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Separation Axioms of Interval-Valued Fuzzy Soft Topology via Quasi-Neighborhood Structure

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Abstract: In this study, we present the concept of the interval-valued fuzzy soft point and then introduce the notions of its neighborhood and quasi-neighborhood in interval-valued fuzzy soft topological spaces. Separation axioms in an interval-valued fuzzy soft topology, so-called $q-T_i$ for $i = 0, 1, 2, 3, 4$, are introduced, and some of their basic properties are also studied.

Keywords: interval-valued fuzzy soft set; interval-valued fuzzy soft topology; interval-valued fuzzy soft point; interval-valued fuzzy soft neighborhood; interval-valued fuzzy soft quasi-neighborhood; interval-valued fuzzy soft separation axioms

1. Introduction

In 1999, Molodtsov [1] proposed a new mathematical approach known as soft set theory for dealing with uncertainties and vagueness. Traditional tools such as fuzzy sets [2] and rough sets [3] cannot clearly define objects. Soft set theory is different from traditional tools for dealing with uncertainties. A soft set was defined by a collection of approximate descriptions of an object based on parameters by a given set-valued map. Maji et al. [4] initiated the research on both fuzzy set and soft set hybrid structures called fuzzy soft sets and presented a concept that was subsequently discussed by many researchers. Different extensions of the classical fuzzy soft sets were introduced, such as generalized fuzzy soft sets [5], intuitionist fuzzy soft sets [6,7], vague soft sets [8], interval-valued fuzzy soft sets [9], and interval-valued intuitive fuzzy soft sets [10]. In particular, to alleviate some disadvantages of fuzzy soft sets, interval-valued fuzzy soft sets were introduced where no objective procedure was available to select the crisp membership degree of elements in fuzzy soft sets. Tanya and Kandemir [11] started topological studies of fuzzy soft sets. They used the classical concept of topology to construct a topological space over a fuzzy soft set and named it the fuzzy soft topology. They also studied some fundamental topological properties for the fuzzy soft topology, such as interior, closure, and base. Later, Simsekler and Yuksel [12] studied the fuzzy soft topological space in the case of Tanay and Kandemir [11]. However, they established the concept of the fuzzy soft topology over a fuzzy soft set with a set of fixed parameters and considered some topological concepts for fuzzy soft topological spaces such as the base, subbase, neighborhood, and Q-neighborhood. Roy and Samanta [13] noted a new concept of the fuzzy soft topology. They suggested the notion of the fuzzy soft topology over an ordinary set by adding fuzzy soft subsets of it, where everywhere, the parameter set is supposed to be fixed. Then, in [14], they continued to study the fuzzy soft topology and established a fuzzy soft point definition and various neighborhood structures. Atmaca and Zorlutuna [15] considered the concept of soft quasi-coincidence for fuzzy soft sets. By applying this new concept, they also studied the basic topological notions such as interior and closure for fuzzy soft sets. The concept of the product fuzzy soft topology and the boundary fuzzy soft topology was introduced by Zahedi et al. [16,17], and they

studied some of their properties. They also suggested a new definition for the fuzzy soft point and then different neighborhood structures. Separation axioms of the fuzzy topological space and fuzzy soft topological space were studied by many authors, see [18–23] and [24–27]. The aim of this work is to develop interval-valued fuzzy soft separation axioms. We start with preliminaries and then give the definition of the interval-valued fuzzy soft point as a generalization of the interval-valued fuzzy point and fuzzy soft point in order to create different neighborhood structures in the interval-valued fuzzy soft topological space in Sections 3 and 4. Finally, in Section 5, the notion of separation axioms q - T_i , $i = 0, 1, 2, 3, 4$ in the interval-valued fuzzy soft topology is introduced, and some of their basic properties are also studied.

2. Preliminaries

Throughout this paper, X is the set of objects and E is the set of parameters. The set of all subsets of X is denoted by $P(X)$ and $A \subset E$, showing a subset of E .

Definition 1 ([1]). A pair (f, A) is called a soft set over X , if f is a mapping given by $f : A \rightarrow P(X)$. For any parameter $e \in A$, $f(e) \subset X$ may be considered as the set e -approximate elements of the soft set (f, A) . In other words, the soft set is not a kind of set, but a parameterized family of subsets of the set X .

Before introducing the notion of the interval-valued fuzzy soft sets, we give the concept of the interval-valued fuzzy set.

Definition 2 ([28]). An interval-valued fuzzy (IVF) set over X is defined by the membership function $f : X \rightarrow \text{int}([0, 1])$, where $\text{int}([0, 1])$ denotes the set of all closed subintervals of $[0, 1]$. Suppose that $x \in X$. Then, $f(x) = [f^-(x), f^+(x)]$ is called the degree of membership of the element $x \in X$, where $f^-(x)$ and f^+ are the lower and upper degrees of the membership of x and $0 < f^-(x) < f^+(x) < 1$.

Yang et al. [9] suggested the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set as below.

Definition 3 ([9]). An interval-valued fuzzy soft (IVFS) set over X denoted by f_E or (f, E) is defined by the mapping $f : E \rightarrow \mathcal{IVF}(X)$, where $\mathcal{IVF}(X)$ is the set of all interval-valued fuzzy sets over X . For any $e \in E$, $f(e)$ can be written as an interval-valued fuzzy set such that $f(e) = \{\langle x, [f_e^-(x), f_e^+(x)] \rangle : x \in X\}$ where $f_e^-(x)$ and $f_e^+(x)$ are the lower and upper degrees of the membership of x with respect to e , where $0 \leq f_e^-(x) \leq f_e^+(x) \leq 1$.

Note that $\mathcal{IVFS}(X, E)$ shows the set of all IVFS sets over X .

Definition 4 ([9]). Let f_A and g_B be two IVFS sets over X . We say that:

1. f_A is an interval-valued fuzzy soft subset of g_B , denoted by $f_A \tilde{\leq} g_B$, if and only if:
 - (i) $A \leq B$,
 - (ii) For all $e \in A$, $f_e^-(x) \leq g_e^-(x)$ and $f_e^+(x) \leq g_e^+(x)$, $\forall x \in X$.
2. $f_A = g_B$ if and only if $f_A \tilde{\leq} g_B$ and $g_A \tilde{\leq} f_B$.
3. The union of two IVFS sets f_A and g_B , denoted by $f_A \tilde{\vee} g_B$, is the IVFS set $(f \vee g, C)$, where $C = A \cup B$, and for all $e \in C$, we have:

$$(f \vee g)_e(x) = \begin{cases} [f_e^-(x), f_e^+(x)], & e \in A - B \\ [g_e^-(x), g_e^+(x)], & e \in B - A \\ [\max(f_e^-(x), g_e^-(x)), \max(f_e^+(x), g_e^+(x))] & e \in A \cap B, \end{cases}$$

for all $x \in X$.

- The intersection of two IVFS sets f_A and g_B , denoted by $f_A \tilde{\wedge} g_B$, is the IVFS set $(f \wedge g, C)$, where $C = A \cap B$, and for all $e \in C$, we have $(f \wedge g)_e(x) = [\min f_e^-(x), g_e^-(x), \min f_e^+(x), g_e^+(x)]$ for all $x \in X$.
- The complement of the IVFS set f_A is denoted by $f_A^c(x)$ where for all $e \in A$, we have $f_e^c(x) = [1 - f_e^+(x), 1 - f_e^-(x)]$.

Definition 5 ([9]). Let f_E be an IVFS set. Then:

- f_E is called the null interval-valued fuzzy soft set, denoted by \emptyset_E , if $f_e^-(x) = f_e^+(x) = 0$, for all $x \in X, e \in E$.
- f_E is called the absolute interval-valued fuzzy soft set, denoted by X_E , if $f_e^-(x) = f_e^+(x) = 1$, for all $x \in X, e \in E$.

Motivated by the definition of the soft mapping, discussed in [29], we define the concept of the IVFS mapping as the following:

Definition 6. Let f_A be an IVFS set over X_1 and g_B be an IVFS set over X_2 , where $A \subseteq E_1$ and $B \subseteq E_2$. Let $\Phi_u : X_1 \rightarrow X_2$ and $\Phi_p : E_1 \rightarrow E_2$ be two mappings. Then:

- The map $\Phi : \mathcal{IVFS}(X_1, E_1) \rightarrow \mathcal{IVFS}(X_2, E_2)$ is called an IVFS map from X_1 to X_2 , and for any $y \in X_2$ and $\varepsilon \in B \subseteq E_2$, the lower image and the upper image of f_A under Φ is the IVFS $\Phi(f_A)$ over X_2 , respectively, defined as below:

$$[\Phi(f^-)](\varepsilon)(y) = \begin{cases} \sup_{x \in \Phi_u^{-1}(y)} [\sup_{e \in \Phi_p^{-1} \cap A} f^-(e)](x), & \text{if } \Phi_p^{-1}(\varepsilon) \cap A \neq \emptyset \text{ and } \Phi_u^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

$$[\Phi(f^+)](\varepsilon)(y) = \begin{cases} \sup_{x \in \Phi_u^{-1}(y)} [\sup_{e \in \Phi_p^{-1} \cap A} f^+(e)](x), & \text{if } \Phi_p^{-1}(\varepsilon) \cap A \neq \emptyset \text{ and } \Phi_u^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

- Let $\Phi : \mathcal{IVFS}(X_1, E_1) \rightarrow \mathcal{IVFS}(X_2, E_2)$ be an IVFS map from X_1 to X_2 . The lower inverse image and the upper inverse image of IVFS g_B under Φ denoted by $\Phi^{-1}(g_B)$ is an IVFS over X_1 , respectively, such that for all $x \in X_1$ and $e \in E_1$, it is defined as below:

$$[\Phi^{-1}(g^-)](e)(x) = \begin{cases} g_{\Phi_p(e)}^-(x), & \text{if } \Phi_p(e) \in B \\ 0, & \text{otherwise,} \end{cases}$$

$$[\Phi^{-1}(g^+)](e)(x) = \begin{cases} g_{\Phi_p(e)}^+(x), & \text{if } \Phi_p(e) \in B \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1. Let $\Phi : \mathcal{IVFS}(X, E) \rightarrow \mathcal{IVFS}(Y, F)$ be an IVFS mapping between X and Y , and let $\{f_{iA}\}_{i \in J} \subset \mathcal{IVFS}(X, E)$ and $\{g_{iB}\}_{i \in J} \subset \mathcal{IVFS}(Y, F)$ be two families of IVFS sets over X and Y , respectively, where $A \subseteq E$ and $B \subseteq F$, then the following properties hold.

- $[\Phi(f_{jA})]^c \leq \Phi(f_{jA})^c$ for each $j \in J$.
- $[\Phi^{-1}(g_{jB})]^c = \Phi^{-1}(g_{jB})^c$ for each $j \in J$.
- If $g_{iB} \leq g_{jB}$, then $\Phi^{-1}(g_{iB}) \leq \Phi^{-1}(g_{jB})$ for each $i, j \in J$.
- If $f_{iA} \leq f_{jA}$, then $\Phi(f_{iA}) \leq \Phi(f_{jA})$ for each $i, j \in J$.
- $\Phi[\bigvee_{j \in J} f_{jA}] = \bigvee_{j \in J} \Phi(f_{jA})$ and $\Phi^{-1}[\bigvee_{j \in J} g_{jB}] = \bigvee_{j \in J} \Phi^{-1}(g_{jB})$.
- $\Phi[\bigwedge_{j \in J} f_{jA}] = \bigwedge_{j \in J} \Phi(f_{jA})$ and $\Phi^{-1}[\bigwedge_{j \in J} g_{jB}] = \bigwedge_{j \in J} \Phi^{-1}(g_{jB})$.

Proof. We only prove Part (5). The other parts follow a similar technique. For any $k \in F$, $y \in Y$, and $a \in A$, then:

$$\begin{aligned}
 \Phi[\tilde{V}_{j \in J} f_{jA}](k)(y) &= \sup_{x \in \Phi_u^-(y)} \left(\sup_{z \in \Phi_p^{-1}(k)} (\tilde{V}_{j \in J} f_{jA})(z)(x) \right) \\
 &= \sup_{x \in \Phi_u^-(y)} \left(\sup_{z \in \Phi_p^{-1}(k)} \left(\max_{j \in J} ([f_{ja}^-, f_{ja}^+]) \right) \right)(k)(y) \\
 &= \sup_{x \in \Phi_u^-(y)} \left(\max_{j \in J} \left(\sup_{z \in \Phi_p^{-1}(k)} [f_{ja}^-(k), f_{ja}^+(k)] \right) \right)(y) \\
 &= \max_{j \in J} \left(\sup_{x \in \Phi_u^-(y)} \left(\sup_{z \in \Phi_p^{-1}(k)} [f_{ja}^-(k)(y), f_{ja}^+(k)(y)] \right) \right) \\
 &= \max_{j \in J} \left(\sup_{x \in \Phi_u^-(y)} \left(\sup_{z \in \Phi_p^{-1}(k)} f_{jA}(k)(y) \right) \right) \\
 &= \max_{j \in J} \Phi(f_{jA})(k)(y) \\
 &= \tilde{V}_{j \in J} \Phi(f_{jA})(k)(y).
 \end{aligned}$$

Now, we prove that $\Phi^{-1}[\tilde{V}_{j \in J} g_{jB}] = \tilde{V}_{j \in J} \Phi^{-1}(g_{jB})$. For any $e \in E$, $x \in X$ and $b \in B$:

$$\begin{aligned}
 \Phi^{-1}[\tilde{V}_{j \in J} g_{jB}](e)(x) &= (\tilde{V}_{j \in J} g_{jB})(\Phi_p(e))(\Phi_u(x)) \\
 &= [\max_{j \in J} g_{jb}^-, \max_{j \in J} g_{jb}^+](\Phi_p(e))(\Phi_u(x)) \\
 &= [[\max_{j \in J} g_{jb}^-(\Phi_p(e))(\Phi_u(x)), \max_{j \in J} g_{jb}^+(\Phi_p(e))(\Phi_u(x))]] \\
 &= [\max_{j \in J} \Phi_u^{-1}(g_{jb}^-)(e)(x), \max_{j \in J} \Phi_u^{-1}(g_{jb}^+)(e)(x)] \\
 &= \max_{j \in J} [\Phi_u^{-1}(g_{jb}^-)(e)(x), \Phi_u^{-1}(g_{jb}^+)(e)(x)] \\
 &= \max_{j \in J} \Phi_u^{-1}(g_{jB})(e)(x) \\
 &= \tilde{V}_{j \in J} \Phi_u^{-1}(g_{jB})(e)(x).
 \end{aligned}$$

□

3. Interval-Valued Fuzzy Soft Topological Spaces

The interval-valued fuzzy topology *IVFT* was discussed by Mondal and Samanta [30]. In this section, we recall their definition and then present different neighborhood structures in the interval-valued fuzzy soft topology (*IVFST*).

Definition 7. Let X be a non-empty set, and let τ be a collection of interval valued fuzzy soft sets over X with the following properties:

- (i) \emptyset_E, X_E belong to τ ,
- (ii) If f_{1E}, f_{2E} are IVFS sets belong to τ , then $f_{1E} \tilde{\wedge} f_{2E}$ belong to τ ,
- (iii) If the collection of IVFS sets $\{f_{jE} | j \in J\}$ where J is an index set, belonging to τ , then $\tilde{V}_{j \in J} f_{jE}$ belong to τ .

Then, τ is called the interval-valued fuzzy soft topology over X , and the triplet (X, E, τ) is called the interval-valued fuzzy soft topological space (*IVFST*).

As the ordinary topologies, the indiscrete *IVFST* over X contains only \emptyset_E and X_E , while the discrete *IVFST* over X contains all IVFS sets. Every member of τ is called an interval-valued fuzzy soft open set (*IVFS-open*) in X . The complement of an *IVFS-open* set is called an *IVFS-closed* set.

Remark 1. If $f_e^-(x) = f_e^+(x) = a \in [0, 1]$, then we put $[f_e^-(x), f_e^+(x)] = [a, a] = a$.

Example 1. Let $X = [0, 1]$ and E be any subset of X . Consider the IVFS set f_E over X by the mapping:

$$f : E \rightarrow \mathcal{IVF}([0, 1])$$

such that for any $e \in E, x \in X$:

$$\tilde{f}_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Then, the collection $\tau = \{\Phi_E, X_E, f_E\}$ is an IVFST over X .

1. Clearly $X_E, \emptyset_E \in \tau$.
2. Let $\{f_{jE}\}_{j \in J}$ be a sub-family of τ where for any $j \in J$ if $x \in X$ such that for all $e \in E$:

$$f_{je}(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Since:

$$\bigvee_j f_{je}(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

then $\bigvee_j f_{jE} \in \tau$.

3. Let $f_E, g_E \in \tau$, where:

$$f_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and:

$$g_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Since:

$$f_e(x) \wedge g_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Thus, $f_E \wedge g_E \in \tau$.

Example 2 ([23]). Let \mathbb{R} be the set of all real numbers with the usual topology τ_u where $\tau_u = \langle \{(a, b), a, b \in \mathbb{R}\} \rangle$ and E is a parameter set. Let $U = (a, b) \subset \mathbb{R}$ be an open interval in \mathbb{R} ; we define IVFS \tilde{U}_E over \mathbb{R} by the mapping:

$$\tilde{U} : E \rightarrow (\text{Int}[0, 1])^{\mathbb{R}}$$

such that for all $x \in \mathbb{R}$:

$$\tilde{U}_e(x) = \begin{cases} 1 & x \in (a, b) \\ 0 & x \notin (a, b). \end{cases}$$

Then, the family $\{\tilde{U}_E : (a, b) \subset \mathbb{R}, \forall a, b \in \mathbb{R}\}$ generates an IVFS over \mathbb{R} , and we denote it by $\tau_u^{(IVFS)}$:

1. Clearly, $\mathbb{R}_E, \emptyset_E \in \tau_u^{(IVFS)}$ where for all $e \in E, k \in \mathbb{R}, \mathbb{R}_E(e)(k) = [1, 1]$, and $\emptyset_e(k) = 0$
2. Let $\{\tilde{U}_{jE}\}_{j \in J}$ be a sub-family of $\tau_u^{(IVFS)}$ where for any $j \in J$ if $x \in (a_j, b_j)$ and interval (a_j, b_j) in \mathbb{R} such that for all $e \in E$:

$$\tilde{U}_{je}(x) = \begin{cases} 1 & x \in (a_j, b_j) \\ 0 & x \notin (a_j, b_j). \end{cases}$$

Since $\bigvee_j \tilde{U}_{jE} = (\bigcup_j \tilde{U}_j, E)$ where $\bigcup_j \tilde{U}_j \in \tau_u$, then $\bigvee_j \tilde{U}_{jE} \in \tau_u^{(IVFS)}$

3. Let $\tilde{U}_E, \tilde{V}_E \in \tau_u^{(IVFS)}$, then $\tilde{U}_E \tilde{\wedge} \tilde{V}_E \in \tau_u^{(IVFS)}$ since $\tilde{U}_E \tilde{\wedge} \tilde{V}_E = (\tilde{U} \cap \tilde{V}, E)$ where $U \cap V \in \tau_u$.

Definition 8. Let interval $[\lambda_e^-, \lambda_e^+] \subseteq [0, 1]$ for all $e \in E$. Then, \tilde{x}_E is called an interval-valued fuzzy soft point (IVFS point) with support $x \in X$ and e lower value λ_e^- and e upper value λ_e^+ , if for each $y \in X$:

$$\tilde{x}(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & y = x \\ 0 & \text{otherwise.} \end{cases}$$

Example 3. Let $X = [0, 1]$ and E be any subset of X . Consider IVFS point \tilde{x}_E with support x , lower value zero, and upper value 0.3, we define IVFS point \tilde{x}_E by:

$$\tilde{x}(e)(c) = \begin{cases} [0, 0.3] & c = x \\ 0 & \text{otherwise,} \end{cases}$$

for any $e \in E$ and $c \in X$.

Definition 9. The IVFS point \tilde{x}_E belongs to IVFS set f_E , denoted by $\tilde{x}_E \tilde{\in} f_E$, whenever for all $e \in E$, we have $\lambda_e^- \leq f_e^-(x)$ and $\lambda_e^+ \leq f_e^+(x)$.

Theorem 1. Let f_E be an IVFS set, then f_E is the union of all its IVFS points, i.e., $f_E = \bigvee_{\tilde{x}_E \tilde{\in} f_E} \tilde{x}_E$.

Proof. Let $x \in X$ be a fixed point, $y \in X$ and $e \in E$. Take all $\tilde{x}_E \tilde{\in} f_E$ with different e lower and e upper values $\lambda_{je}^-, \lambda_{je}^+$ where $j \in J$. Then, there exists $\lambda_{je}^- = f_e^-(x)$, $\lambda_{je}^+ = f_e^+(x)$ where:

$$\begin{aligned} \bigvee_{\tilde{x}_E \tilde{\in} f_E} \tilde{x}_e(y) &= [\sup \tilde{x}_e^-(y), \sup \tilde{x}_e^+(y)] \\ &= [\sup_{\lambda_{je}^- \leq f_e^-(x)} \lambda_{je}^-, \sup_{\lambda_{je}^+ \leq f_e^+(x)} \lambda_{je}^+] \\ &= [f_e^-(x), f_e^+(x)]. \end{aligned}$$

□

Proposition 2. Let $\{f_{jE}\}_{j \in J}$ be a family of IVFS sets over X , where J is an index set and \tilde{x}_E is an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ . If $\tilde{x}_E \tilde{\in} \bigwedge_{j \in J} \{f_{jE}\}$, then $\tilde{x}_E \tilde{\in} \{f_{jE}\}$ for each $j \in J$.

Proof. Let \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ , and let $\tilde{x}_E \tilde{\in} \bigwedge_{j \in J} \{f_{jE}\}$. Then, $\lambda_e^- \leq \bigwedge_{j \in J} \{f_{je}^-(x)\} \leq \{f_{je}^-(x)\}$ for each $e \in E$, $x \in X$ and $\lambda_e^+ \leq \bigwedge_{j \in J} \{f_{je}^+(x)\} \leq \{f_{je}^+(x)\}$ for each $e \in E$, $x \in X$. Thus, $[\lambda_e^-, \lambda_e^+] \leq [\{f_{je}^-(x)\}, \{f_{je}^+(x)\}]$, for each $e \in E$, $x \in X$. Hence, $\tilde{x}_E \tilde{\in} \{f_{jE}\}_{j \in J}$. □

Remark 2. If $\tilde{x}_E \tilde{\in} f_E \bigvee g_E$ does not imply $\tilde{x}_E \tilde{\in} f_E$ or $\tilde{x}_E \tilde{\in} g_E$.

This is shown in the following example.

Example 4. Let τ be an IVFST over X , where $\tau = \{\emptyset_E, X_E, f_E, g_E, f_E \wedge g_E\}$, and \tilde{x}_E be the absolute IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ . If f_E and g_E are two IVFS sets in X defined as below:

$$f : E \rightarrow \mathcal{IVF}([0, 1])$$

and:

$$g : E \rightarrow \mathcal{IVF}([0, 1])$$

such that for any $e \in E, x \in X$:

$$f_e(x) = \begin{cases} [1, 0.5] & 0 \leq x \leq e \\ 0 & e < x \leq 1 \end{cases}$$

and:

$$g_e(x) = \begin{cases} [0.2, 1] & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Since:

$$f_e(x) \vee g_e(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq e \\ 0 & \text{if } e < x \leq 1, \end{cases}$$

then $\tilde{x}_E \in f_E \tilde{\vee} g_E$, but $\tilde{x}_E \notin f_E$ and $\tilde{x}_E \notin g_E$.

Theorem 2. Let \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ and f_E and g_E be IVFS sets. If $\tilde{x}_E \in f_E \tilde{\vee} g_E$, then there exists IVFS point $\tilde{x}_{1E} \in f_E$ and IVFS point $\tilde{x}_{2E} \in g_E$ such that $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.

Proof. Let $\tilde{x}_E \in f_E \tilde{\vee} g_E$. Then, $\lambda_e^- \leq f_e^-(x) \vee g_e^-(x)$ and $\lambda_e^+ \leq f_e^+(x) \vee g_e^+(x)$, for each $e \in E, x \in X$. Let us choose

$$E_1 = \{e \in E \mid \lambda_e^- \leq f_e^-(x), \lambda_e^+ \leq f_e^+(x) : x \in X\},$$

$$E_2 = \{e \in E \mid \lambda_e^- \leq g_e^-(x), \lambda_e^+ \leq g_e^+(x) : x \in X\}$$

and:

$$\tilde{x}_1(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & \text{if } y = x_1, e \in E_1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{x}_2(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+], & \text{if } y = x_2, e \in E_2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $x_{1e}^- \leq f_{1e}^-(x)$ and $x_{1e}^+ \leq f_{1e}^+(x)$ for each $e \in E_1, x \in X$, that implies $\tilde{x}_{1E} \in f_{1E}$ and also $x_{2e}^- \leq f_{2e}^-(x)$, and $x_{2e}^+ \leq f_{2e}^+(x)$ for each $e \in E_2, x \in X$, that implies $\tilde{x}_{2E} \in f_{2E}$. Consequently, $E_1 \tilde{\vee} E_2 = E$ and $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$. \square

Definition 10. Let (X, E, τ) be an IVFST space and \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ . The IVFS set g_E is called the interval-valued fuzzy soft neighborhood (IVFSN) of IVFS point \tilde{x}_E , if there exists the IVFS-open set f_E in X such that $\tilde{x}_E \in f_E \tilde{\vee} g_E$. Therefore, the IVFS-open set f_E is an IVFSN of the IVFS point \tilde{x}_E if $\forall e \in E, x \in X$ such that $\lambda_e^- < f_e^-(x)$ and $\lambda_e^+ < f_e^+(x)$.

Definition 11. Let (X, E, τ) be an IVFST space and \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ and \tilde{x}_E^* be an IVFS point with support x^* , e lower value ϵ_e^- , and e upper value ϵ_e^+ . \tilde{x}_E^* is said to be compatible with λ_e^-, λ_e^+ , if \tilde{x}_E^* provides that $0 \leq \epsilon_e^- \leq \lambda_e^-$ and $0 \leq \epsilon_e^+ \leq \lambda_e^+$ for each $e \in E$.

Proposition 3.

1. If f_E is an IVFSN of the IVFS point \tilde{x}_E and $f_E \tilde{\leq} h_E$, then h_E is also an IVFSN of \tilde{x}_E .
2. If f_E and g_E are two IVFSN of the IVFS point \tilde{x}_E , then $f_E \wedge g_E$ is also the IVFSN of \tilde{x}_E .
3. If f_E is an IVFSN of the IVFS point \tilde{x}_E^* with support x^* , e lower value $\lambda_e^- - \epsilon_e^-$, and e upper value $\lambda_e^+ - \epsilon_e^+$, for all ϵ_e^- compatible with λ_e^- and ϵ_e^+ compatible with λ_e^+ , then f_E is an IVFSN of the IVFS point \tilde{x}_E .
4. If f_E is an IVFSN of the IVFS point \tilde{x}_{1E} and g_E is an IVFSN of the IVFS point \tilde{x}_{2E} , then $f_E \tilde{\vee} g_E$ is also an IVFSN of \tilde{x}_{1E} and \tilde{x}_{2E} .
5. If f_E is an IVFSN of the IVFS point \tilde{x}_E , then there exists IVFSN g_E of \tilde{x}_E such that $g_E \tilde{\leq} f_E$ and g_E is IVFSN of IVFS point \tilde{y} with support y , e lower value γ_e^- , and e upper value γ_e^+ , for all $\tilde{y}_E \in g_E$.

Proof.

1. Let f_E be an IVFSN of the IVFS point \tilde{x} . Then, there exists the IVFS-open set g_E in X such that $\tilde{x}_E \in g_E \leq f_E$. Since $f_E \leq h_E$, $\tilde{x}_E \in g_E \leq f_E \leq h_E$. Thus, h_E is an IVFSN of \tilde{x}_E .
2. Let f_E and g_E be two IVFSN of the IVFS point \tilde{x}_E . Then, there exists two IVFS-open sets h_E, k_E in X such that $\tilde{x}_E \in h_E \leq f_E$ and $\tilde{x}_E \in k_E \leq g_E$. Thus, $\tilde{x}_E \in h_E \wedge k_E \leq f_E \wedge g_E$. Since $h_E \wedge k_E$ is an IVFS-open set, $g_E \wedge f_E$ is an IVFSN of \tilde{x}_E .
3. Let f_E be an IVFSN of the IVFS point \tilde{x}_E^* with support x^* , e lower value $\lambda_e^- - \varepsilon_e^-$, and e upper value $\lambda_e^+ - \varepsilon_e^+$, for all ε_e^- compatible with λ_e^- and ε_e^+ compatible with λ_e^+ . Then, there exists IVFS-open set $g_E^{x^*}$ such that $\tilde{x}_E^* \in g_E^{x^*} \leq f_E$. Let $g_E = \bigvee_{x^*} g_E^{x^*}$, then g_E is IVFS-open in X and $g_E \leq f_E$. By Theorem 1 and since for all $e \in E$, $\bigvee \tilde{x}_E^* = \tilde{x}_E \leq \bigvee_{x^*} g_E^{x^*} = g_E \leq f_E$. Hence, $\tilde{x}_E \in g_E \leq f_E$, i.e., f_E is an IVFSN of \tilde{x}_E .
4. Let f_E be an IVFSN of the IVFS point \tilde{x}_{1E} with support x_1 , e lower value λ_{1e}^- , and e upper value λ_{1e}^+ and g_E be an IVFSN of the IVFS point \tilde{x}_{2E} with support x_2 , e lower value λ_{2e}^- , and e upper value λ_{2e}^+ . Then, there exists IVFS-open sets h_{1E}, h_{2E} such that $\tilde{x}_{1E} \in h_{1E} \leq f_E$ and $\tilde{x}_{2E} \in h_{2E} \leq g_E$, respectively. Since $\tilde{x}_{1E} \in h_{1E}$, $\lambda_{1e}^- \leq h_{1e}^-(x)$, $\lambda_{1e}^+ \leq h_{1e}^+(x)$ for each $e \in E$ and $x \in X$. Since $\tilde{x}_{2E} \in h_{2E}$, $\lambda_{2e}^- \leq h_{2e}^-(x)$, $\lambda_{2e}^+ \leq h_{2e}^+(x)$ for each $e \in E$ and $x \in X$. Thus, we have:

$$\max\{[\lambda_{1e}^-, \lambda_{1e}^+], [\lambda_{2e}^-, \lambda_{2e}^+]\} \leq \max\{[h_{1e}^-(x), h_{1e}^+(x)], [h_{2e}^-(x), h_{2e}^+(x)]\}$$

for each $e \in E$, $x \in X$. Therefore, $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E} \in h_{1E} \tilde{\vee} h_{2E}$, $h_{1E} \tilde{\vee} h_{2E} \in \tau$, and $h_{1E} \tilde{\vee} h_{2E} \leq f_E \tilde{\vee} g_E$. Consequently, $f_E \tilde{\vee} g_E$ is an IVFSN of $x_{1E} \tilde{\vee} x_{2E}$.

5. Let f_E be an IVFSN of the IVFS point \tilde{x}_E , with support x , e lower value λ_e^- , and e upper value λ_e^+ . Then, there exists IVFS-open set g_E such that $\tilde{x}_E \in g_E \leq f_E$. Since g_E is an IVFS-open set, g_E is a neighborhood of its points, i.e., g_E is an IVFSN of IVFS point \tilde{y}_E with support y , e lower value γ_e^- , and e upper value γ_e^+ , for all $e \in E$. Furthermore, g_E is an IVFSN of IVFS point \tilde{x}_E since $\tilde{x}_E \in g_E$. Therefore, there exists g_E that is an IVFSN of \tilde{x}_E such that $g_E \leq f_E$ and g_E is an IVFSN of \tilde{y}_E ; since f_E is an IVFSN of \tilde{x}_E .

□

Definition 12. Let (X, E, τ) be an IVFST space and f_E be an IVFS set. The IVFS-closure of f_E denoted by $Cl f_E$ is the intersection of all IVFS-closed super sets of f_E . Clearly, $Cl f_E$ is the smallest IVFS-closed set over X that contains f_E .

Example 5 ([23]). Consider IVFST τ_u^{IVFS} over \mathbb{R} as introduced in Example 2, and if \tilde{H}_E is an IVFS over \mathbb{R} related of the open interval $H = (a, b) \subset \mathbb{R}$ by mapping:

$$\tilde{H} : E \rightarrow (Int[0, 1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in (a, b) \\ 0 & x \notin (a, b), \end{cases}$$

where $e \in E$ and $x \in \mathbb{R}$, then the closure of \tilde{H}_E is defined as:

$$Cl \tilde{H} : E \rightarrow (Int[0, 1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b]. \end{cases}$$

Remark 3. By replacing \tilde{x}_E for f_E , the IVFS-closure of \tilde{x}_E denoted by $Cl \tilde{x}_E$ is the intersection of all IVFS-closed super sets of \tilde{x}_E .

Proposition 4. Let (X, E, τ) be an IVFST space and f_E and g_E be two IVFSS over X . Then:

1. $Cl\emptyset_E = \emptyset_E$ and $Cl\tilde{X}_E = \tilde{X}_E$,
2. $f_E \leq Clf_E$, and Clf_E is the smallest IVFS-closed set containing $IVFSf_E$,
3. $Cl(Cl f_E) = Cl f_E$,
4. if $f_E \leq g_E$, then $(Cl f_E) \leq Cl g_E$.
5. f_E is an IVFS-closed set if and only if $f_E = Cl f_E$,
6. $Cl(f_E \tilde{\vee} g_E) = Cl f_E \tilde{\vee} Cl g_E$,
7. $Cl(f_E \tilde{\wedge} g_E) \leq Cl f_E \tilde{\wedge} Cl g_E$.

Proof. We only prove Part (6). A similar technique is used to show the other parts.

Since $f_E \leq f_E \tilde{\vee} g_E$ and $g_E \leq f_E \tilde{\vee} g_E$, by Part (4), we have $Cl f_E \leq Cl(f_E \tilde{\vee} g_E)$ and $Cl g_E \leq Cl(f_E \tilde{\vee} g_E)$. Then, $Cl f_E \tilde{\vee} Cl g_E \leq Cl(f_E \tilde{\vee} g_E)$.

Conversely, we have $f_E \leq Cl f_E$ and $g_E \leq Cl g_E$, by Part (2). Then, $f_E \tilde{\vee} g_E \leq Cl f_E \tilde{\vee} Cl g_E$ where $Cl f_E \tilde{\vee} Cl g_E$ is an IVFS-closed set. Thus, $Cl(f_E \tilde{\vee} g_E) \leq Cl f_E \tilde{\vee} Cl g_E$.

Therefore, $Cl(f_E \tilde{\vee} g_E) = Cl f_E \tilde{\vee} Cl g_E$. \square

Definition 13. Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two IVFSTS and:

$$\Phi : (X_1, E_1, \tau_1) \rightarrow (X_2, E_2, \tau_2)$$

be an IVFS map. Then, Φ is called an:

1. interval-valued fuzzy soft continuous (IVFSC) map if and only if for each $g_{E_2} \in \tau_2$, we have $\Phi^{-1}(g_{E_2}) \in \tau_1$,
2. interval-valued fuzzy soft open (IVFSO) map if and only if for each $f_E \in \tau_1$, we have $\Phi(f_E) \in \tau_2$.

Theorem 3. Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two IVFST and Φ be an IVFS mapping from X_1 to X_2 , then the following statements are equivalent:

1. Φ is IVFC,
2. For each IVFS point \tilde{x}_E on X_1 , the inverse of every neighborhood of $\Phi(\tilde{x}_E)$ under Φ is a neighborhood of \tilde{x}_E ,
3. For each IVFS point \tilde{x}_E on X_1 and each neighborhood g_E of $\Phi(\tilde{x}_E)$, there exists a neighborhood f_E of \tilde{x}_E such that $\Phi(f_E) \leq g_E$.

Proof.

(1) \Rightarrow (2) Let g_E be an IVFSN of $\Phi(\tilde{x}_E)$ in τ_2 . Then, there exists an IVFS-open set f_E in τ_2 such that $\Phi(\tilde{x}_E) \in f_E \leq g_E$. Since Φ is IVFSC, $\Phi^{-1}(f_E)$ is an IVFS-open in τ_1 , and we have $\tilde{x}_E \in \Phi^{-1}(f_E) \leq \Phi^{-1}(g_E)$.

(2) \Rightarrow (3) Let g_E be an IVFSN of $\Phi(\tilde{x}_E)$. By the hypothesis, $\Phi^{-1}(g_E)$ is an IVFSN of \tilde{x}_E . Consider $f_E = \Phi^{-1}(g_E)$ to be an IVFSN of \tilde{x}_E . Then, we have $\Phi(f_E) = \Phi(\Phi^{-1}(g_E)) \leq g_E$.

(3) \Rightarrow (1) Let g_E be an IVFS-open set in τ_2 . We must show that $\Phi^{-1}(g_E)$ is an IVFS-open set in τ_1 . Now, let $\tilde{x}_E \in \Phi^{-1}(g_E)$. Then, $\Phi(\tilde{x}_E) \in g_E$. Since g_E is an IVFS-open set in τ_2 , we get that g_E is an IVFSN $\Phi(\tilde{x}_E)$ in τ_2 . By the hypothesis, there exists IVFS-open set f_E that is an IVFSN of \tilde{x}_E such that $\Phi(f_E) \leq g_E$. Thus, $f_E \leq \Phi^{-1}[\Phi(f_E)] \leq \Phi^{-1}(g_E)$ for f_E is an IVFSN of \tilde{x}_E . From here, $f_E \leq \Phi^{-1}(g_E)$, as f_E is an IVFSN of \tilde{x}_E . Hence, $\Phi^{-1}(g_E) \in \tau_1$. \square

4. Quasi-Coincident Neighborhood Structure of Interval-Valued Fuzzy Soft Topological Spaces

In this section, we present the quasi-coincident neighborhood structure in the interval-valued fuzzy soft topology (IVFST) and its properties.

Definition 14. The IVFS point \tilde{x}_E is called soft quasi-coincident with IVFS f_E , denoted by $\tilde{x}_E \tilde{q} f_E$, if there exists $e \in E$ such that $\lambda_e^- + f_e^-(x) > 1$ and $\lambda_e^+ + f_e^+(x) > 1$. If f_E is not soft quasi-coincident with f_E , we write $f_E \neg \tilde{q} g_E$.

Definition 15. The IVFS set f_E is called soft quasi-coincident with IVFS g_E , denoted by $f_E \tilde{q} g_E$, if there exists $e \in E$ such that $f_e^-(x) + g_e^-(x) > 1$ and $f_e^+(x) + g_e^+(x) > 1$.

Proposition 5. Let \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ and f_E, g_E two IVFS sets. Then:

- (i) $f_E \tilde{\leq} g_E \Leftrightarrow f_E \neg \tilde{q} g_E^c$,
- (ii) $\tilde{x}_E \tilde{\in} f_E \Leftrightarrow \tilde{x}_E \neg \tilde{q} f_E^c$.

Proof. We just prove Part (i). A similar technique is used to show Part (ii). For two IVFS sets f_E, g_E , we have:

$$\begin{aligned} f_E \tilde{\leq} g_E &\Leftrightarrow \forall e \in E : [f_e^-(x), f_e^+(x)] \leq [g_e^-(x), g_e^+(x)], \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) \leq g_e^-(x) \text{ and } f_e^+(x) \leq g_e^+(x), \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) + 1 - g_e^-(x) \leq 1 \text{ and } f_e^+(x) + 1 - g_e^+(x) \leq 1, \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) + g_e^{-c}(x) \leq 1 \text{ and } f_e^+(x) + g_e^{+c}(x) \leq 1, \forall x \in X \\ &\Leftrightarrow f_E \neg \tilde{q} g_E^c. \end{aligned}$$

□

Proposition 6. Let $\{f_{jE} : j \in J\}$ be a family of IVFS sets over X and \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ . If $\tilde{x}_E \tilde{q} (\bigwedge f_{jE})$, then $\tilde{x}_E \tilde{q} f_{jE}$ for each $j \in J$.

Proof. Let $\tilde{x}_E \tilde{q} (\bigwedge f_{jE})$. Then, $\lambda_e^- \tilde{q} (\bigwedge_j f_{je}^-)(x)$, $\lambda_e^+ \tilde{q} (\bigwedge_j f_{je}^+)(x)$ for $e \in E$, and $x \in X$. This implies that $\lambda_e^- > 1 - \bigwedge_j (f_{je}^-)(x)$ and $\lambda_e^+ > 1 - \bigwedge_j (f_{je}^+)(x)$, $x \in X$. Since $\bigwedge_j f_{je}^-(x) \leq f_{je}^-(x)$ and $\bigwedge_j f_{je}^+(x) \leq f_{je}^+(x)$, then $\lambda_e^- > 1 - \bigwedge_j (f_{je}^-)(x) > 1 - f_{je}^-(x)$ for each $e \in E, x \in X$ and $\lambda_e^+ > 1 - \bigwedge_j (f_{je}^+)(x) > 1 - f_{je}^+(x)$ for each $e \in E, x \in X$. Hence, $\lambda_e^- > 1 - f_{je}^-(x)$ and $\lambda_e^+ > 1 - f_{je}^+(x)$. Therefore, $[\lambda_e^-, \lambda_e^+] > [1, 1] - [f_{je}^-(x), f_{je}^+(x)]$ implies that $\tilde{x}_E > 1 - f_{je}^-$ and $\tilde{x}_E \tilde{q} f_{jE}$ for each $j \in J$. □

Remark 4. $\tilde{x}_E \tilde{q} (f_E \vee g_E)$ does not imply $\tilde{x}_E \tilde{q} f_E$ or $\tilde{x}_E \tilde{q} g_E$. This is shown in the following example.

Example 6. Let us consider Example 4 where $\tilde{x}_E \tilde{q} (f_E \vee g_E)$, but $\tilde{x}_E \neg \tilde{q} f_E$ and $\tilde{x}_E \neg \tilde{q} g_E$.

Theorem 4. Let \tilde{x}_E be an IVFS point \tilde{x}_E with support x , e lower value λ_e^- , and e upper value λ_e^+ and f_E, g_E be IVFS sets over X . If $\tilde{x}_E \tilde{q} (f_E \vee g_E)$, then there exists $\tilde{x}_{1E} \tilde{q} f_E$ and $\tilde{x}_{2E} \tilde{q} g_E$ such that $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.

The proof is very similar to the proof of Theorem 2.

Definition 16. Let (X, E, τ) be an IVFSTS and \tilde{x}_E be an IVFS point with support x , e lower values λ_e^- , and e upper values λ_e^+ . The IVFS set g_E is called a quasi-soft neighborhood (QIVFSN) of IVFS point \tilde{x}_E if there exists the IVFS-open set f_E in X such that $\tilde{x}_E \tilde{q} f_E \tilde{\leq} g_E$. Thus, the IVFS-open set f_E is a QIVFSN of the IVFS point \tilde{x}_E if and only if $\exists e \in E, x \in X$ such that $\lambda_e^- + f_e^-(x) > 1$ and $\lambda_e^+ + f_e^+(x) > 1$.

Remark 5. A quasi-coincident soft neighborhood of an IVFS point generally does not contain the point itself. This is shown by the following:

Example 7. Let $X = [0, 1]$ and E be any subset of X . Consider two IVFS sets f_E, g_E over X by the mapping $f : E \rightarrow \mathcal{IVF}([0, 1])$ and $g : E \rightarrow \mathcal{IVF}([0, 1])$ such that for any $e \in E, x \in X$:

$$\tilde{f}_e(x) = \begin{cases} [0.4, 0.5] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and:

$$\tilde{g}_e(x) = \begin{cases} [0.6, 0.7] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and \tilde{x}_E be any IVFS point defined by:

$$\tilde{x}_e(c) = \begin{cases} [0.4, 0.5] & c = x \\ 0 & c \neq x. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly, τ is an IVFST over X . Since $f_E \tilde{\leq} g_E$ and $\tilde{x} \tilde{q} f_E$, thus g_E is a QIVFSN of \tilde{x}_E . However, $\tilde{x}_E \notin g_E$.

Proposition 7.

- (1) If $f_E \tilde{\leq} g_E$ and f_E is a QINVSN of \tilde{x}_E , then g_E is also a QINVSN of \tilde{x}_E ,
- (2) If f_E, g_E are QINVSN of \tilde{x}_E , then $f_E \tilde{\wedge} g_E$ is also a QINVSN of \tilde{x}_E .
- (3) If f_E is a QINVSN of \tilde{x}_{1E} and g_E is a QINVSN of \tilde{x}_{2E} , then $f_E \tilde{\vee} g_E$ is also a QINVSN of $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.
- (4) If f_E is a QINVSN of \tilde{x}_E , then there exists g_E that is a QINVSN of \tilde{x}_E , such that $g_E \tilde{\leq} f_E$, and g_E is a QINVSN of $y_E, \forall y_E \tilde{q} g_E$.

Proof. (1) and (2) are straightforward.

- (3) Let f_E be a QINVSN of \tilde{x}_{1E} and g_E be a QINVSN of \tilde{x}_{2E} . Then, there exists an IVFS-open set h_{1E} in X such that $\tilde{x}_{1E} \tilde{q} h_{1E} \tilde{\leq} f_E$ and g_E is a QINVSN of \tilde{x}_{2E} . Thus, there exists an IVFS-open set h_{2E} in X such that $\tilde{x}_{2E} \tilde{q} h_{2E} \tilde{\leq} g_E$. Since $\tilde{x}_{1E} \tilde{q} h_{1E}$ for each $e \in E, x \in X, \lambda_{1e}^- + h_{1e}^- > 1, \lambda_{1e}^+ + h_{1e}^+ > 1$, this implies that $\lambda_{1e}^- > 1 - h_{1e}^-, \lambda_{1e}^+ > 1 - h_{1e}^+$ for each $e \in E$. Since $\tilde{x}_{2E} \tilde{q} h_{2E}$, for each $e \in E, \lambda_{2e}^- + h_{2e}^- > 1, \lambda_{2e}^+ + h_{2e}^+ > 1$, this implies that $\lambda_{2e}^- > 1 - h_{2e}^-, \lambda_{2e}^+ > 1 - h_{2e}^+$ for each $e \in E, x \in X$. From here,

$$\max(\lambda_{1e}^-, \lambda_{2e}^-) > \max(1 - h_{1e}^-(x), (1 - h_{2e}^-(x)), \max(\lambda_{1e}^+, \lambda_{2e}^+) > \max(1 - h_{1e}^+(x), (1 - h_{2e}^+(x)).$$

Therefore, $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E} \tilde{q} (h_{1E} \tilde{\vee} h_{2E}) \tilde{\leq} f_E \tilde{\vee} g_E$. Consequently, $f_E \tilde{\vee} g_E$ is a QINVSN of $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.

- (4) Let f_E be a QINVSN of \tilde{x}_E . Then, there exists g_E that is a QINVSN of \tilde{x}_E such that $\tilde{x}_E \tilde{q} g_E \tilde{\leq} f_E$. Consider the $g_E = h_E$. Indeed, since $\tilde{x}_E \tilde{q} h_E$ and h_E is an IVFS-open set, h_E is a QINVSN of \tilde{x}_E . Thus, we obtain h_E that is a QINVSN of \tilde{y}_E .

□

Theorem 5. In IVFST(X, E, τ), the IVFS point \tilde{x}_E belongs to $Cl f_E$ if and only if each QIVFS of \tilde{x}_E is soft quasi-coincident with f_E .

Proof. Let IVFS point \tilde{x}_E with support x , e lower value λ_e^- , and e upper value λ_e^+ belong to $Cl f_E$, i.e., $\tilde{x}_E \tilde{\in} Cl f_E$. For any IVFS-closed g_E containing f_E , $\tilde{x}_E \tilde{\in} g_E$, which implies that $\lambda_e^- \leq g_e^-(x)$ and $\lambda_e^+ \leq g_e^+(x)$, for all $x \in X, e \in E$. Consider h_E to be an QIVFN of the IVFS point \tilde{x}_E and $h_E \neg \tilde{q} f_E$. Then, for any $e \in E$ and $x \in X, h_e^-(x) + f_e^-(x) \leq 1, h_e^+(x) + f_e^+(x) \leq 1$, and so, $f_E \tilde{\leq} h_E^c$. Since h_E is a QIVFSN of the IVFS point \tilde{x}_E , by \tilde{x}_E , it does not belong to h_E^c . Therefore, we have that \tilde{x}_E does not belong to $Cl f_E$. This is a contradiction.

Conversely, let any QIVFSN of the IVFS point \tilde{x}_E be soft quasi-coincident with f_E . Consider that \tilde{x}_E does not belong to Clf_E , i.e., $\tilde{x}_E \notin Clf_E$. Then, there exists an IVFS-closed set g_E , which contains f_E such that \tilde{x}_E does not belong to g_E . We have $\tilde{x}_E \tilde{q} g_E^c$. Then, g_E^c is an QIVFSN of the IVFS point \tilde{x}_E and $f_E \neg \tilde{q} g_E^c$. This is a contradiction with the hypothesis. \square

5. IVFS Quasi-Separation Axioms

In this section, we develop the separation axioms to IVFST, so-called IVFSQ separation axioms (IVFSq- T_i axioms) for $i = 0, 1, 2, 3, 4$, and consider some of their properties.

Definition 17. Let (X, E, τ) be an IVFST space. Let \tilde{x}_E and \tilde{y}_E be IVFS points over X , where:

$$\tilde{x}(e)(z) = \begin{cases} [\lambda_e^-, \lambda_e^+] & z = x \\ 0 & \text{otherwise} \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} [\gamma_e^-, \gamma_e^+] & z = y \\ 0 & \text{otherwise}, \end{cases}$$

then \tilde{x}_E and \tilde{y}_E are said to be distinct if and only if $\tilde{x}_E \tilde{\wedge} \tilde{y}_E = \emptyset_E$, which means $x \neq y$.

Definition 18. Let (X, E, τ) be an IVFST space. The IVFS point \tilde{x}_E is called a crisp IVFS point $x_E^{[1,1]}$, if $\lambda_e^- = \lambda_e^+ = 1$ for all $e \in E$.

Definition 19. Let (X, E, τ) be an IVFST space and \tilde{x}_E and \tilde{y}_E be two IVFS points. If there exists IVFS open sets f_E and g_E such that:

- when \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+ , respectively, and f_E is an IVFSN of the IVFS point \tilde{x}_E and $\tilde{y}_E \neg \tilde{q} f_E$ or g_E is an IVFSN of the IVFS point \tilde{y}_E and $\tilde{x}_E \neg \tilde{q} g_E$,
- when \tilde{x}_E and \tilde{y}_E are two IVFS points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$ and f_E is a QIVFSN of the IVFS point \tilde{y}_E such that $\tilde{x}_E \neg \tilde{q} f_E$,

then (X, E, τ) is an interval-valued fuzzy soft quasi- T_0 space (IVFSq- T_0 space).

Example 8. Consider the IVFS set defined in Example 3.1 and \tilde{x}_E, \tilde{y}_E to be any two distinct IVFS points in X defined by:

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{if } z \neq y. \end{cases}$$

Then, f_E is an IVFSN of \tilde{x}_E and $\tilde{y}_E \neg \tilde{q} f_E$. Thus, X is an IVFSq- T_0 space.

Theorem 6. (X, E, τ) is an IVFSq- T_0 space if and only if for every two IVFS points \tilde{x}_E, \tilde{y}_E and $\tilde{x}_E \notin Cl\tilde{y}_E$ or $\tilde{y}_E \notin Cl\tilde{x}_E$.

Proof. Let (X, E, τ) be an IVFSq- T_0 space and \tilde{x}_E and \tilde{y}_E be two IVFS points in X .

First consider that \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively. Then, a crisp IVFS point $\tilde{x}_E^{[1,1]}$ has an IVFSN f_E such that $\tilde{y}_E \neg \tilde{q} f_E$ or a crisp IVFS point $\tilde{y}_E^{[1,1]}$ has an IVFSN g_E such that $\tilde{x}_E \neg \tilde{q} g_E$. Consider that the crisp IVFS point $\tilde{x}_E^{[1,1]}$ has an IVFSN f_E such that $\tilde{y}_E \neg \tilde{q} f_E$. Moreover, f_E is an QIVFSN of \tilde{x}_E

and $\tilde{y}_E \neg \tilde{q} f_E$. Hence, $\tilde{x}_E \notin Cl \tilde{y}_E$. Next, we consider the case \tilde{x}_E and \tilde{y}_E to be two IVFS points with the same supports $x = y$, e lower value $\lambda_e^- < \gamma_e^-$, and e upper value $\lambda_e^+ < \gamma_e^+$. Then, \tilde{y}_E has a QIVFSN that is not quasi-coincident with \tilde{x}_E , and so, by Theorem 5, $\tilde{x}_E \notin Cl \tilde{y}_E$.

Conversely, let \tilde{x}_E and \tilde{y}_E be two IVFS points in X . Consider without loss of generality that $\tilde{x}_E \notin Cl \tilde{y}_E$. First, consider that \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively, since $\tilde{x}_E \notin Cl \tilde{y}_E$ for any $e \in E$, $f_e^-(y) = f_e^+(y) = 0$ and $f_e^-(x) = f_e^+(x) = 1$. Then, $Cl(\tilde{y}_E)^c$ is an IVFSN of \tilde{x}_E such that $Cl(\tilde{y}_E)^c \neg \tilde{q} \tilde{y}_E$. Next, let \tilde{x}_E and \tilde{y}_E be two IVFS points with the same supports $x = y$, and we must have e lower value $\lambda_e^- > \gamma_e^-$ and e upper value $\lambda_e^+ > \gamma_e^+$, then \tilde{x}_E has a QIVFSN that is not quasi-coincident with \tilde{y}_E . \square

Definition 20. Let (X, E, τ) be an IVFST and \tilde{x}_E and \tilde{y}_E be two IVFS points, if there exists IVFS open sets f_E and g_E such that:

- (a) when \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively, f_E is an IVFSN of IVFS points \tilde{x}_E and $\tilde{y}_E \neg \tilde{q} f_E$, and g_E is an IVFSN of IVFS points \tilde{y}_E and $\tilde{x}_E \neg \tilde{q} g_E$,
- (b) when \tilde{x}_E and \tilde{y}_E are two IVFS points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$, f_E is an QIVFSN of the IVFS point \tilde{y}_E such that $\tilde{x}_E \neg \tilde{q} f_E$,

then (X, E, τ) is an interval-valued fuzzy soft quasi- T_1 space (IVFSq- T_1 space).

Theorem 7. (X, E, τ) is an IVFSq- T_1 space if and only if any IVFS point \tilde{x}_E in X is an IVFS-closed set.

Proof. Suppose that each IVFS point \tilde{x}_E in X is an IVFS-closed set, i.e., $g_E = \tilde{x}_E^c$. Then, g_E is an IVFS-open set. Let \tilde{x}_E and \tilde{y}_E be two IVFS points as follows: First, consider that \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively. Then, g_E is an IVFS-open set such that g_E is an IVFSN of IVFS point \tilde{y}_E and $\tilde{x}_E \neg \tilde{q} g_E$. Similarly, $f_E = \tilde{y}_E^c$ is an IVFS-open set and f_E is an IVFSN of the IVFS points \tilde{x}_E and $\tilde{y}_E \neg \tilde{q} f_E$. Next, we consider the case \tilde{x}_E and \tilde{y}_E to be two IVFS points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$. Then, \tilde{y}_E has a QIVFSN g_E , which is not quasi-coincident with \tilde{x}_E . Thus, X is an IVFSq- T_1 space.

Conversely, Let (X, E, τ) be an IVFSq- T_1 space. Suppose that any IVFS point \tilde{x}_E is not an IVFS-closed set in X , i.e., $f_E \neq \tilde{x}_E^c$. Then, $\tilde{f}_E \neq Cl \tilde{f}_E$, and there exists $\tilde{y}_E \in Cl \tilde{f}_E$ such that $\tilde{x}_E \neq \tilde{y}_E$.

First, consider that \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively. Suppose that e lower value $\lambda_e^- \leq 0.5$ and e upper value $\lambda_e^+ \leq 0.5$. Since $\tilde{y}_E \in Cl f_E$, by Theorem 4.1, any f_E is a QIVFSN of \tilde{y}_E and $\tilde{x}_E \neg \tilde{q} f_E$. Then, there exists IVFS-open set h_E such that $\tilde{y}_E \tilde{q} h_E \leq f_E$. Hence, $h_e^-(y) + \gamma_e^- > 1$. Next, let \tilde{x}_E and \tilde{y}_E be two IVFS points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$. Since $\tilde{y}_E \in Cl x_E$, by Theorem 5, each f_E is a QIVFSN of IVFS points \tilde{y}_E , $\tilde{x}_E \neg \tilde{q} f_E$. This is a contradiction. \square

Definition 21. Let (X, E, τ) be an IVFST and \tilde{x}_E and \tilde{y}_E be two IVFS points, if there exists IVFS open sets f_E and g_E such that:

- (a) when \tilde{x}_E and \tilde{y}_E are two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively, f_E is an IVFSN of the IVFS point \tilde{x}_E and g_E is an IVFSN of the IVFS point \tilde{y}_E , such that $f_E \neg \tilde{q} g_E$,
- (b) when \tilde{x}_E and \tilde{y}_E are two IVFS points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$, f_E is an IVFSN of IVFS point \tilde{x}_E and g_E is a QIVFSN of IVFS point \tilde{y}_E ,

then (X, E, τ) is an interval-valued fuzzy soft quasi- T_2 space (IVFS q- T_2 space).

Example 9. Suppose that $X = [0, 1]$ and E are any proper ($E \subset X$). Consider IVFS sets f_E and g_E over X defined as below: $f : E \rightarrow \mathcal{IVF}([0, 1])$ and $g : E \rightarrow \mathcal{IVF}([0, 1])$, such that for any $e \in E, x \in X$:

$$f(e)(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1 \end{cases}$$

and:

$$g(e)(x) = \begin{cases} 0 & 0 \leq x \leq e \\ 1 & e \leq x \leq 1. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly, τ is an IVFST over X . Therefore, for any two absolute distinct IVFS points \tilde{x}_E, \tilde{y}_E in X defined by:

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{if } z \neq y. \end{cases}$$

Then, f_E is an IVFSN of \tilde{x}_E , and g_E is an IVFSN of \tilde{y}_E , such that $f_E \neg \tilde{q} g_E$. Then, X is an IVFS q - T_2 space.

Theorem 8. IVFST(X, E, τ) is an IVFS q - T_2 space if and only if for any $x \in X$, we have $\tilde{x}_E = \tilde{\bigwedge}\{Cl f_E : f_E \in \text{IVFSN of } \tilde{x}_E\}$.

Proof. Let (X, E, τ) be a crisp IVFS q - T_2 space and \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value γ_e^+ . Let \tilde{y}_E be a crisp IVFS point with support y , e lower value γ_e^- , and e upper value λ_e^+ . If \tilde{x}_E and \tilde{y}_E are two IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+ , respectively, then there exist two IVFS-open sets f_E and g_E containing IVFS points \tilde{y}_E and \tilde{x}_E , respectively, such that $f_E \neg \tilde{q} g_E$. Then, g_E is an IVFSN of IVFS point \tilde{x}_E and f_E is a QIVFSN of \tilde{y}_E such that $f_E \neg \tilde{q} g_E$. Hence, $\tilde{y}_E \notin Cl g_E$. If \tilde{x}_E and \tilde{y}_E are two IVFS points with the same supports $x = y$, then $\gamma_e^- > \lambda_e^-$ and $\gamma_e^- > \lambda_e^+$. Thus, there are QIVFSN f_E of IVFS points \tilde{y}_E and IVFSN g_E such that $f_E \neg \tilde{q} g_E$. Hence, $\tilde{y}_E \notin Cl g_E$.

Conversely, let \tilde{x}_E and \tilde{y}_E be two distinct IVFS points with different supports x and y , e lower values, and e upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+ , respectively. Since:

$$\tilde{x}_E = \tilde{\bigwedge}\{Cl f_E : f_E \in \text{IVFSN of } \tilde{x}_E\}, \text{ and } \tilde{\bigwedge}\{Cl([f_e^-, f_e^+])(y) : f_E \in \text{IVFSN of } \tilde{x}_E\} = 0.$$

Thus, $\tilde{y}_E \neg \tilde{q} \tilde{\bigwedge}\{Cl f_E : f_E \in \text{IVFSN of } \tilde{x}_E\}$. Therefore, there exists f_E that is an IVFSN of \tilde{x} and $\tilde{y}_E \neg \tilde{q} Cl f_E$. Take two τ -IVFS-open sets f_E and $(Cl f_E)^c$. Therefore, f_E is an IVFSN of IVFS point \tilde{x}_E , $(Cl f_E)^c$ an IVFSN of IVFS point \tilde{y}_E , and $f_E \neg \tilde{q} (Cl f_E)^c$. \square

Definition 22. Let (X, E, τ) be an IVFST. If for any IVFS point \tilde{x}_E with support x , e lower values λ_e^- , and e upper values λ_e^+ and any IVFS-closed set f_E in X such that $\tilde{x}_E \neg \tilde{q} f_E$, there exists two IVFS-open sets h_E and k_E such that $\tilde{x}_E \tilde{\in} h_E$ and $f_E \tilde{\leq} k_E, h_E \neg \tilde{q} k_E$, then (X, E, τ) is called an interval-valued fuzzy soft quasi regular space (IVFS q -regular space).

(X, E, τ) is called an interval-valued fuzzy soft quasi- T_3 space, if it is an IVFS q -regular space and an IVFS q - T_1 space.

Theorem 9. IVFST(X, E, τ) is an IVFS q - T_3 space if and only if for any IVFSN g_E of IVFS point \tilde{x}_E there exists an IVFS-open set f_E in X such that $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} Cl f_E \tilde{\leq} g_E$.

Proof. Let g_E be an IVFS set in X and \tilde{x}_E be an IVFS point with support x , e lower value λ_e^- , and e upper value λ_e^+ such that $\tilde{x}_E \in g_E$. Then, clearly, g_E^c is an IVFS-closed set. Since X is an IVFS q - T_3 space, there exist two IVFS-open sets f_E, h_E such that $\tilde{x}_E \in f_E, g_E^c \leq h_E, h_E$ and $f_E \neg \tilde{q} h_E$. Thus, $f_E^c \leq h_E^c$. Therefore, $Cl f_E \leq h_E^c$ implies $Cl f_E \leq g_E$. Hence, $\tilde{x}_E \in f_E \leq Cl f_E \leq g_E$.

Conversely, let \tilde{x}_E be an IVFS point with different support x , e lower value λ_e^- , and e upper value λ_e^+ , and let g_E be an IVFS-closed set such that $\tilde{x}_E \neg \tilde{q} g_E$. Then, g_E^c is an IVFS-open set containing the IVFS point \tilde{x}_E , i.e., $\tilde{x}_E \in g_E^c$. Thus, there exists an IVFS-open set f_E containing \tilde{x}_E such that $\tilde{x}_E \in f_E \leq Cl f_E \leq g_E, g_E \leq (Cl f_E)^c$. Therefore, clearly, $(Cl f_E)^c$ is an IVFS-open set containing g_E and $f_E \neg \tilde{q} (Cl f_E)^c$. Hence, X is an IVFS q - T_3 space. \square

Definition 23. Let (X, E, τ) be an IVFST. If for any two IVFS-closed sets f_E and g_E such that $f_E \neg \tilde{q} g_E$, there exists two IVFS-open sets h_E and k_E such that $f_E \leq h_E$ and $g_E \leq k_E$, then (X, E, τ) is called an interval-valued fuzzy soft quasi-normal space (IVFS q -normal space).

(X, E, τ) is called an interval-valued fuzzy soft quasi T_4 space if it is an IVFS q -normal space and an IVFS q - T_1 space.

Theorem 10. IVFST (X, E, τ) is an IVFS q - T_4 space if and only if for any IVFS-closed set f_E and IVFS-open set containing f_E , there exists an IVFS-open set h_E in X such that $f_E \leq h_E \leq Cl h_E \leq g_E$.

Proof. Let f_E be an IVFS-closed set in X and g_E be an IVFS-open set in X containing f_E , i.e., $f_E \leq g_E$. Then, g_E^c is an IVFS-closed set such that $f_E \neg \tilde{q} g_E^c$.

Since X is an IVFS q - T_4 space, there exist two IVFS-open sets h_E, k_E such that $f_E \leq h_E, g_E^c \leq k_E$, and $h_E \neg \tilde{q} k_E$. Thus, $h_E \leq k_E^c$, but $Cl h_E \leq Cl k_E^c = k_E$. Furthermore, $g_E^c \leq k_E$ implies $k_E^c \leq g_E$. That is an IVFS-closed set over X . Therefore, $Cl h_E \leq k_E$. Hence, we have $f_E \leq h_E \leq Cl h_E \leq g_E$.

Conversely, let \tilde{f}_E and g_E be any IVFS-closed sets such that $\tilde{f}_E \neg \tilde{q} g_E$. Then, $\tilde{f}_E \leq g_E^c$. Thus, there exists an IVFS-open set h_E such that $\tilde{f}_E \leq h_E \leq Cl h_E \leq g_E$. Therefore, there are two IVFS-open sets h_E and $(Cl h_E)^c$ such that $\tilde{f}_E \leq h_E, g_E \leq (Cl h_E)^c$. This shows that X is an IVFS q - T_4 space. \square

Theorem 11. If $\Phi : (X_1, E_1, \tau_1) \rightarrow (X_2, E_2, \tau_2)$ is an IVFSC and IVFSO map where $\Phi_u X_1 \rightarrow X_2$ and $\Phi_p E_1 \rightarrow E_2$ are two ordinary bijections, then X_1 is an IVFS q - T_i space if and only if X_2 is an IVFS q - T_i space for $i = 0, 1, 2, 3, 4$.

Proof. We just prove when $i = 2$. The other parts are similar.

Suppose that we have two IVFS points \tilde{k}_{E_2} and \tilde{s}_{E_2} with different supports k and s , e lowers value, and e upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+ , respectively, for any $e \in E_2$. Then, the inverse lower and upper image of IVFS point \tilde{k}_{E_2} under the IVFSO map Φ is an IVFS point in X_1 with different support $\Phi^{-1}(k)$ as below:

$$\Phi^{-1}(\tilde{k}^-)(e)(x) = \tilde{k}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{k}^+)(e)(x) = \tilde{k}^+(\Phi_p(e))(\Phi_u(x)).$$

Furthermore, the inverse lower and upper image of IVFS point \tilde{s}_{E_2} under the IVFSO map Φ is an IVFS point in X_1 with different support $\Phi^{-1}(s)$ as below:

$$\Phi^{-1}(\tilde{s}^-)(e)(x) = \tilde{s}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{s}^+)(e)(x) = \tilde{s}^+(\Phi_p(e))(\Phi_u(x)).$$

Since (X_1, E_1, τ_1) is an IVFS q - T_2 space, there exist two IVFS-open sets f_E and g_E in X_1 such that $\Phi^{-1}(\tilde{k}_{E_2}) \in f_E, \Phi^{-1}(\tilde{s}_{E_2}) \in g_E$, and $f_E \neg \tilde{q} g_E$. Thus, $\tilde{k}_{E_2} \in f_E$ and $\tilde{s}_{E_2} \in g_E$, while $\Phi(f_E) \neg \tilde{q} \Phi(g_E)$. Therefore, (X_2, E_2, τ_2) is an IVFS q - T_2 space.

Conversely, suppose that we have two IVFS points \tilde{x}_E and \tilde{y}_E with different supports $x, y \in X_1$, e lower value, and e upper value λ_e^-, λ_e^+ and γ_e^-, γ_e^+ , respectively. Then, the lower and upper image

of an *IVFS* point \tilde{x}_E under the *IVFSC* map Φ is an *IVFS* point in X_2 with different support $\Phi_u(x)$ as below:

$$\begin{aligned}\Phi(\tilde{x}^-)(\varepsilon)(k) &= \sup_{z \in \Phi^{-1}(k)} [\sup_{e \in \Phi_p^{-1}(\varepsilon)} (\tilde{x}^-)(e)](z) \\ &= \begin{cases} \lambda_e^- & \text{if } k = \Phi_u(x) \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

and:

$$\begin{aligned}\Phi(\tilde{x}^+)(\varepsilon)(k) &= \sup_{z \in \Phi^{-1}(k)} [\sup_{e \in \Phi_p^{-1}(\varepsilon)} (\tilde{x}^+)(e)](z) \\ &= \begin{cases} \lambda_e^+ & \text{if } k = \Phi_u(x) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and the lower and upper image of an *IVFS* point \tilde{y}_E under the *IVFSC* map Φ is an *IVFS* point in X_2 with different support $\Phi_u(y)$ as below:

$$\begin{aligned}\Phi(\tilde{y}^-)(\varepsilon)(k) &= \sup_{z \in \Phi^{-1}(k)} [\sup_{e \in \Phi_p^{-1}(\varepsilon)} (\tilde{y}^-)(e)](z) \\ &= \begin{cases} \gamma_e^- & \text{if } k = \Phi_u(y) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and:

$$\begin{aligned}\Phi(\tilde{y}^+)(\varepsilon)(k) &= \sup_{z \in \Phi^{-1}(k)} [\sup_{e \in \Phi_p^{-1}(\varepsilon)} (\tilde{y}^+)(e)](z) \\ &= \begin{cases} \gamma_e^+ & \text{if } k = \Phi_u(y) \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Since (X_2, E_2, τ_2) is an *IVFSq-T₂* space, there exist two *IVFS*-open sets f_{E_2} and g_{E_2} in X_2 such that $\Phi(\tilde{x}) \tilde{\in} f_{E_2}$, $\Phi(\tilde{y}) \tilde{\in} g_{E_2}$, and $f_{E_2} \neg \tilde{q} g_{E_2}$. Clearly, $\tilde{x}_E \tilde{\in} \Phi^{-1}(f_{E_2})$, $\tilde{y}_E \tilde{\in} \Phi^{-1}(g_{E_2})$ and $\Phi^{-1}(f_{E_2}) \neg \tilde{q} \Phi^{-1}(g_{E_2})$. Then, (X_1, E_1, τ_1) is an *IVFSq-T₂* space. \square

6. Conclusions

The aim of this study was to develop the interval-valued fuzzy soft separation axioms in order to build a framework that will provide a method for object ranking. Thus, in this paper, we introduced a new definition of the interval-valued fuzzy soft point and then considered some of its properties, and different types of neighborhoods of the *IVFS* point were studied in interval-valued fuzzy soft topological spaces. The separation axioms of interval-valued fuzzy soft topological spaces were presented, and furthermore, the basic properties were also studied.

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