. Hence, these iterates also depend on the first derivative (see Remark 1 (d)). Our techniques can be used on other solvers to extend their applicability in a similar fashion.

2. Local Convergence

Here, we present a study of local convergence for solver (2). For this, we consider a function $\varphi_0 : [0, \infty) \to [0, \infty)$ which is nondecreasing and continuous such that $\varphi_0(0) = 0$. We assume

$$\varphi_0(\zeta) = 1 \tag{4}$$

has a minimal positive solution r_0.

Define functions g_1, g_2, h_1 and h_2 on the interval $[0, r_0)$ by

$$\begin{aligned} g_1(\zeta) &= \frac{\int_0^1 \varphi\Big((1-\theta)\zeta\Big)d\theta}{1-\varphi_0(\zeta)}, \\ g_2(\zeta) &= \psi(\zeta, g_1(\zeta)\zeta)\zeta^{\lambda-1}, \\ h_1(\zeta) &= g_1(\zeta) - 1. \\ &\text{and} \\ h_2(\zeta) &= g_2(\zeta) - 1, \end{aligned}$$

where $v, \varphi : [0, r_0) \to [0, \infty)$ and functions $\psi : [0, r_0) \times [0, r_0) \to [0, \infty)$ are also nondecreasing and continuous, satisfying $\varphi(0) = 0$. We have that $h_1(0) = h_2(0) = -1$ and $h_1(\zeta) \to \infty$, $h_2(\zeta) \to \infty$ as $\zeta \to r_0^-$. Then, by the intermediate value theorem, we notice that the functions h_1 and h_2 have solutions in the interval $(0, r_0)$. Call as r_1 and r_2 the smallest such solutions in $(0, r_0)$ of the functions h_1 and h_2, respectively. Assume $p(t) = 1$ has minimal positive solution r_p. Consider functions

$$\begin{aligned} p(\zeta) &= \frac{1}{2}\Big[3\varphi_0\Big(g_1(\zeta)\zeta\Big) + \varphi_0(\zeta)\Big], \\ h_p(\zeta) &= p(\zeta) - 1. \end{aligned}$$

These functions are defined in the interval $[0, \bar{r})$, where $\bar{r} = \min\{r_0, r_p\}$. Consider functions $g^{(i)}, h^{(i)}, i = 1, 2, \ldots, m$ on $[0, \bar{r})$ as

$$\begin{aligned} g^{(i)}(\zeta) &= \left(1 + q(\zeta)\int_0^1 v(\theta g^{(i-1)}(\zeta)\zeta)d\theta\right)^{i-1} g_2(\zeta), \quad g^{(-1)}(\zeta) = g_2(\zeta), \\ &\text{and} \\ h^{(i)}(\zeta) &= g^{(i)}(\zeta) - 1, \end{aligned}$$

where

$$q(\zeta) = \frac{1}{2}\left(\frac{\varphi_0(\zeta) + \varphi_0\Big(g_1(\zeta)\zeta\Big)}{1 - p(\zeta)}\right).$$

Then, $h^{(i)}(0) = -1$ and $h^{(i)}(\zeta) \to \infty$ as $\zeta \to \bar{r}^-$. Defined by $r^{(i)}$ be the minimal solutions of corresponding to functions $h^{(i)}$ in $(0, \bar{r})$.

Set r as

$$r = \min\{r_1, r_2, r^{(i)}\}. \tag{5}$$

Then, it follows

$$0 < r < r_0 \tag{6}$$

and for all $t \in [0,\ r)$

$$0 \leq g_1(\zeta) < 1, \tag{7}$$

$$0 \leq g_2(\zeta) < 1, \tag{8}$$

$$0 \leq p(\zeta) < 1, \tag{9}$$

$$0 \leq q(\zeta) \tag{10}$$

and

$$0 \leq g^{(i)}(\zeta) < 1. \tag{11}$$

Let $U(\xi,\ \rho)$, $\bar{U}(\xi,\ \rho)$ be, respectively, open and closed balls in $\mathbb{E}_1$ centered at $\xi \in \mathbb{E}_1$ and of radius $\rho > 0$. Next, the local convergence analysis of solver (2) follows.

Theorem 1. *Let $F : \mathbb{A} \subseteq \mathbb{E}_1 \to \mathbb{E}_2$ be a differentiable operator. Let $v,\ \varphi_0,\ \varphi : [0,\ \infty) \to [0,\ \infty)$ and $\psi : [0,\ \infty) \times [0,\ \infty) \to [0,\ \infty)$ be a nondecreasing continuous function such that $\varphi_0(0) = \varphi(0) = 0$. The parameter r_0 be defined by (4). Suppose that there exists $\mu \in \mathbb{A}$ such that*

$$F(\mu) = 0, \quad F'(\mu)^{-1} \in \ell B(\mathbb{E}_2,\ \mathbb{E}_1) \tag{12}$$

and

$$\|F'(\mu)^{-1}(F'(x) - F'(\mu)\| \leq \varphi_0(\|x - \mu\|), \text{ for all } x \in \mathbb{A}. \tag{13}$$

Moreover, suppose that for all $x,\ y \in \mathbb{A}_0 = \mathbb{A} \cap U(\mu,\ r_0)$

$$\|F'(\mu)^{-1}\big(F'(x) - F'(y)\big)\| \leq \varphi(\|x - y\|), \tag{14}$$

$$\|F'(\mu)^{-1}F'(x)\| \leq v(\|x - \mu\|), \tag{15}$$

$$\|\phi_1(x,\ y) - \mu\| \leq \psi(\|x - \mu\|,\ \|y - \mu\|)\|x - \mu\|^{\lambda} \tag{16}$$

and

$$\bar{U}(\mu, r) \subseteq \mathbb{A}. \tag{17}$$

Then, $\{x_\sigma\}$ generated for $x_0 \in U(\mu,\ r) - \{x^*\}$ by solver (2) is well defined, remains in $U(\mu,\ r)$ for all $\sigma = 0, 1, 2, 3, 4, \ldots$ and converges to μ, so that

$$\|y_\sigma - \mu\| \leq g_1(\|x_\sigma - \mu\|)\|x_\sigma - \mu\| \leq \|x_\sigma - \mu\| < r, \tag{18}$$

$$\|z_\sigma - \mu\| \leq g_2(\|x_\sigma - \mu\|)\|x_\sigma - \mu\| \leq \|x_\sigma - \mu\|, \tag{19}$$

$$\|z_\sigma^{(i)} - \mu\| \leq g^{(i)}(\|x_\sigma - \mu\|)\|x_\sigma - \mu\| \leq \|x_\sigma - \mu\|,\ i = 1,\ 2,\ \ldots,\ m-1 \tag{20}$$

and

$$\|x_{\sigma+1} - \mu\| \leq g^{(m)}(\|x_\sigma - \mu\|)\|x_\sigma - \mu\| \leq \|x_\sigma - \mu\|. \tag{21}$$

Further, if

$$\int_0^1 \varphi_0(\theta R) d\theta < 1 \ \text{ for } \ R \geq r, \tag{22}$$

then, μ is the only solution of equation $F(x) = 0$ in $\mathbb{A}_1 := \mathbb{A} \cap \bar{U}(\mu,\ R)$.

Proof. We select mathematical induction to show that expressions (18)–(21) are satisfied. Using hypotheses $x_0 \in U(\mu, r) - \{x^*\}$, (4), (5) and (13), we yield

$$\|F'(\mu)^{-1}(F'(x_0) - F'(\mu))\| \leq \varphi_0(\|x_0 - \mu\|) < \varphi_0(r) < 1. \tag{23}$$

Therefore, $F'(x_0)^{-1} \in \ell B(\mathbb{E}_2,\ \mathbb{E}_2)$, $y_0,\ z_0$ are well defined, and

$$\|F'(x_0)^{-1}F'(\mu)\| \leq \frac{1}{1-\varphi_0(\|x_0-\mu\|)}. \tag{24}$$

By adopting (2), (5), (7), (14) and (24), we have

$$\begin{aligned}
\|y_0-\mu\| &= \|x_0-\mu-F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(\mu)\| \left\| \int_0^1 F'(\mu)^{-1}\left(F'(\mu+\theta(x_0-\mu))-F'(x_0)\right)(x_0-\mu)d\theta \right\| \\
&\leq \frac{\int_0^1 w\Big((1-\theta)\|x_0-\mu\|\Big)d\theta\|x_0-\mu\|}{1-\varphi_0(\|x_0-\mu\|)} \\
&= g_1(\|x_0-\mu\|)\|x_0-\mu\| \leq \|x_0-\mu\| < r,
\end{aligned} \tag{25}$$

showing (18) for $\sigma = 0$ and $y_0 \in U(\mu,\ r)$.

By (2), (5), (8), (16) and (25), we yield

$$\begin{aligned}
\|z_0-\mu\| &= \|\phi_1(x_0,y_0)-\mu\| \\
&\leq \psi\Big(\|x_0-\mu\|,\ \|y_0-\mu\|\Big)\|x_0-\mu\|^{\lambda} \\
&\leq \psi\Big(\|x_0-\mu\|,\ g_1(\|x_0-\mu\|)\|x_0-\mu\|\Big)\|x_0-\mu\|^{\lambda} \\
&= g_2(\|x_0-\mu\|)\|x_0-\mu\| \leq \|x_0-\mu\| < r,
\end{aligned} \tag{26}$$

showing (19) (for $\sigma = 0$) and $z_0 \in U(\mu,\ r)$. We can write by (12)

$$F(z_0) = F(z_0)-F(\mu) = \int_0^1 F'\Big(\mu+\theta(z_0-\mu)\Big)d\theta(z_0-\mu). \tag{27}$$

Then, from (15), (26) and (27), we obtain

$$\begin{aligned}
\|F'(\mu)^{-1}F(z_0)\| &\leq \int_0^1 v(\theta\|z_0-\mu\|)\|z_0-\mu\|d\theta \\
&\leq \int_0^1 v(\theta g_1(\|x_0-\mu\|)\|x_0-\mu\|)d\theta g_1(\|x_0-\mu\|)\|x_0-\mu\|.
\end{aligned} \tag{28}$$

We must show that $\phi(x_0,\ y_0) \neq 0$. In view of (5), (9), (13) and (25), we get

$$\begin{aligned}
&\left\|(2F'(\mu))^{-1}\Big[3F'(y_0)-F'(\mu)-2F'(\mu)\Big]\right\| \\
&\qquad \leq \frac{1}{2}\Big[3\varphi_0(\|y_0-\mu\|)+\varphi_0(\|x_0-\mu\|)\Big] \\
&\qquad \leq \frac{1}{2}\Big[3\varphi_0\Big(g_1(\|x_0-\mu\|)\|x_0-\mu\|\Big)+\varphi_0(\|x_0-\mu\|)\Big] \\
&\qquad = p(\|x_0-\mu\|) < p(r) < 1,
\end{aligned} \tag{29}$$

so $z_0,\ z_0^{(1)},\ \ldots z_0^{(m-1)},\ x_1$ exist

$$\left\|\left[3F'(y_0)-F'(\mu)\right]^{-1}F'(\mu)\right\| \leq \frac{1}{2(1-p(\|x_0-\mu\|))} \tag{30}$$

and

$$\begin{aligned}
\|\phi(x, y_0)F'(\mu)\| &\le \left\|\frac{1}{3}\left[4\left(3F'(y_0)-F'(x_0)\right)^{-1}-F'(x_0)^{-1}\right]F'(\mu)\right\| \\
&\le \left\|\left(3F'(y_0)-F'(x_0)\right)^{-1}\left(F'(x_0)-F'(y_0)\right)F'(\mu)\right\|.
\end{aligned} \tag{31}$$

Using (2), (5), (8), (9), (11) (for $i=2$), (28) and (31), we obtain

$$\begin{aligned}
\|z_0^{(1)}-\mu\| &= \|z_0-\mu\| + \|\phi(x_0, y_0)F'(\mu)\|\|F'(\mu)^{-1}F(z_0)\| \\
&\le \left(1+\|\phi(x_0, y_0)F'(\mu)\|\int_0^1 v(\theta\|z_0-\mu\|)d\theta\right)\|z_0-\mu\| \\
&\le g^{(1)}(\|x_0-\mu\|)\|x_0-\mu\| \le \|x_0-\mu\| < r,
\end{aligned} \tag{32}$$

so (20) holds for $\sigma=0,\ i=1$ and $z_0^{-1}\in U(\mu,\ r)$. In an analogous way, we obtain for $i=2,\ 3,\ \ldots, m-1$ that

$$\begin{aligned}
\|z_0^{(i-1)}-\mu\| &= \|z_0^{(i-2)}-\mu\| + q(\|x_0-\mu\|)\int_0^1 v(\theta\|z_0^{(i-2)}-\mu\|)d\theta\|z_0-\mu\| \\
&\le g^{(i-1)}(\|x_0-\mu\|)\|x_0-\mu\| \le \|x_0-\mu\| < r,
\end{aligned} \tag{33}$$

which implies (20) holds for $\sigma=0,\ i=1,\ 2,\ \ldots,\ m-1$, and $z_0^m\in U(\mu,\ r)$.

In view of solver (2), (5), (11) (for $i=m$) and the proceeding estimates

$$\begin{aligned}
\|x_1-\mu\| &\le \|z_0^{(m-1)}-\mu\| + \|\phi(x_0, y_0)F'(\mu)\|\|F'(\mu)^{-1}F(z_0^{(m-1)})\| \\
&\le \left(1+q(\|x_0-\mu\|)\int_0^1 v(\theta\|z_0^{(m-1)}-\mu\|)d\theta\right)\|z_0^{(m-1)}-\mu\| \\
&= g^{(m)}(\|x_0-\mu\|)\|x_0-\mu\| \le \|x_0-\mu\| < r,
\end{aligned} \tag{34}$$

showing (21) (for $\sigma=0$) with $x_1\in U(\mu,\ r)$. Now, change $x_0, y_0,\ z_0,\ z_0^{(i)}\,(i=1,\ 2,\ \ldots, m)$ and x_1 by $x_\sigma, y_\sigma,\ z_\sigma,\ z_\sigma^{(i)}$ and $x_{\sigma+1}$ in the preceding estimates. Hence, we attain (18)–(21). By adopting

$$\|x_{\sigma+1}-\mu\| \le c\|x_\sigma-\mu\| < r,\ c = g^{(m)}(\|x_0-\mu\|)\in[0,\ 1), \tag{35}$$

we have $\lim_{\sigma\to\infty} x_\sigma = \mu$ with $x_{\sigma+1}\in U(\mu,\ r)$. Finally, for the uniqueness of required solution, we assume that $y^*\in A_1$ satisfying $F(y^*)=0$. Set $Q=\int_0^1 F'(\mu+\theta(\mu-y^*))d\theta$, so

$$\begin{aligned}
&\|F'(\mu)^{-1}(Q-F'(\mu))\| \le \|\textstyle\int_0^1 \varphi_0(\theta\|y^*-\mu\|)d\theta \\
&\le \textstyle\int_0^1 \varphi_0(\theta R)d\theta < 1.
\end{aligned} \tag{36}$$

Hence, Q is invertible. Then,

$$0 = F(\mu)-F(y^*) = Q(\mu-y^*), \tag{37}$$

yields $y^*=\mu$. □

Remark 1.

(a) *It is clear from* (13) *that we can drop the hypothesis* (15) *and choose*

$$v(\zeta) = 1+\varphi_0(\zeta) \text{ or } v(\zeta) = 1+\varphi_0(r_0). \tag{38}$$

Indeed, we have

$$\begin{aligned}\|F'(\mu)^{-1}\left[\left(F'(x)-F'(\mu)\right)+F'(\mu)\right]\| &= 1+\|F'(\mu)^{-1}(F'(x)-F'(\mu))\| \\ &\leq 1+\varphi_0(\|x-\mu\|) \\ &= 1+\varphi_0(\zeta) \quad \textit{for} \quad \|x-\mu\| \leq r_0.\end{aligned} \tag{39}$$

(b) *We can set*

$$r_0 = \varphi_0^{-1}(1) \tag{40}$$

instead (4) *provided that function* φ_0 *is strictly increasing.*

(c) *If* $\varphi_0,\ w,\ v$ *are constants functions, then*

$$r_1 = \frac{2}{2\varphi_0 + w} \tag{41}$$

and

$$r \leq r_1, \tag{42}$$

where r_1 *is the radius for Newton's solver [14].*

$$x_{\sigma+1} = x_\sigma - F'(x_\sigma)^{-1}F(x_\sigma). \tag{43}$$

Rheinboldt [26] and Traub [6] also provided radius of convergence instead of r_1

$$r_{TR} = \frac{2}{3\varphi_1} \tag{44}$$

and by Argyros [1,2]

$$r_A = \frac{2}{2\varphi_0 + \varphi_1}, \tag{45}$$

where φ_1 *is a constant for* (9) *on D, so*

$$w \leq \varphi_1,\ \varphi_0 \leq \varphi_1, \tag{46}$$

so

$$r_{TR} \leq r_A \leq r_1 \tag{47}$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \quad \textit{as} \quad \frac{\varphi_0}{w} \to 0. \tag{48}$$

(d) *By adopting conditions to the 7th-order derivative of operator F, the order of the convergence of solver* (2) *was given in Reference [7]. We assume hypotheses only on the 1st-order derivative of operator F. For obtaining the order of convergence, we adopted*

$$\xi = \frac{\ln \frac{\|x_{\sigma+2}-\mu\|}{\|x_{\sigma+1}-\mu\|}}{\ln \frac{\|x_{\sigma+1}-\mu\|}{\|x_\sigma-\mu\|}}, \quad \text{for each } \sigma = 0,1,2,3,4,\ldots \tag{49}$$

or

$$\xi^* = \frac{\ln \frac{\|x_{\sigma+2}-x_{\sigma+1}\|}{\|x_{\sigma+1}-x_\sigma\|}}{\ln \frac{\|x_{\sigma+1}-x_\sigma\|}{\|x_\sigma-x_{\sigma-1}\|}}, \quad \text{for each } \sigma = 1,2,3,4,\ldots, \tag{50}$$

the computational order of convergence COC and the approximate computational order of convergence ACOC [28,29], respectively. These definitions can also be found in Reference [27]. They do not require derivatives higher than one. Indeed, notice that to generate iterates x_n *and therefore compute* ξ *and* ξ^**, we* *need to use the formula (2) using only the first derivatives. It is vital to note that ACOC does not need the prior information of exact root μ.*

(e) *Consider F satisfying the autonomous differential equation [1,2] of*

$$F'(x) = P(F(x)) \tag{51}$$

where P is a given and continuous operator. Then, $F'(x^*) = P(F(x^*)) = P(0)$, our results apply but without knowledge of x^* and choose $F(x) = e^x - 1$. Hence, we select $P(x) = x + 1$.

3. Concrete Applications

Here, we illustrate the theoretical consequences suggested in Section 2. We choose $\lambda = 1$ and $\varphi_1(x_\sigma, y_\sigma) = y_\sigma - F'(y_\sigma)^{-1}F(y_\sigma)$, in all examples. Next, we provide numerical examples given as follows:

Example 1. *Choose $\mathbb{E}_1 = \mathbb{E}_2 = A$, where $A = C[0, 1]$. We study the mixed Hammerstein-like equation [4,18], defined as follows:*

$$x(s) = 1 + \int_0^1 H(s, \zeta)\left(x(\zeta)^{\frac{3}{2}} + \frac{x(\zeta)^2}{2}\right) d\zeta, \tag{52}$$

where

$$H(s,\zeta) = \begin{cases} (1-s)\zeta, \ \zeta \le s, \\ s(1-\zeta), \ s \le \zeta, \end{cases} \tag{53}$$

defined in $[0,1] \times [0,1]$. The solution $\mu(s) = 0$ is the same as zero of (1), where $F : A \to A$, given as:

$$F(x)(s) = x(s) - \int_0^\zeta H(s, \zeta)\left(x(\zeta)^{\frac{3}{2}} + \frac{x(\zeta)^2}{2}\right) d\zeta. \tag{54}$$

But

$$\left\| \int_0^\zeta H(s, \zeta) d\zeta \right\| \le \frac{1}{8}, \tag{55}$$

and

$$F'(x)y(s) = y(s) - \int_0^\zeta G(s, \zeta)\left(\frac{3}{2}x(\zeta)^{\frac{1}{2}} + x(\zeta)\right) d\zeta,$$

so since $F'(\mu(s)) = I$,

$$\left\| F'(\mu)^{-1}(F'(x) - F'(y)) \right\| \le \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{\frac{1}{2}} + \|x-y\|\right). \tag{56}$$

Then, we consider

$$\varphi_0(\zeta) = \varphi(\zeta) = \frac{1}{8}\left(\frac{3}{2}\zeta^{\frac{1}{2}} + \zeta\right)$$

and

$$v(\zeta) = 1 + \varphi_0(\zeta),$$

by Remark 1. But F' is not Lipschitz, so earlier studies [4,7] are not applicable to solving this problem. On the other hand, our technique does not exhibit this kind of behavior. The different radii of convergence mentioned in Table 1.

Table 1. Distinct radii of convergence.

i	r_1	r_2	$r^{(i)}$	r
1	2.6303	0.816299	0.816299	0.816299
2	2.6303	0.816299	0.677029	0.677029

We notice that the radius of convergence decreases as "i" increases as expected, since we trade higher order convergence, with a smaller domain of convergence of initial points.

Example 2. *Describing the movement of a particle in 3-D by the following system of differential equations*

$$\begin{aligned} f_1'(x) - f_1(x) - 1 = 0 \\ f_2'(y) - (e-1)y - 1 = 0 \\ f_3'(z) - 1 = 0 \end{aligned} \tag{57}$$

with $x,\ y,\ z \in A$ *for* $f_1(0) = f_2(0) = f_3(0) = 0$. *Define* $v = (x,\ y,\ z)^T$ *by function* $F := (f_1,\ f_2,\ f_3) : A \to \mathbb{R}^3$ *given as follows:*

$$F(v) = \left(e^x - 1,\ \frac{e-1}{2}y^2 + y,\ z \right)^T. \tag{58}$$

So, we obtain

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have for $\mu = (0,0,0)^T$ *that* $\varphi_0(\zeta) = (e-1)\zeta,\ \ \varphi(\zeta) = e^{\frac{1}{e-1}}\zeta$, *and* $v(\zeta) = e^{\frac{1}{e-1}}$. *The different radii of convergence mentioned in Table 2.*

Table 2. Distinct radii of convergence.

i	r_1	r_2	$r^{(i)}$	r
1	0.377542	0.416275	0.416275	0.416275
2	0.377542	0.416275	0.272799	0.272799
3	0.377542	0.416275	0.227777	0.227777
4	0.377542	0.416275	0.198038	0.198038

We notice that the radius of convergence decreases as "i" increases as expected, since we trade higher order convergence, with a smaller domain of convergence of initial points.

Example 3. *Let us choose* $\mathbb{E}_1 = \mathbb{E}_2 = S$, *facilitated by the max norm. Set* $A = \bar{U}(0,\ 1)$ *and choose a function* F *on* A

$$F(\Gamma)(x) = \phi(x) - 5\int_0^1 x\theta\Gamma(\theta)^3 d\theta. \tag{59}$$

We have that

$$F'(\Gamma(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\Gamma(\theta)^2\xi(\theta)d\theta,\ \text{for each } \xi \in A. \tag{60}$$

Then, we have that $\varphi_0(\zeta) = 15\zeta,\ \ \varphi(\zeta) = 30\zeta$ *and* $v(\zeta) = 2$. *So, we yield the Table 3, where we calculated distinct radii of convergence.*

Table 3. Distinct radii of convergence.

i	r_1	r_2	$r^{(i)}$	r
1	0.0333333	0.0625	0.0625	0.0625
2	0.0333333	0.0625	0.0324524	0.0324524
3	0.0333333	0.0625	0.0296809	0.0296809
4	0.0333333	0.0625	0.0270781	0.0270781

We notice that the radius of convergence decreases as "i" increases as expected, since we trade a higher order convergence with a smaller domain of convergence of initial points.

Example 4. *By the academic problem that we considered in the introduction, we yield* $\varphi_0(\zeta) = \varphi(\zeta) = 96.662907t$ *and* $v(\zeta) = 2$. *So, we have the different radii of convergence depicted in Table 4.*

Table 4. Distinct radii of convergence.

i	r_1	r_2	$r^{(i)}$	r
1	0.00689682	0.0102917	0.0102917	0.0102917
2	0.00689682	0.0102917	0.00623774	0.00623774
3	0.00689682	0.0102917	0.00599906	0.00599906
4	0.00689682	0.0102917	0.00565863	0.00565863

We notice that the radius of convergence decreases as "i" increases as expected, since we trade a higher order convergence with a smaller domain of convergence of initial points.

4. Application of Our Scheme on Large System of Nonlinear Equations

We cited the (j), $(\|F(x_j)\|)$, $\|x_{j+1} - x_j\|$ and $\xi^* \approx \dfrac{\log\left[\|x_{j+1}-x_j\|/\|x_j-x_{j-1}\|\right]}{\log\left[\|x_j-x_{j-1}\|/\|x_{j-1}-x_{j-2}\|\right]}$ as the index of number of iteration, absolute residual errors, errors among two iterations and computational convergence order, respectively, in Tables 5–7.

The whole calculation is performed in the Mathematica software (Version-9, Wolfram Research, Champaign, IL, USA). We consider at least 1000 digits of mantissa in order to minimize the round-off errors. The notation $a_1\ (\pm a_2)$ employs $a_1 \times 10^{(\pm a_2)}$.

Example 5. *We assume here a boundary value problem [30], which is given by*

$$v'' = \frac{1}{2}v^3 + 3v' - \frac{3}{2-x} + \frac{1}{2}, \quad v(0) = 0, \ v(1) = 1. \tag{61}$$

Further, we chosen a σ-point partition of $[0,\ 1]$ in the following way:

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_\sigma, \text{ where } x_{i+1} = x_i + k, \ k = \frac{1}{\sigma}.$$

Furthermore, we assume that $v_0 = v(x_0) = 0,\ v_1 = v(x_1),\ \ldots,\ v_{\sigma-1} = v(x_{\sigma-1}),\ v_\sigma = v(x_\sigma) = 1.$ *By adopting the following technique for removing derivatives for problem* (61)

$$v_j' = \frac{v_{j+1} - v_{j-1}}{2k}, \ v_j'' = \frac{v_{j-1} - 2v_j + v_{j+1}}{k^2}, \ j = 1,\ 2,\ \ldots,\ \sigma - 1.$$

We have

$$v_{j+1} - 2v_j + v_{j-1} - \frac{k^2}{2}v_j^3 - \frac{3}{2-x_j}k^2 - \frac{1}{k^2} = 0, \ j = 1, 2, \ldots, \sigma - 1.$$

a system of nonlinear equations (SNE) of order $(\sigma-1)\times(\sigma-1)$. *We choose the starting approximation* $y_h^{(0)} = (1.5,\ 1.5,\ 1.5,\ 1.5,\ 1.5,\ 1.5)^T$. *We solved the problem for a* 6×6 *SNE by choosing* $\sigma = 7$. *We obtained the following solution*

$$\mu = \big(0.0765439\ldots,\ 0.165874\ldots,\ 0.271521\ldots,\ 0.398454\ldots,\ 0.553886\ldots,\ 0.748688\ldots\big)^T.$$

We depicted the numerical out comes in Table 5.

Table 5. Computational results on a boundary value problem 5.

Cases of (2)	j	$\lVert F(x_j)\rVert$	$\lVert x_{j+1}-x_j\rVert$	ξ^*
	0	1.9	2.8	
$i=1$	1	8.5 (−6)	2.2 (−5)	
	2	9.0 (−38)	1.7 (−37)	
	3	9.6 (−231)	1.5 (−230)	6.0097
	0	1.9	2.8	
$i=2$	1	6.8 (−8)	2.9 (−7)	
	2	6.0 (−66)	6.5 (−66)	
	3	2.0 (−534)	5.0 (−534)	7.9819

We have computed ACOC and observed that as we increases "i" so does the ACOC.

Example 6. *We choose a prominent 2D Bratu problem [31,32], which is given by*

$$\begin{aligned}
&u_{xx}+u_{tt}+Ce^{u}=0, on\\
&A:(x,t)\in 0\le x\le 1, 0\le t\le 1,\\
&along\ boundary\ hypothesis\ u=0\ on\ A.
\end{aligned} \tag{62}$$

Let us assume that $\Theta_{i,j}=u(x_i,t_j)$ *is a numerical result over the grid points of the mesh. In addition, we consider that* τ_1 *and* τ_2 *are the number of steps in the direction of* x *and* t*, respectively. Moreover, we choose that* h *and* k *are the respective step sizes in the direction of* x *and* y*, respectively. In order to find the solution of PDE (62), we adopt the following approach*

$$u_{xx}(x_i,t_j)=\frac{\Theta_{i+1,j}-2\Theta_{i,j}+\Theta_{i-1,j}}{h^2},\ C=0.1,\ t\in[0,1], \tag{63}$$

which further yields the succeeding SNE

$$\Theta_{i,j+1}+\Theta_{i,j-1}-\Theta_{i,j}+\Theta_{i+1,j}+\Theta_{i-1,j}+h^2C\exp\left(\Theta_{i,j}\right)\quad i=1,2,3,\ldots,\tau_1, j=1,2,3,\ldots,\tau_2. \tag{64}$$

By choosing $\tau_1=\tau_2=11,\ h=\frac{1}{11}$, *and* $C=0.1$, *we get a large SNE of order* 100×100. *The starting point is*

$$x_0=0.1(\sin(\pi h)\sin(\pi k),\ \sin(2\pi h)\sin(2\pi k),\ \ldots,\ \sin(10\pi h)\sin(10\pi k))^T$$

and results are depicted in Table 6.

Table 6. Computational results of 2D Bratu problem in Example 6.

Cases of (2)	j	$\|F(x_j)\|$	$\|x_{j+1} - x_j\|$	ξ^*
$i = 1$	0	8.1 (−2)	5.0 (−1)	
	1	1.2 (−21)	4.9 (−21)	
	2	3.3 (−141)	5.7 (−141)	
	3	4.3 (−860)	1.6 (−569)	5.9911
$i = 2$	0	8.1 (−2)	5.0 (−1)	
	1	9.6 (−30)	1.2 (−29)	
	2	1.3 (−256)	6.4 (−256)	
	3	2.0 (−2068)	2.4 (−2068)	8.0096

We have computed ACOC and observed that, as "i" increases, so does the ACOC.

Example 7. *Finally, we deal with succeeding SNE*

$$F(X) = \begin{cases} -1 + x_j^2 x_{j+1} = 0, \ 1 \le j \le \sigma - 1, \\ -1 + x_\sigma^2 x_1 = 0. \end{cases} \tag{65}$$

In order to access a giant system of nonlinear equations of order 200×200*, we pick* $\sigma = 200$*. In addition, we consider the following starting approximation for this problem:*

$$x^{(0)} = \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \cdots, \frac{5}{4}(200 \ times)\right)^T,$$

and converges to $\mu = (1, 1, 1, 1, \cdots, 1(200 \ times))^T$*. The attained computation outcomes are illustrated in Table 7.*

Table 7. Computational results on Example 7.

Cases of (2)	j	$\|F(x_j)\|$	$\|x_{j+1} - x_j\|$	ξ^*
$i = 1$	0	1.3 (+1)	3.5	
	1	9.8 (−4)	3.3 (−4)	
	2	2.0 (−31)	6.6 (−32)	
	3	2.7 (−225)	9.0 (−226)	7.0000
$i = 2$	0	1.3 (+1)	3.5	
	1	1.3 (−5)	4.3 (−6)	
	2	5.5 (−64)	1.8 (−64)	
	3	9.5 (−648)	3.2 (−648)	10.000

We have computed ACOC and observed that, as "i" increases, so does the ACOC.

5. Concluding Remarks

Recently, there has been a surge in the development of multi-step solvers for nonlinear equations. In this article, we present a unifying local convergence of solver (2), relying only on the first derivative. This way, we expand the applicability of these solvers. Notice that in earlier studies that are special cases of (2), higher than one derivatives are used, which do not appear in the solver. Moreover, no bounds on the distances $\|x_\sigma - \mu\|$ are provided, nor uniqueness theorems. Furthermore, we provide computable bounds and uniqueness of solutions. This is where the novelty of our article lies. Numerical and applications are also given to test the convergence conditions. In our application, we solve the 2D-Bratu, BVP problems as well as a system of nonlinear equations of 200×200.

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References

1. Argyros, I.K. *Convergence and Application of Newton-type Iterations*; Springer: New York, NY, USA, 2008.
2. Argyros, I.K.; Hilout, S. *Computational Methods in Nonlinear Analysis*; World Scientific Publishing Company: Hackensack, NJ, USA, 2013.
3. Cordero, A.; Torregrosa, J.R.; Vassileva, M.P. Increasing the order of convergence of iterative schemes for solving nonlinear system. *J. Comput. Appl. Math.* **2012** , *252*, 86–94. [CrossRef]
4. Hernández, M.A.; Martinez, E. On the semilocal convergence of a three steps Newton-type process under mild convergence conditions. *Numer. Algor.* **2015**, *70*, 377–392. [CrossRef]
5. Sharma, J.R.; Ghua, R.K.; Sharma, R. An efficient fourth-order weighted-Newton method for system of nonlinear equations. *Numer. Algor.* **2013**, *62*, 307–323. [CrossRef]
6. Traub, J.F. *Iterative Methods for the Solution of Equations*; Prentice- Hall: Englewood Cliffs, NJ, USA, 1964.
7. Xiao, X.; Yin, H. Achieving higher order of convergence for solving systems of nonlinear equations. *Appl. Math. Comput.* **2017**, *311*, 251–261. [CrossRef]
8. Potra, F.A.; Pták, V. *Nondiscrete introduction and iterative process*; Research Notes in Mathematics Volume 103; Pitman Advanced Publishing Program: Boston, MA, USA, 1984.
9. Montazeri, H.; Soleymani, F.; Shateyi, S.; Motsa, S.S. On a new method for computing the numerical solution of systems of nonlinear equations. *J. Appl. Math.* **2012**,*2012*, 751975. [CrossRef]
10. Amat, S.; Busquier, S.; Plaza, s.; Guttiérrez, J.M. Geometric constructions of iterative functions to solve nonlinear equations. *J. Comput. Appl. Math.* **2003**, *157*, 197–205. [CrossRef]
11. Amat, S.; Busquier, S.; Plaza, S. Dynamics of the King and Jarratt iterations. *Aequ. Math.* **2005**, *69*, 212–223. [CrossRef]
12. Amat, S.; Hernández, M.A.; Romero, N. A modified Chebyshev's iterative method with at least sixth order of convergence. *Appl. Math. Comput.* **2008**, *206*, 164–174. [CrossRef]
13. Argyros, I.K.; George, S. Local convergence of some higher-order Newton-like method with frozen derivative. *SeMA J.* **2015**, *70*, 47–59. [CrossRef]
14. Argyros, I.K.; Magreñán, Á.A. Ball convergence theorems and the convergence planes of an iterative methods for nonlinear equations. *SeMA J.* **2015**, *71*, 39–55.
15. Argyros, I.K.; Magreñán, Á.A. On the convergence of an optimal fourth-order family of methods and its dynamics. *Appl. Math. Comput.* **2015**, *252*, 336–346. [CrossRef]
16. Cordero, A.; Torregrosa, J.R. Variants of Newton's method using fifth-order quadrature formulas. *Appl. Math. Comput.* **2007**, *190*, 686–698. [CrossRef]
17. Cordero, A.; Torregrosa, J.R. Variants of Newton's method for functions of several variables. *Appl. Math. Comput.* **2006** , *183*, 199–208. [CrossRef]
18. Ezquerro, J.A.; Hernández, M.A. New iterations of R-order four with reduced computational cost. *BIT Numer. Math.* **2009** , *49*, 325–342. [CrossRef]
19. Ezquerro, J.A.; Hernández, M.A. A uniparametric Halley type iteration with free second derivative. *Int. J. Pure and Appl. Math.* **2003**, *6*, 99–110.
20. Gutiérrez, J.M.; Hernández, M.A. Recurrence relations for the super-Halley method. *Comput. Math. Appl.* **1998**, *36*, 1–8. [CrossRef]
21. Gutiérrez, J.M.; Magreñán, Á.A.; Romero, N. On the semilocal convergence of Newton–Kantorovich method under center-Lipschitz conditions. *Appl. Math. Comput.* **2013**, *221*, 79–88. [CrossRef]
22. Kou, J. A third-order modification of Newton method for systems of nonlinear equations. *Appl. Math. Comput.* **2007**, *191*, 117–121.
23. Magreñán, Á.A. Different anomalies in a Jarratt family of iterative root-finding methods. *Appl. Math. Comput.* **2014**, *233*, 29–38.

24. Magreñán, Á.A. A new tool to study real dynamics: The convergence plane. *Appl. Math. Comput.* **2014**, *248*, 215–224. [CrossRef]
25. Petkovic, M.S.; Neta, B.; Petkovic, L.; Džunič, J. *Multipoint Methods for Solving Nonlinear Equations*; Academic Press: New York, NY, USA, 2013.
26. Rheinboldt, W.C. An adaptive continuation process for solving systems of nonlinear equations. *Banach Ctr. Publ.* **1978**, *3*, 129–142. [CrossRef]
27. Weerakoon, S.; Fernando, T.G.I. A variant of Newton's method with accelerated third order convergence. *Appl. Math. Lett.* **2000**, *13*, 87–93. [CrossRef]
28. Beyer, W.A.; Ebanks, B.R.; Qualls, C.R. Convergence rates and convergence-order profiles for sequences. *Acta Appl. Math.* **1990**, *20*, 267–284. [CrossRef]
29. Potra, F.A. On Q-order and R-order of convergence. *J. Optim. Theory Appl.* **1989**, *63*, 415–431. [CrossRef]
30. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
31. Kapania, R.K. A pseudo-spectral solution of 2-parameter Bratu's equation. *Comput. Mech.* **1990**, *6*, 55–63. [CrossRef]
32. Simpson, R.B. A method for the numerical determination of bifurcation states of nonlinear systems of equations. *SIAM J. Numer. Anal.* **1975**, *12*, 439–451. [CrossRef]

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