

On Differential Equations Characterizing Legendrian Submanifolds of Sasakian Space Forms

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Abstract: In this paper, we give an estimate of the first eigenvalue of the Laplace operator on minimally immersed Legendrian submanifold N^n in Sasakian space forms $\tilde{N}^{2n+1}(\epsilon)$. We prove that a minimal Legendrian submanifolds in a Sasakian space form is isometric to a standard sphere \mathbb{S}^n if the Ricci curvature satisfies an extrinsic condition which includes a gradient of a function, the constant holomorphic sectional curvature of the ambient space and a dimension of N^n . We also obtain a Simons-type inequality for the same ambient space forms $\tilde{N}^{2n+1}(\epsilon)$.

Keywords: legendrian submanifolds; sasakian space forms; obata differential equation; isometric immersion

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1. Introduction and Motivations

In 1959, Yano and Nagano [1] proved that if a complete Einstein space of dimension strictly greater than 2 admits a 1-parameter group of non-homothetic conformal transformations, then it is isometric to a sphere. Later, Obata [2] gave a simplified proof of the result of Yano and Nagano by analyzing a differential equation, nowadays known as Obata equation. Recall that a complete manifold (N^n, g) admits a non-constant function ψ satisfying the Obata differential equation

$$\text{Hess}(\psi) + \psi g = 0, \quad (1)$$

if and only if (N^n, g) is isometric to the standard sphere \mathbb{S}^n . Such characterizations of complete spaces are of great interest and they were investigated by many geometers (see [3–12]). For example, Tashiro [13] has shown that the Euclidean spaces \mathbb{R}^n are characterized by a differential equation $\nabla^2 \psi = c g$, where c is a positive constant. Utilizing Obata Equation (1), Barros et al. [14] have shown that a compact gradient almost Ricci soliton $(N^n, g, \nabla \psi, \lambda)$ with the Codazzi Ricci tensor and constant sectional curvature is isometric to the Euclidean sphere, and then ψ is a height function in this case. For more terminologies related to the Obata equation, see [8]. In [15], Lichnerowicz proved that, if the first non-zero eigenvalue μ_1 of the Laplacian on a compact manifold (M^n, g) with $\text{Ric} \geq n - 1$, is not less than n , while $\mu_1 = n$, then (M^n, g) is isometric to the sphere \mathbb{S}^n . This means that the Obata's rigidity theorem could be used to analyze the equality case of Lichnerowicz's eigenvalue estimates

in [15]. In the sequel, inspired by ideas developed in [16–18], we derive some rigidity theorems in the present paper.

On the other hand, by considering N^n as a compact submanifold immersed in Euclidean space \mathbb{R}^{n+p} or the standard Euclidean sphere \mathbb{S}^{n+p} , Jiancheng Zhang [17] derived the Simons-type [18] inequalities of the first eigenvalue μ_1 and the squared norm of the second fundamental form S without need of minimality. In addition, a lower bound of S can be provided if it is constant. Similar results can be found in [14,16]. As a generalization in the case of an odd-dimensional sphere, a minimally immersed Legendrian submanifold into a Sasakian space form of constant holomorphic sectional curvature ϵ should be considered in order to obtain Simon's-like inequality theorem.

2. Preliminaries and Notations

An odd-dimensional C^∞ -manifold (\tilde{N}, g) is said to be an *almost contact metric manifold* if it is equipped with almost contact structure (ϕ, η, ζ) satisfying following properties:

$$\phi^2 = -I + \zeta \otimes \eta, \quad \eta(\zeta) = 1, \quad \phi(\zeta) = 0, \quad \eta \circ \phi = 0, \quad (2)$$

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad \& \quad \eta(V_1) = g(V_1, \zeta), \quad (3)$$

$\forall V_1, V_2 \in \Gamma(T\tilde{N})$, where ϕ, ζ and η are a tensor field of type $(1, 1)$, a structure vector field and a dual 1-form, respectively. Moreover, an almost contact metric manifold \tilde{N}^{2m+1} is referred to as a *Sasakian manifold* if it fulfills the following relation

$$(\tilde{\nabla}_{V_1}\phi)V_2 = g(V_1, V_2)\zeta - \eta(V_2)V_1. \quad (4)$$

It follows that

$$\tilde{\nabla}_{V_1}\zeta = -\phi V_1, \quad (5)$$

for any $V_1, V_2 \in \Gamma(T\tilde{N})$, where $\tilde{\nabla}$ stands for the Riemannian connection in regard to g . A Sasakian manifold \tilde{N}^{2m+1} equipped with constant ϕ -sectional curvature ϵ is referred to as *Sasakian space form* and denoted by $\tilde{N}^{2m+1}(\epsilon)$. Then, the following formula for the curvature tensor \tilde{R} of $\tilde{N}^{2m+1}(\epsilon)$ can be expressed as:

$$\begin{aligned} \tilde{R}(V_1, V_2, V_3, V_4) = & \frac{\epsilon + 3}{4} \left\{ g(V_2, V_3)g(V_1, V_4) - g(V_1, V_3)g(V_2, V_4) \right\} \\ & + \frac{\epsilon - 1}{4} \left\{ \eta(V_1)\eta(V_3)g(V_2, V_4) + \eta(V_4)\eta(V_2)g(V_1, V_3) \right. \\ & \quad - \eta(V_2)\eta(V_3)g(V_1, V_4) - \eta(V_1)g(V_2, V_3)\eta(V_4) \\ & \quad + g(\phi V_2, V_3)g(\phi V_1, V_4) - g(\phi V_1, V_3)g(\phi V_2, V_4) \\ & \quad \left. + 2g(V_1, \phi V_2)g(\phi V_3, V_4) \right\}, \quad (6) \end{aligned}$$

$\forall V_1, V_2, V_3, V_4 \in \Gamma(T\tilde{N})$. Moreover, \mathbb{R}^{2m+1} and \mathbb{S}^{2m+1} with standard Sasakian structures can be given as typical examples of Sasakian space forms. An n -dimensional Riemannian submanifold N^n of $\tilde{N}^{2m+1}(\epsilon)$ is referred to as *totally real* if the standard almost contact structure ϕ of $\tilde{N}^{2m+1}(\epsilon)$ maps any tangent space of N^n into its corresponding normal space (see [4,19–21]). Now, let N^n be an isometric immersed submanifold of dimension n in $\tilde{N}^{2m+1}(\epsilon)$. Then N^n is referred to as a *Legendrian submanifold* if ζ is a normal vector field on N^n , i.e., N^n is a C -totally real submanifold, and $m = n$ [22]. Legendrian submanifolds play a substantial role in contact geometry. From Riemannian geometric perspective, studying Legendrian submanifolds of Sasakian manifolds was initiated in

1970's. Many geometers have drawn significant attention to minimal Legendrian submanifolds in particular. In order to proceed let us recall the definition of the curvature tensor \tilde{R} for Legendrian submanifold in $\tilde{N}^{2n+1}(\epsilon)$ which is given by

$$\tilde{R}(V_1, V_2, V_3, V_4) = \left(\frac{\epsilon+3}{4}\right) \left\{ g(V_2, V_3)g(V_1, V_4) - g(V_1, V_3)g(V_2, V_4) \right\}. \quad (7)$$

Let $\{e_1, \dots, e_n\}$ be an adapted orthogonal frame to N^n . Then, the second fundamental form h associated to N^n is defined as

$$h(e_i, e_j) = \sum_{\gamma=1}^n \sigma_{ij}^\gamma e_\gamma,$$

where $\sigma_{ij}^\gamma = \langle A_\gamma e_i, e_j \rangle$ and A_γ is the shape operator in the direction of e_γ . Hence, the Gauss formula for Legendrian submanifold N^n in $\tilde{N}^{2n+1}(\epsilon)$ in the local coordinates has the form

$$R_{ijkl}^i = \left(\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}\right) \left(\frac{\epsilon+3}{4}\right) + \sum_{\gamma=1}^n (\sigma_{ik}^\gamma \sigma_{jl}^\gamma - \sigma_{il}^\gamma \sigma_{jk}^\gamma).$$

Therefore, we have

$$R_{jij}^i = \left(\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}\right) \left(\frac{\epsilon+3}{4}\right) + \sum_{\gamma=1}^n (\sigma_{ii}^\gamma \sigma_{jj}^\gamma - \sigma_{ij}^\gamma \sigma_{ji}^\gamma). \quad (8)$$

We should note that Ψ is a C-totally real minimal immersion. Then, (8) yields

$$Ric(e_i, e_j) = (n-1) \left(\frac{\epsilon+3}{4}\right) \delta_{ij} - \sum_{\gamma=1}^n \sigma_{ir}^\gamma \sigma_{jr}^\gamma. \quad (9)$$

Now, we recall that Bochner formula [4] as follows: if $\psi : N^n \rightarrow \mathbb{R}$ is a function defined on a Riemannian manifold N^n , then we have

$$\frac{1}{2} \Delta |\nabla \psi|^2 = |Hess(\psi)|^2 + Ric_{N^n}(\nabla \psi, \nabla \psi) + g(\nabla \psi, \nabla(\Delta \psi)), \quad (10)$$

where, Ric denotes the Ricci tensor of N^n and $|A|$ stands for the norm of an operator A which is given by $|A|^2 = tr(AA^*)$; A^* is the transpose of A .

3. The Main Results

Now, we give a proof of the following essential proposition that we need later to prove our main Theorems 1 and 2.

Proposition 1. Let $\Psi : N^n \rightarrow \tilde{N}^{2n+1}(\epsilon)$ be a minimal immersion of a compact Legendrian submanifold into the Sasakian space form $\tilde{N}^{2n+1}(\epsilon)$ and ψ be a first eigenfunction associated to the Laplacian of N^n . Then if $\{e_1, \dots, e_n\}$ is an orthonormal tangent basis on N^n , we have

$$\left\{ (n-1) \left(\frac{\epsilon+3}{4}\right) - \mu_1 \right\} \int_N |\nabla \psi|^2 dV + \int_N |Hess(\psi)|^2 dV = \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV, \quad (11)$$

and particularly, we get

$$\begin{aligned} \int_N \sum_{i=1}^n |h(\nabla\psi, e_i)|^2 dV &= \int_N \left| \text{Hess}(\psi) + \frac{\mu_1}{n} \psi I \right|^2 dV \\ &+ \left\{ (n-1) \left(\frac{\epsilon+3}{4} \right) - \frac{\mu_1}{n} \right\} \int_N |\nabla\psi|^2 dV, \end{aligned} \quad (12)$$

where I denotes the identity operator on TN , μ_1 is an eigenvalue of the eigenfunction ψ such that $\Delta\psi + \mu_1\psi = 0$, and $\text{Hess}(\psi)$ is the squared norm of the Hessian of ψ

Proof. Let I be the identity operator on TN . Then we have

$$\left| \text{Hess}(\psi) - t\psi I \right|^2 = |\text{Hess}(\psi)|^2 - 2t\psi g(I, \text{Hess}(\psi)) + |I|^2 t^2 \psi^2. \quad (13)$$

It should be noted that $|I|^2 = \text{trace}(II^*) = n$, and

$$g(\text{Hess}(\psi), I) = \text{trace}(\text{Hess}(\psi)I^*) = \text{trace}(\text{Hess}(\psi)) = \Delta\psi.$$

Therefore, if $\Delta\psi + \mu_1\psi = 0$, we derive it for any $t \in \mathbb{R}$. Integrating Equation (13), and using the above equation and Stokes theorem, we get

$$\int_N \left| \text{Hess}(\psi) - t\psi I \right|^2 dV = \int_N |\text{Hess}(\psi)|^2 dV + \left(2t + \frac{n}{\mu_1} t^2 \right) \int_N |\nabla\psi|^2 dV. \quad (14)$$

Setting $t = -\frac{\mu_1}{n}$ in (14), we get

$$\int_N |\text{Hess}(\psi)|^2 dV = \int_N \left| \text{Hess}(\psi) + \frac{\mu_1}{n} \psi I \right|^2 dV + \frac{\mu_1}{n} \int_N |\nabla\psi|^2 dV. \quad (15)$$

On other hand, Equation (9) yields

$$\text{Ric}(\psi_i e_i, \psi_j e_j) = (n-1) \left(\frac{\epsilon+3}{4} \right) \delta_{ij} \psi_i \psi_j - \sum_{\gamma=1}^{2n+1} \sum_{r=1}^n \sigma_{ir}^\gamma \sigma_{jr}^\gamma \psi_i \psi_j.$$

Tracing the above equation, we obtain

$$\text{Ric}(\nabla\psi, \nabla\psi) = \left(\frac{\epsilon+3}{4} \right) (n-1) |\nabla\psi|^2 - \sum_{i=1}^n |h(\nabla\psi, e_i)|^2. \quad (16)$$

As we consider that $\Delta\psi = -\mu_1\psi$, combining the integration of Bochner formula with utilizing Stokes theorem, one arrives

$$\int_N |\text{Hess}(\psi)|^2 dV + \int_N \text{Ric}_{N^n}(\nabla\psi, \nabla\psi) dV = \mu_1 \int_N |\nabla\psi|^2 dV. \quad (17)$$

From (16) and (17), we conclude

$$\begin{aligned} \left\{ \left(\frac{\epsilon+3}{4} \right) n - \mu_1 \right\} \int_N |\nabla\psi|^2 dV &= \int_N \sum_{i=1}^n |h(\nabla\psi, e_i)|^2 dV \\ &+ \left(\frac{\epsilon+3}{4} \right) \int_N |\nabla\psi|^2 dV - \int_N |\text{Hess}(\psi)|^2 dV. \end{aligned}$$

This is the first result (11) of proposition. On the other hand, using (15) in the last equality, we obtain

$$\left\{ \left(\frac{\epsilon+3}{4} \right) n - \mu_1 \right\} \int_N |\nabla \psi|^2 dV = \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV + \left(\frac{\epsilon+3}{4} \right) \int_N |\nabla \psi|^2 dV - \int_N \left| \text{Hess}(\psi) + \frac{\mu_1}{n} \psi I \right|^2 dV - \frac{\mu_1}{n} \int_N |\nabla \psi|^2 dV.$$

The above formula can written as

$$\left\{ \left(\frac{\epsilon+3}{4} \right) n - \left(\frac{\epsilon+3}{4} \right) - \mu_1 + \frac{\mu_1}{n} \right\} \int_N |\nabla \psi|^2 dV = \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV - \int_N \left| \text{Hess}(\psi) + \frac{\mu_1}{n} \psi I \right|^2 dV$$

After some computation, we get

$$\int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV = \int_N \left| \text{Hess}(\psi) + \frac{\mu_1}{n} \psi I \right|^2 dV + \left\{ \frac{(n-1)}{n} \left(\left(\frac{\epsilon+3}{4} \right) n - \mu_1 \right) \right\} \int_N |\nabla \psi|^2 dV,$$

which completes the proof of the proposition. \square

The first result of our study can be given as follows.

Theorem 1. Suppose that $\Psi : N^n \rightarrow \tilde{N}^{2n+1}(\epsilon)$ is a minimal immersion of a compact Legendrian submanifold into Sasakian space form $\tilde{N}^{2n+1}(\epsilon)$ and ψ is a first eigenfunction of the Laplacian of N^n associated to the first eigenvalue μ_1 . Then, we have

(i) The second fundamental form satisfies the following

$$\int_N |\text{Hess}(\psi)|^2 dV \leq \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV + \left(\frac{\epsilon+3}{4} \right) \int_N |\nabla \psi|^2 dV, \quad (18)$$

provided that the inequality $n \left(\frac{\epsilon+3}{4} \right) \geq \mu_1$ holds, where $\text{Hess}(\psi)$ denotes the squared norm of the Hessian of ψ and $\{e_1, \dots, e_n\}$ is an orthonormal frame tangent to N^n . Moreover, the equality holds if and only if

$$\mu_1 = \left(\frac{\epsilon+3}{4} \right) n. \quad (19)$$

(ii) Furthermore, if the inequality

$$\int_N |\text{Hess}(\psi)|^2 dV \geq \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV \quad (20)$$

holds, then we have lower bound for eigenvalue μ_1 , that is,

$$\mu_1 \geq \left(\frac{\epsilon+3}{4} \right) (n-1).$$

(iii) In particular, if the following inequality

$$\frac{\mu_1}{n} \int_N |\nabla \psi|^2 dV \geq \left(\frac{\epsilon+3}{4} \right) \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV \quad (21)$$

holds, then the eigenvalue μ_1 satisfies the following inequality

$$\mu_1 \geq \left(\frac{\epsilon+3}{4}\right)(n-1).$$

Proof. We proceed as follows. Let

$$n\left(\frac{\epsilon+3}{4}\right) \geq \mu_1.$$

We point out that (11) of Proposition 1 is non-negative. Therefore, we can write

$$\int_N \sum_{i=1}^n |h(\nabla\psi, e_i)|^2 dV + \left(\frac{\epsilon+3}{4}\right) \int_N |\nabla\psi|^2 dV \geq \int_N |Hess(\psi)|^2 dV.$$

Furthermore, the equality sign of the above inequality holds if and only if

$$\mu_1 = n\left(\frac{\epsilon+3}{4}\right).$$

Moreover, the first equation of Proposition 1 can take the form

$$\begin{aligned} \int_N |Hess(\psi)|^2 dV &= \int_N \sum_{i=1}^n |h(\nabla\psi, l_i)|^2 dV \\ &+ \left\{ \mu_1 - \left(\frac{\epsilon+3}{4}\right)(n-1) \right\} \int_N |\nabla\psi|^2 dV. \end{aligned} \quad (22)$$

Now, if we consider the following inequality

$$\int_N |Hess(\psi)|^2 dV \geq \int_N \sum_{i=1}^n |h(\nabla\psi, l_i)|^2 dV,$$

then Equation (22) yields that

$$\left\{ \mu_1 - \left(\frac{\epsilon+3}{4}\right)(n-1) \right\} \geq 0.$$

Finally, we note that

$$\int_N |\nabla\psi|^2 dV \geq \frac{n}{\mu_1} \left(\frac{\epsilon+3}{4}\right) \int_N \sum_{i=1}^n |h(\nabla\psi, l_i)|^2 dV.$$

This implies that

$$\int_N |Hess(\psi)|^2 dV \geq \int_N \sum_{i=1}^n |h(\nabla\psi, l_i)|^2 dV,$$

which completes the proof of the theorem. \square

Now, we recall the following lemma which would help us to prove the next Theorem.

Lemma 1 ([16]). Let $T : U \rightarrow U$ be a trace-less non-null symmetric linear operator defined over a finite dimensional vector space U . Let $\{e_1, \dots, e_n\}$ be an orthonormal frame diagonalizing T , i.e., $Te_i = \mu_i e_i$. If $\dim \text{Ker} T = q$, then we get

$$\mu_j^2 \leq \frac{(n-q-1)|T|^2}{(n-q)}, \quad \forall j.$$

Now, we give the second result of the study as follows.

Theorem 2. Let $\Psi : N^n \rightarrow \tilde{N}^{2n+1}(\epsilon)$ be a minimal immersion of a compact Legendrian submanifold into a Sasakian space form $\tilde{N}^{2n+1}(\epsilon)$, μ_1 be the first eigenvalue of the Laplacian of N^n and $\dim \text{Ker}(h) = q$. Then, we have

$$\int_N S |Hess(\psi)|^2 dV \geq \left\{ \frac{(n-q)(n\beta-1)(n\beta-\mu_1)}{(n-q-1)n\beta} \right\} \int_N |\nabla \psi|^2 dV,$$

where $\beta = \frac{\epsilon+3}{4}$ and S is the squared norm of the second fundamental form h . Moreover, if S is constant, we get

$$S \geq \frac{(n-q)(n\beta-1)}{n\beta(n-q-1)(n\beta-\mu_1)},$$

where $\Delta \psi + \mu_1 \psi = 0$.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthogonal referential diagonalizing T , i.e., $Te_i = k_i e_i$ and let θ_i be the angle between $\nabla \psi$ and e_i . Then, we have

$$|h(\nabla \psi, e_i)|^2 = g(T\nabla \psi, e_i)^2 = g(\nabla \psi, Te_i)^2 = k_i^2 \cos^2 \theta_i |\nabla \psi|^2.$$

By virtue of (11) in Proposition 1, we obtain

$$\begin{aligned} \int_N \left(\sum_{i=1}^n k_i^2 \cos^2 \theta_i \right) |\nabla \psi|^2 dV &= \int_N |Hess(\psi)|^2 dV \\ &+ \left\{ \left(\frac{\epsilon+3}{4} \right) (n-1) - \mu_1 \right\} \int_N |\nabla \psi|^2 dV. \end{aligned}$$

Utilizing Lemma 1, the above equation gives

$$\begin{aligned} \left(\frac{n-q-1}{n-q} \right) \int_N S |\nabla \psi|^2 dV &\geq \int_N |Hess(\psi)|^2 dV \\ &+ \left\{ \left(\frac{\epsilon+3}{4} \right) (n-1) - \mu_1 \right\} \int_N |\nabla \psi|^2 dV. \end{aligned} \quad (23)$$

Let us assume the following inequality

$$\int_N |Hess(\psi)|^2 dV \geq \left(\frac{4\mu_1}{\epsilon+3} \right) \int_N |\nabla \psi|^2 dV,$$

holds. Using this assumption with fixing $\beta = \frac{\epsilon+3}{4}$, then (23) becomes

$$\left(\frac{n-q-1}{n-q} \right) \int_N S |\nabla \psi|^2 dV \geq \left(\frac{n^2 \beta^2 - n\beta \mu_1 - n\beta^2 + \mu_1}{n\beta} \right) \int_N |\nabla \psi|^2 dV.$$

After some computations, we get

$$\int_N S |Hess(\psi)|^2 dV \geq \left\{ \frac{(n-q)(n\beta-1)(n\beta-\mu_1)}{(n-q-1)n\beta} \right\} \int_N |\nabla\psi|^2 dV.$$

This completes the proof. \square

The following theorem gives the characterization Theorem as follows.

Theorem 3. Let $\Psi : N^n \rightarrow \tilde{N}^{2n+1}(\epsilon)$ be a minimal immersion of a compact Legendrian submanifold into Sasakian space form $\tilde{N}^{2n+1}(\epsilon)$ and ψ be a first eigenfunction associated to the Laplacian of N^n . Then, we have

- (i) If $\nabla\psi \in \text{Ker}(h)$, then $\Psi(N^n)$ is isometric to the standard sphere \mathbb{S}^n with $\mu_1 > 0$ and $n = 1$.
- (ii) If following Ricci inequality holds

$$\text{Ric}_{N^n}(\nabla\psi, \nabla\psi) \geq (n-1) \left(\frac{\epsilon+3}{4} \right) |\nabla\psi|^2,$$

then $\Psi(N^n)$ is isometric to a sphere \mathbb{S}^n with $\epsilon > -3$ and $n \geq 2$.

Proof. At first, we provide the state of Obata Theorem [2] as follows: a Riemannian manifold M^n is isometric to a unit sphere \mathbb{S}^n if and only if it is equipped with a differentiable function ψ such that $Hess(\psi) = -\psi$, where $Hess(\psi)$ is the Hessian form. Now, we assume that $\nabla\psi \in \text{ker}(h)$, i.e.,

$$h(\nabla\psi, e_i) = 0, \quad \forall e_i.$$

Then by using Equation (12), we attain

$$\int_N \left| Hess(\psi) + \frac{\mu_1}{n} \psi \right|^2 dV = \frac{(n-1)(\mu_1 - n\beta)}{n} \int_N |\nabla\psi|^2 dV.$$

Using the fact that the right-hand side of the above equation is non-positive leads to

$$0 < \mu_1 = n \left(\frac{\epsilon+3}{4} \right).$$

Therefore, $Hess(\psi) = -\mu_1\psi$, as $\mu_1 > 0$ and $n = 1$. Now, utilizing Obata Theorem [2], we conclude that $\Phi(N^n)$ is isometric to \mathbb{S}^n with $\mu_1 = n$. Thus, we have gotten the first part of Theorem 3. To prove the second statement of the theorem, let us consider that

$$\text{Ric}_{N^n}(\nabla\psi, \nabla\psi) \geq \left(\frac{\epsilon+3}{4} \right) (n-1) |\nabla\psi|^2.$$

According to Equation (16), we find that

$$\int_N (n-1) \left(\frac{\epsilon+3}{4} \right) |\nabla\psi|^2 dV \geq \sum_{i=1}^n |h(\nabla\psi, e_i)|^2 dV + \left(\frac{\epsilon+3}{4} \right) (n-1) \int_N |\nabla\psi|^2 dV.$$

This leads to

$$\sum_{i=1}^n |h(\nabla\psi, e_i)|^2 dV \leq 0. \quad (24)$$

Hence, we conclude that $h(\nabla\psi, e_i) = 0$, i.e., $\nabla\psi \in \text{ker}(h)$. The proof is now complete. \square

Tashiro [13] has proved more general results than of Obata and Kanai. The following theorem is of interest in characterizing the Euclidean space in terms of a certain differential equation. Therefore, we are able to prove the following result.

Theorem 4. Let $\Psi : N^n \rightarrow \tilde{N}^{2n+1}(\epsilon)$ be a minimal immersion of a compact Legendrian submanifold into Sasakian space form $\tilde{N}^{2n+1}(\epsilon)$. Then N^n is isometric to Euclidean space \mathbb{R}^n if and only if the following equation is satisfied

$$\int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV + \int_N \frac{\mu_1^2}{n} dV = \left\{ \mu_1 - \left(\frac{\epsilon+3}{4} \right) (n-1) \right\} \int_N |\nabla \psi|^2 dV, \quad (25)$$

where ψ is a first eigenfunction associated to the Laplacian of N^n with first non-zero eigenvalue μ_1 .

Proof. Let us consider the equation

$$|Hess(\psi) + tI|^2 = |Hess(\psi)|^2 + t^2|I|^2 + 2tg(Hess(\psi), I),$$

which implies that

$$|Hess(\psi) + tI|^2 = |Hess(\psi)|^2 + t^2n - 2t\Delta\psi.$$

Putting $t = -\frac{\mu_1}{n}$ and integrating the above equation along volume element dV , we obtain

$$\int_N \left| Hess(\psi) - \frac{\mu_1}{n} I \right|^2 dV = \int_N \left(|Hess(\psi)|^2 + \frac{\mu_1^2}{n} \right) dV.$$

Using (16) and (17), we get

$$\begin{aligned} \int_N \left| Hess(\psi) - \frac{\mu_1}{n} I \right|^2 dV &= \int_N \sum_{i=1}^n |h(\nabla \psi, e_i)|^2 dV - \left\{ \mu_1 - \left(\frac{\epsilon+3}{4} \right) (n-1) \right\} \int_N |\nabla \psi|^2 dV \\ &\quad + \int_N \frac{\mu_1^2}{n} dV. \end{aligned} \quad (26)$$

If (25) is satisfied, then (26) implies that

$$\left| Hess(\psi) - \frac{\mu_1}{n} I \right|^2 = 0.$$

Hence, we get

$$Hess(\psi)(X, X) = \frac{\mu_1}{n} g(X, X), \quad (27)$$

for any $X \in \Gamma(N)$. Therefore, by applying Tashiro Theorem [13], we conclude that N^n is isometric to the Euclidean space \mathbb{R}^n . The converse part can be proved easily from (26) if N^n is isometric to Euclidean space \mathbb{R}^n . \square

We provide an interesting application of Theorem 3 in the following corollary by choosing $\epsilon = 1$ (see [19]).

Corollary 1. Let $\Psi : N^n \rightarrow \mathbb{S}^{2n+1}$ be a minimal immersion of a compact Legendrian submanifold into the sphere \mathbb{S}^{2n+1} and ψ be a first eigenfunction associated to the Laplacian of N^n . Then, we get the following

- (i) If $\nabla \psi \in \text{Ker}(h)$, then $\Psi(N^n)$ is isometric to standard sphere \mathbb{S}^n .
- (ii) If $\text{Ric}_{N^n}(\nabla \psi, \nabla \psi) \geq (n-1)|\nabla \psi|^2$, then $\Psi(N^n)$ is isometric to the sphere \mathbb{S}^n .

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