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Gamma-Bazilevič Functions

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Abstract: For $\gamma \geq 0$ and $\alpha \geq 0$, we introduce the class $\mathcal{B}_1^{\gamma}(\alpha)$ of Gamma–Bazilevič functions defined for $z \in \mathbb{D}$ by $Re\left\{\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}} + \frac{zf''(z)}{f'(z)} + (\alpha-1)\left(\frac{zf'(z)}{f(z)} - 1\right)\right]^{\gamma}\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}\right]^{1-\gamma}\right\} > 0$. We shown that $\mathcal{B}_1^{\gamma}(\alpha)$ is a subset of $\mathcal{B}_1(\alpha)$, the class of $B_1(\alpha)$ Bazilevič functions, and is therefore univalent in $\mathbb{D}$. Various coefficient problems for functions in $\mathcal{B}_1^{\gamma}(\alpha)$ are also given.

Keywords: univalent; Bazilevič functions; coefficients

1. Introduction and Definitions

Denote by $\mathcal{A}$ the class of normalized analytic functions f, defined in the unit disk $\mathbb{D}$, and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

and by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}$.

A function $f \in \mathcal{S}$ is said to be convex if f maps $\mathbb{D}$ onto a convex set, and starlike if f maps $\mathbb{D}$ onto a set star-shaped with respect to the origin. Let $\mathcal{C}$ and $\mathcal{S}^*$ denote the classes of convex and starlike functions in $\mathcal{S}$ respectively. Then $f \in \mathcal{C}$ if and only if $Re\,(1 + (zf''(z)/f'(z))) > 0$ for $z \in \mathbb{D}$. Similarly, $f \in \mathcal{S}^*$ if and only if $Re\,(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$.

For $\alpha \in \mathbb{R}$, the class $\mathcal{M}_{\alpha}$ of α-convex functions defined by,

$$Re\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\right\} > 0,$$

for $z \in \mathbb{D}$ and $\frac{f(z)}{z}f'(z) \neq 0$ is well known. Introduced by Miller, Mocanu and Reade [1], many interesting properties for functions in $\mathcal{M}_{\alpha}$ have been found (See e.g., [2,3]).

Denote by $\mathcal{M}^{\gamma}$ the analogue of $\mathcal{M}_{\alpha}$ in term of powers, defined for $\gamma \in \mathbb{R}$ by

$$Re\left\{\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\right\} > 0,$$

for $z \in \mathbb{D}$. The class $\mathcal{M}^{\gamma}$ was introduced in [4], and many interesting properties of functions in $\mathcal{M}^{\gamma}$ have been found. It was shown in [4] that $\mathcal{M}^{\gamma}$ is a subset of $\mathcal{S}^*$. Further, sharp bounds for $|a_2| and |a_3|$ were obtained, together with the sharp Fekete–Szegö theorem. Other result can be found in [5,6].

The purpose of this paper is to introduce an analogue of $\mathcal{M}^{\gamma}$ for Bazilevič functions. We first recall the Bazilevič functions $\mathcal{B}_1(\alpha)$ introduced by Singh in 1973, which form a natural subset of $\mathcal{S}$ as follows [7].

Definition 1. *Let $f \in \mathcal{A}$. Then for $\alpha \geq 0$, $f \in \mathcal{B}_1(\alpha)$ if, and only if, for $z \in \mathbb{D}$,*

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] > 0.$$

We next introduce the Gamma–Bazilevič functions as follows, noting that we restrict our definition $\gamma \geq 0$ merely for convenience.

Definition 2. *Let $f \in \mathcal{A}$, with $f(z) \neq 0$ and $f'(z) \neq 0$. For $\gamma \geq 0$ and $\alpha \geq 0$, a function $f \in \mathcal{A}$ is said to be Gamma–Bazilevič if, for $z \in \mathbb{D}$,*

$$Re\left\{\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}} + \frac{zf''(z)}{f'(z)} + (\alpha-1)\left(\frac{zf'(z)}{f(z)} - 1\right)\right]^{\gamma}\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}\right]^{1-\gamma}\right\} > 0.$$

We denote this class by $\mathcal{B}_1^{\gamma}(\alpha)$.

Clearly $\mathcal{B}_1^{0}(\alpha) = \mathcal{B}_1(\alpha)$, and $\mathcal{B}_1^{\gamma}(0) = \mathcal{M}^{\gamma}$. We also note that when $\alpha = 1$ and $\gamma = 0$, we obtain the class $\mathcal{R}$ of functions whose derivative has a positive real part, and that when $\alpha = 0$ and $\gamma = 0$ we obtain the starlike functions, and when $\alpha = 0$ and $\gamma = 1$ we obtain the convex functions.

We also note that when $\gamma = 1$, we obtain the following new class $\mathcal{B}_1^{\gamma}(1)$, which forms a subset of $\mathcal{R}$.

$$Re\left\{f'(z) + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

2. Preliminaries

We begin by stating two Lemmas which we will use in what follows.

Lemma 1 (Nunokawa, [8]). *Let p be analytic in $\mathbb{D}$, with $p(z) \neq 0$ and $p(0) = 1$. If there exists $z_0 \in \mathbb{D}$, such that $|\arg p(z_0)| < \dfrac{\alpha\pi}{2}$ for $|z| < |z_0|$, and $|\arg p(z_0)| = \dfrac{\alpha\pi}{2}$ for some $\alpha > 0$, then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg p(z_0) = \alpha\frac{\pi}{2},$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg p(z_0) = -\alpha\frac{\pi}{2},$$

and where $p^{1/\alpha}(z_0) = \pm ia$ for $a > 0$.

Let $\mathcal{P}$ be the class of function h satisfying $Re\, h(z) > 0$ for $z \in \mathbb{D}$, with expansion

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2}$$

We shall use the following results concerning the coefficients c_n of $h \in \mathcal{P}$, which can be found in [9].

Lemma 2. *If $h \in \mathcal{P}$ and be given by (2), then $|c_n| \leq 2$ for $n \geq 1$, and*

$$\left|c_2 - \frac{\mu}{2}c_1^2\right| \leq \max\{2,\ 2|\mu-1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2, \\ 2|\mu-1|, & \textit{elsewhere}. \end{cases}$$

3. Gamma-Bazilevič Functions

We first show $\mathcal{B}_1^\gamma(\alpha) \subset \mathcal{B}_1(\alpha)$, so that functions in $\mathcal{B}_1^\gamma(\alpha)$ are univalent in $\mathbb{D}$.

Theorem 1. *Let $f \in \mathcal{A}$. Then for $\gamma \geq 0$ and $\alpha \geq 0$,*

$$Re\left\{\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha} + \frac{zf''(z)}{f'(z)} + (\alpha-1)\left(\frac{zf'(z)}{f(z)} - 1\right)\right]^\gamma \left[\frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha}\right]^{1-\gamma}\right\} > 0$$

implies

$$Re\left\{\frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha}\right\} > 0,$$

for $z \in \mathbb{D}$. Thus $\mathcal{B}_1^\gamma(\alpha) \subset \mathcal{B}_1(\alpha)$.

Proof. Let $p(z) = \dfrac{zf'(z)}{f(z)^{1-a}z^a}$, then

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha} + \frac{zf''(z)}{f'(z)} + (\alpha-1)\left(\frac{zf'(z)}{f(z)} - 1\right).$$

Now note that $p(z)$ is analytic in $\mathbb{D}$ with $p(z) \neq 0$ and $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathbb{D}$, such that $|\arg p(z_0)| < \dfrac{\pi}{2}$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \dfrac{\pi}{2}$. Then by Lemma 1,

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2},$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2},$$

and where $p(z_0) = \pm ia$ for $a > 0$.

There are two cases.

Case 1. If $\arg p(z_0) = \dfrac{\pi}{2}$, then

$$\begin{aligned}
&\arg\left\{\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)^\gamma p(z_0)^{1-\gamma}\right\} \\
&= \gamma \arg\left[p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right] + (1-\gamma)\arg p(z_0) \\
&= \gamma \arg(ia + ik) + (1-\gamma)\frac{\pi}{2} \\
&= \gamma\frac{\pi}{2} + (1-\gamma)\frac{\pi}{2} \\
&= \frac{\pi}{2},
\end{aligned}$$

where $p(z_0) = ia$ and $k \geq \dfrac{1}{2}\left(a + \dfrac{1}{a}\right)$.

Case 2. If $\arg p(z_0) = -\dfrac{\pi}{2}$, then

$$
\begin{aligned}
&\arg\left\{\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)^{\gamma}p(z_0)^{1-\gamma}\right\}\\
&=\gamma\arg\left[p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right]+(1-\gamma)\arg p(z_0)\\
&=\gamma\arg(-ia+ik)-(1-\gamma)\frac{\pi}{2}\\
&=-\gamma\frac{\pi}{2}-(1-\gamma)\frac{\pi}{2}\\
&=-\frac{\pi}{2},
\end{aligned}
$$

where $p(z_0) = -ia$ and $k \le -\frac{1}{2}\left(a+\frac{1}{a}\right)$. Therefore, we have a contradiction. There is thus no point $z_0 \in \mathbb{D}$ such that $|\arg p(z_0)| < \frac{\pi}{2}$ for $|z| < |z_0|$, and $|\arg p(z_0)| = \frac{\pi}{2}$. $\square$

4. Initial Coefficients

We first find expressions for a_2 and a_3 in terms of the coefficients of $h \in \mathcal{P}$.

It follows from Definition 2 that we can write,

$$
\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}+\frac{zf''(z)}{f'(z)}+(\alpha-1)\left(\frac{zf'(z)}{f(z)}-1\right)\right]^{\gamma}\left[\frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}}\right]^{1-\gamma}=h(z), \tag{3}
$$

where $h \in \mathcal{P}$.

Equating coefficients in (3) gives

$$
\begin{aligned}
a_2 &= \frac{c_1}{(1+\alpha)(1+\gamma)},\\
a_3 &= \frac{1}{(2+\alpha)(1+2\gamma)}\left(c_2-\frac{(\alpha^2\gamma^2-\alpha^2\gamma+\alpha^2+2\alpha\gamma^2-4\alpha\gamma+\alpha+\gamma^2-7\gamma-2)}{2(1+\alpha^2)(1+\gamma)^2}c_1^2\right).
\end{aligned} \tag{4}
$$

We now extend coefficient results given in [6] for the coefficients of $\mathcal{M}^{\gamma}$ and the results of Singh [7] for $\mathcal{B}_1(\alpha)$, noting that the bounds for $|a_2|$ and $|a_3|$ hold for all $\gamma \ge 0$ and $\alpha \ge 0$.

Theorem 2. *If $f \in \mathcal{B}_1^{\gamma}(\alpha)$ and is given by (1), then*

$$
|a_2| \le \frac{2}{(1+\alpha)(1+\gamma)},
$$

$$
|a_3| \le \frac{2}{(2+\alpha)(1+2\gamma)},
$$

when $0 \le \gamma \le \frac{1}{2}\left(7+\sqrt{57}\right)$ *and* $\alpha \ge \dfrac{-1+4\gamma-2\gamma^2}{2\left(1-\gamma+\gamma^2\right)}+\dfrac{1}{2}\sqrt{\dfrac{9+12\gamma-4\gamma^2+16\gamma^3}{\left(1-\gamma+\gamma^2\right)^2}}$,

and when $\gamma > \frac{1}{2}\left(7+\sqrt{57}\right)$.

Also

$$
|a_3| \le \frac{2(3+\alpha+9\gamma+8\alpha\gamma+3\alpha^2\gamma)}{(1+\alpha)^2(2+\alpha)(1+\gamma)^2(1+2\gamma)},
$$

when $0 \le \gamma < \frac{1}{2}\left(7+\sqrt{57}\right)$ *and* $0 \le \alpha < \dfrac{-1+4\gamma-2\gamma^2}{2\left(1-\gamma+\gamma^2\right)}+\dfrac{1}{2}\sqrt{\dfrac{9+12\gamma-4\gamma^2+16\gamma^3}{\left(1-\gamma+\gamma^2\right)^2}}$,

all the inequalities are sharp.

Proof. The first inequality in Theorem 2 follows at once from (4) since $|c_1| \leq 2$.

For $|a_3|$, from (4) we use Lemma 2, and write

$$|a_3| = \frac{1}{(2+\alpha)(1+2\gamma)} \left| c_2 - \frac{(\alpha^2\gamma^2 - \alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 - 4\alpha\gamma + \alpha + \gamma^2 - 7\gamma - 2)}{2(1+\alpha^2)(1+\gamma)^2} c_1^2 \right|.$$

Then in Lemma 2, let

$$\mu = \frac{(\alpha^2\gamma^2 - \alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 - 4\alpha\gamma + \alpha + \gamma^2 - 7\gamma - 2)}{(1+\alpha^2)(1+\gamma)^2},$$

so that applying Lemma 2 gives the inequalities for $|a_3|$.

The inequality for $|a_2|$ is sharp when $c_1 = 2$. The first inequality for $|a_3|$ is sharp when $c_1 = 0$ and $c_2 = 2$, and the second inequality for $|a_3|$ is sharp when $c_1 = c_2 = 2$, which completes the proof of Theorem 2. □

5. Fekete–Szegö Theorem

We next establish sharp Fekete–Szegö inequalities for $\mathcal{B}_1^\gamma(\alpha)$, which extends those given in [7] for $\mathcal{B}_1(\alpha)$, and in [4] for $\mathcal{M}^\gamma$.

Theorem 3. *Let $f \in \mathcal{B}_1^\gamma(\alpha)$. Then for $\nu \in \mathbb{R}$,*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{2\left(3\alpha^2\gamma + \alpha(-4\gamma(\nu-2) - 2\nu + 1) + \gamma(9-8\nu) - 4\nu + 3\right)}{(1+\alpha)^2(2+\alpha)(1+\gamma)^2(1+2\gamma)} \\[2ex] \qquad \textit{if } \nu \leq \dfrac{\alpha^2\left(-\gamma^2\right) + \alpha^2\gamma - \alpha^2 - 2\alpha\gamma^2 + 4\alpha\gamma - \alpha - \gamma^2 + 7\gamma + 2}{4\alpha\gamma + 2\alpha + 8\gamma + 4}, \\[2ex] \dfrac{2}{(2+\alpha)(1+2\gamma)}, \\[2ex] \qquad \textit{if } \dfrac{\alpha^2\left(-\gamma^2\right) + \alpha^2\gamma - \alpha^2 - 2\alpha\gamma^2 + 4\alpha\gamma - \alpha - \gamma^2 + 7\gamma + 2}{4\alpha\gamma + 2\alpha + 8\gamma + 4} \leq \nu \\ \qquad\qquad \leq \dfrac{\alpha^2\gamma^2 + 5\alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 + 12\alpha\gamma + 3\alpha + \gamma^2 + 11\gamma + 4}{4\alpha\gamma + 2\alpha + 8\gamma + 4}, \\[2ex] -\dfrac{2\left(3\alpha^2\gamma + \alpha(-4\gamma(\nu-2) - 2\nu + 1) + \gamma(9-8\nu) - 4\nu + 3\right)}{(1+\alpha)^2(2+\alpha)(1+\gamma)^2(1+2\gamma)} \\[2ex] \qquad \textit{if } \nu \geq \dfrac{\alpha^2\gamma^2 + 5\alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 + 12\alpha\gamma + 3\alpha + \gamma^2 + 11\gamma + 4}{4\alpha\gamma + 2\alpha + 8\gamma + 4}. \end{cases}$$

All the inequalities are sharp.

Proof. From (4) we obtain

$$|a_3 - \nu a_2^2| = \frac{2}{(2+\alpha)(1+2\gamma)} \left| c_2 - \frac{\mu}{2} c_1^2 \right|,$$

with

$$\mu = \frac{-2 + \alpha + \alpha^2 - 7\gamma - 4\alpha\gamma - \alpha^2\gamma + \gamma^2 + 2\alpha\gamma^2 + \alpha^2\gamma^2 + 4\nu + 2\alpha\nu + 8\gamma\nu + 4\alpha\gamma\nu}{(1+\alpha)^2(1+\gamma)^2}.$$

Applying Lemma 2, $\mu \in [0,2]$ whenever

$$\frac{\alpha^2\left(-\gamma^2\right)+\alpha^2\gamma-\alpha^2-2\alpha\gamma^2+4\alpha\gamma-\alpha-\gamma^2+7\gamma+2}{4\alpha\gamma+2\alpha+8\gamma+4} \leq \nu \leq \frac{\alpha^2\gamma^2+5\alpha^2\gamma+\alpha^2+2\alpha\gamma^2+12\alpha\gamma+3\alpha+\gamma^2+11\gamma+4}{4\alpha\gamma+2\alpha+8\gamma+4},$$

gives the second inequality.

When μ outside $[0,2]$, Lemma 2 gives the first inequality when

$$\nu \leq \frac{\alpha^2\left(-\gamma^2\right)+\alpha^2\gamma-\alpha^2-2\alpha\gamma^2+4\alpha\gamma-\alpha-\gamma^2+7\gamma+2}{4\alpha\gamma+2\alpha+8\gamma+4},$$

and the third inequality when

$$\nu \geq \frac{\alpha^2\gamma^2+5\alpha^2\gamma+\alpha^2+2\alpha\gamma^2+12\alpha\gamma+3\alpha+\gamma^2+11\gamma+4}{4\alpha\gamma+2\alpha+8\gamma+4}.$$

The second inequality is sharp when $c_1 = 0$ and $c_2 = 2$. The first and third inequalities are sharp when $c_1 = c_2 = 2$. This completes the proof of Theorem 3. □

6. Logarithmic Coefficients

The logarithmic coefficients g_n of f are defined in $\mathbb{D}$ by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} g_n z^n. \tag{5}$$

Differentiating (5) and equating coefficients gives

$$\begin{aligned} g_1 &= \frac{1}{2}a_2, \\ g_2 &= \frac{1}{2}(a_3 - \frac{1}{2}a_2^2), \\ g_3 &= \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3). \end{aligned}$$

For $f \in \mathcal{B}_1^{\gamma}(\alpha)$, we give sharp bounds for $|g_n|$ when $n = 1,2$, which extend those given in [10] and [6].

Theorem 4. *Let $f \in \mathcal{B}_1^{\gamma}(\alpha)$, then*

$$|g_1| \leq \frac{1}{(1+\alpha)(1+\gamma)} \text{ when } \gamma \geq 0 \text{ and } \alpha \geq 0,$$

$$|g_2| \leq \frac{1}{(2+\alpha)(1+2\gamma)}, \text{ when } 0 \leq \gamma \leq 3 \text{ and } \alpha \geq -1+\sqrt{\frac{1+2\gamma}{1-\gamma+\gamma^2}},$$

and when $\gamma > 3$ and $\alpha \geq 0$.

Further,

$$|g_2| \leq \frac{1+(5+6\alpha+3\alpha^2)\gamma}{(1+\alpha)^2(2+\alpha)(1+\gamma)^2(1+2\gamma)}, \text{ when } 0 < \gamma < 3 \text{ and } 0 \leq \alpha < -1+\sqrt{\frac{1+2\gamma}{1-\gamma+\gamma^2}}.$$

All the inequalities are sharp.

Proof. We note first that since $|c_1| \leq 2$, the inequality $|g_1| \leq \dfrac{1}{(1+\alpha)(1+\gamma)}$ is trivial.

The result for $|g_2|$ follows at once from the above Fekete–Szegö theorem in the case $\mu = 1/2$. For the first inequality, we use the second inequality in Theorem 3, and for the second inequality we use the first inequality in Theorem 3.

We note that the inequality for $|g_1|$ is sharp when $c_1 = 2$. The first inequality for $|g_2|$ is sharp when $c_2 = 2$ and $c_1 = 0$, and the second inequality is sharp when choosing $c_1 = c_2 = 2$. This completes the proof of Theorem 4. □

Remark 1. *Finding sharp upper bounds for $|g_n|$ for all $n \geq 3$ when $f \in \mathcal{B}_1^{\gamma}(\alpha)$ remains an open problem. In the case $\alpha = 0$, sharp results for $n = 1, 2, 3$ have been obtained in [6]. For $\gamma = 0$, it was shown in [10] that*

$$|g_n| \leq \frac{1}{n+\alpha},$$

for $n = 1, 2, 3$.

7. Inverse Coefficients

For any univalent function f there exists an inverse function f^{-1} defined on some disc $|\omega| < r_0(f)$, with Taylor expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots \tag{6}$$

Suppose that $\mathcal{B}_1^{\gamma}(\alpha)^{-1}$ is the set of inverse functions f^{-1} of $\mathcal{B}_1^{\gamma}(\alpha)$, given by (6). Then $f(f^{-1}(\omega)) = \omega$, and equating coefficients gives

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3. \end{aligned}$$

We prove the following, noting again that the inequalities for $|A_2|$ and $|A_3|$ hold for all $\gamma \geq 0$ and $\alpha \geq 0$ thus extending results extend in [10] and [6].

Theorem 5. *Let $f \in \mathcal{B}_1^{\gamma}(\alpha)$ and f^{-1} be given by (6), then*

$$|A_2| \leq \frac{2}{(1+\alpha)(1+\gamma)},$$

$$|A_3| \leq \frac{2}{(2+\alpha)(1+2\gamma)},$$

when $0 \leq \gamma \leq \frac{1}{2}\left(5+\sqrt{41}\right)$ and $\alpha \geq \dfrac{1-4\gamma-2\gamma^2}{2\,(1+5\gamma+\gamma^2)} + \dfrac{1}{2}\sqrt{\dfrac{17+92\gamma+124\gamma^2+16\gamma^3}{(1+5\gamma+\gamma^2)^2}}$,
and when $\gamma > \frac{1}{2}\left(5+\sqrt{41}\right)$.

Further,

$$|A_3| \leq \frac{10+6\alpha+14\gamma-6\alpha^2\gamma}{(1+\alpha)^2(2+\alpha)(1+\gamma)^2(1+2\gamma)},$$

when $0 \leq \gamma < \frac{1}{2}\left(5+\sqrt{41}\right)$ and $0 \leq \alpha < \dfrac{1-4\gamma-2\gamma^2}{2\,(1+5\gamma+\gamma^2)} + \dfrac{1}{2}\sqrt{\dfrac{17+92\gamma+124\gamma^2+16\gamma^3}{(1+5\gamma+\gamma^2)^2}}$.
All the inequalities are sharp.

Proof. We again use the expressions for the coefficients given in (4).
Since $(1+\alpha)(1+\gamma)a_2 = c_1$ and $|c_1| \leq 2$, the first inequality is trivial.

Next we note that from (4)

$$|A_3| = \frac{1}{(2+\alpha)(1+2\gamma)}\left|c_2 - \frac{(\alpha^2\gamma^2 - \alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 + 4\alpha\gamma + 5\alpha + \gamma^2 + 9\gamma + 6)}{2(1+\alpha)^2(1+\gamma)^2}c_1^2\right|.$$

Let

$$\mu = \frac{(\alpha^2\gamma^2 - \alpha^2\gamma + \alpha^2 + 2\alpha\gamma^2 + 4\alpha\gamma + 5\alpha + \gamma^2 + 9\gamma + 6)}{(1+\alpha)^2(1+\gamma)^2},$$

and applying Lemma 2 gives the required inequalities.

The inequality for $|A_2|$ is sharp when $c_1 = 2$. The first inequality for $|A_3|$ is sharp on choosing $c_1 = 0$ and $c_2 = 2$, and the second inequality is sharp when $c_1 = c_2 = 2$. This completes the proof of Theorem 5. □

Remark 2. *Clearly finding sharp bounds for $|a_4|$ and $|A_4|$ appears to be far more difficult, and requires significantly more analysis. We note that applying the often used lemmas in [9] fails to give sharp results.*

We also note that even when $\gamma = 1$, the analysis for $|a_4|$ and $|A_4|$ is far from simple, and appears to require methods deeper than those used or mentioned in this paper.

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