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A Navier–Stokes-Type Problem with High-Order Elliptic Operator and Applications

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Abstract: The existence, uniqueness and uniformly L^p estimates for solutions of a high-order abstract Navier–Stokes problem on half space are derived. The equation involves an abstract operator in a Banach space E and small parameters. Since the Banach space E is arbitrary and A is a possible linear operator, by choosing spaces E and operators A , the existence, uniqueness and L^p estimates of solutions for numerous classes of Navier–Stokes type problems are obtained. In application, the existence, uniqueness and uniformly L^p estimates for the solution of the Wentzell–Robin-type mixed problem for the Navier–Stokes equation and mixed problem for degenerate Navier–Stokes equations are established.

Keywords: stokes systems; Navier–Stokes equations; differential equations with small parameters; semigroups of operators; differential-operator equations; maximal L^p regularity

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1. Introduction

We consider the mixed problem for Navier–Stokes (N–S)-type equation with small parameter

$$\frac{\partial u}{\partial t} + B_\varepsilon u + (u, B_{1\varepsilon})u + B_{1\varepsilon}\varphi + Au = f(x, t), \quad B_{2\varepsilon}u = 0, \quad (1a)$$

$$L_k u = \sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0, t) = 0, \quad v_k \in \{0, 1, \dots, 2m-1\}, \quad k = 1, 2, \dots, m, \quad (1b)$$

$$u(x, 0) = a(x), \quad x \in \mathbb{R}_+^n, \quad t \in (0, T), \quad (1c)$$

where

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n, x_n > 0, x = (x', x_n), x' = (x_1, x_2, \dots, x_{n-1})\},$$

$$\sigma_i = \frac{1}{2m} \left(i + \frac{1}{q} \right), \quad q \in (1, \infty), \quad B_\varepsilon u = \sum_{k=1}^n (-1)^m \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}},$$

$$B_{1\varepsilon}\varphi = - \left(\varepsilon_1^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_1^m}, \varepsilon_2^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_2^m}, \dots, \varepsilon_n^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_n^m} \right), \quad B_{2\varepsilon}u = - \sum_{k=1}^n \varepsilon_k^{\frac{1}{2}} \frac{\partial^m u}{\partial x_k^m}$$

α_{ki} are complex numbers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, ε_k are small positive parameters, m is a positive integer with $m \geq 1$ and A is a linear operator in a Banach space E . Here, $u = u_\varepsilon(x, t) =$

$(u_1(x, t), u_2(x, t), \dots, u_n(x, t)), u_k(x, t) = u_{k\varepsilon}(x, t)$ and $\varphi = \varphi(x, t)$ are represent the E -valued unknown velocity and pressure like functions, respectively, $f = (f_1(x, t), f_2(x, t), \dots, f_n(x, t))$ and a represent a given E -valued external force and the initial velocity.

Remark 1. When we consider the N–S problem 1a–1c, it means that the solution u belongs to space $X_q^{1,2m}(A)$. Using Lions and Peetre result (see for example [1], § 1.8.2) then the trace operator $u \rightarrow \partial_t u(x, \cdot)$ is bounded from $X_q^{1,2m}(A)$ to $\mathbb{Z}(\mathbb{E}_0, \mathbb{E})$. We assume A is to be such that $(u, \nabla)u \in E$ for $u \in X_q^{1,2m}(A)$, where $(\mathbb{E}_0, \mathbb{E})_{\theta, p}$ denotes real interpolation spaces between $\mathbb{E}_0, \mathbb{E}$ (see e.g., [1], § 1.3.2),

$$X_q^{1,2m}(A) = \left(W^{1,2m,p}(\not\leq_+; E(A), E) \right)^n, \mathbb{E} = (L^p(\mathbb{R}^n; E))^n, \theta = \frac{1}{2p},$$

$$\mathbb{E}_0 = \left(W^{2m,p}(\mathbb{R}^n; E(A), E) \right)^n, \mathbb{Z}(\mathbb{E}_0, \mathbb{E}) = \left(C(\mathbb{R}^n; (\mathbb{E}_0, \mathbb{E})_{\theta, p}) \right)^n,$$

here, $L^p(\mathbb{R}_+^n; E)$, $W^{2m,p}(\mathbb{R}^n; E(A), E)$, $W^{1,2m,p}(\not\leq_+; E(A), E)$ and $(\mathbb{E}_0, \mathbb{E})_{\theta, p}$ will be defined in the sequel.

Boundary value problems (BVPs) for differential-operator equations (DOEs) in classes of functions such as Lebesgue ones have been object of interest of a lots of scientists (see, for example [1–3]). Presentations to differential-operator equations have been done by several authors [4–6]. Regularity results for differential-operator equations are contained in [7–9]. In the present note, authors study degenerate parameter-dependent Boundary Value Problems for arbitrary order differential-operator equations. These kinds of problems have been applied in several fields which are useful in lots of fluid mechanics models.

The focus of our work was to prove uniform existence and uniqueness of the stronger local and global solution of the Navier–Stokes problem with a small parameter (1a)–(1c). This problem is characterized by the presence of an abstract operator A and a small parameter ε_k that, respectively, corresponds to the inverse of a Reynolds number Re that is very large for the N–S equations. Regularity results of N–S equations were obtained, for example, by the authors in [4–6,10–17]. The N–S equations with small viscosity when the boundary is either characteristic or non-characteristic have been well-studied; see for example in the papers [3,14,16]. In addition, regularity properties of abstract differential equation (ADE) were deeply studied in [2,7–9,18–22]. Here, the authors study abstract N–S equations with a high elliptic part in a Banach space E with operator coefficient A . In [22], we derived the L^p –regularity properties of the abstract Stokes problem. For $E = \mathbb{C}$, $A = a > 0$, $m = 1$, $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n = 1$ the problem (1a)–(1c) state to be usual N–S problem. In this paper, the authors prove that the corresponding Stokes type problem

$$\begin{aligned} \frac{\partial u}{\partial t} + B_\varepsilon u + Au + B_{1\varepsilon} \varphi &= f(x, t), \quad B_{2\varepsilon} u = 0, \quad x \in \mathbb{R}_+^n, \quad t \in (0, T), \\ \sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0, t) &= 0, \quad v_k \in \{0, 1, \dots, m-1\}, \quad k = 1, 2, \dots, m, \\ u(x, 0) &= a(x) \end{aligned} \quad (1d)$$

has a unique solution $(u, B_{1\varepsilon} \varphi)$ for $f \in L^p(0, T; X_q) = B(p, q)$, $p, q \in (1, \infty)$ and the following uniform estimate holds

$$\begin{aligned} \left\| \frac{\partial u}{\partial t} \right\|_{B(p, q)} + \sum_{k=1}^n \left\| \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}} \right\|_{B(p, q)} + \|Au\|_{B(p, q)} + \|B_{1\varepsilon} \varphi\|_{B(p, q)} \leq \\ C \left(\|f\|_{B(p, q)} + \|a\|_{(X_q^{2m}(A), X_q)_{\frac{1}{p}, p}} \right) \end{aligned} \quad (1e)$$

with $C = C(T, p, q)$ independent of f and ε , where $\left(X_q^{2m}(A), X_q\right)_{\frac{1}{p}, p}$ denote the real interpolation space between $X_q^{2m}(A) = W^{2m, q}(\mathbb{R}_+^n; E(A), E)^n$ and $X_q = (L^q(\mathbb{R}_+^n; E))^n$ defined by the K -method (see e.g., [1], §1.3.2). Then, by following Kato-Fujita [13] method, by using (1e) we derive a local a priori estimates for solutions of (1a)–(1c), i.e., we prove that for $\gamma < 1$ and $\delta \geq 0$ such that $\frac{n}{2q} - \frac{1}{2} \leq \gamma$, $-\gamma < \delta < 1 - |\gamma|$, $a \in D(Q_{\varepsilon q}^\gamma)$ there exists $T_* \in (0, T)$ independent of $\varepsilon_k \in (0, 1]$ such that $\|Q_{\varepsilon q}^{-\delta} P_q f(t)\|$ is continuous on $(0, T)$ and satisfies $\|Q_{\varepsilon q}^{-\delta} P_q f(t)\| = o(t^{\gamma+\delta-1})$ as $t \rightarrow 0$; there exists a local solution of (1a)–(1c) such that $u \in C([0, T_*]; D(Q_{\varepsilon q}^\gamma))$, $u(0) = a$, $u \in C((0, T_*]; D(Q_{\varepsilon q}^\alpha))$ for some $T_* > 0$ and $\|Q_{\varepsilon q}^\alpha u(t)\| = o(t^{\gamma-\alpha})$ as $t \rightarrow 0$ uniformly in ε for α with $\gamma < \alpha < 1 - \delta$. Moreover, the solution of (1a)–(1c) is unique for some β with $\beta > |\gamma|$. For sufficiently small date we show that there exists a global solution of (1a)–(1c). Particularly, we prove that there is a $\delta > 0$ such that if $\|a\|_{X_q} < \delta$, then there exists a global solution u_ε of (1a)–(1c) so that

$$t^{(1-\frac{n}{q})/2} u_\varepsilon, t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \in C([0, \infty); X_q) \text{ for } n \leq q \leq \infty.$$

Moreover, the following uniform estimates hold

$$\sup_{t, \varepsilon_j} \left\| t^{(1-\frac{n}{q})/2} u_\varepsilon \right\|_{X_q} \leq C, \sup_{t, \varepsilon_j} \left\| t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \right\|_{X_q} \leq C, j = 1, 2, \dots, n,$$

where $Q_{\varepsilon q}$ denotes the corresponding Stokes operator and P_q is a projection operator in X_q .

In application, we put $E = L^{p_1}(0, 1)$ and A to be differential operator in (1a) and (1b), with generalized Wentzell–Robin boundary condition defined by

$$D(A) = \left\{ u \in \left(W_{p_1}^2(0, 1) \right)^n, B_j u = A u(j), j = 0, 1 \right\},$$

$$A u = a u^{(2)} + b u^{(1)},$$

where a, b are complex-valued functions. Then, we obtain the existence, uniqueness and uniformly $L^p(\bar{\Omega})$ estimates for solutions the following Wentzell–Robin type mixed problem for the N–S equation

$$\frac{\partial u}{\partial t} - B_\varepsilon u + (u \cdot B_{1\varepsilon}) u + B_{1\varepsilon} \varphi + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} = f(x, y, t), \quad (1f)$$

$$B_{2\varepsilon} u = 0, u = u(x, y, t), x \in \mathbb{R}_+^n,$$

$$\sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0, t) = 0, v_k \in \{0, 1, \dots, 2m-1\}, k = 1, 2, \dots, m, \quad (1g)$$

$$A u(x, j, t) = 0, j = 0, 1, u(x, 0) = a(x), x' \in \mathbb{R}^{n-1}, y \in (0, 1). \quad (1h)$$

Note that the regularity properties of Wentzell–Robin-type boundary value problems (BVP) for elliptic equations were studied e.g., in [23,24] and the references therein. Here,

$$\tilde{\Omega} = \mathbb{R}_+^n \times (0, 1), \bar{\Omega} = \mathbb{R}_+^n \times (0, b), \mathbf{p} = (p_1, p), p_1, p \in (1, \infty)$$

$L^p(\tilde{\Omega})$ denotes the space of all $\mathbf{p}$ -summable complex-valued functions with mixed norm i.e., the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L^p(\tilde{\Omega})} = \left(\int_{\mathbb{R}_+^n} \left(\int_0^1 |f(x, y)|^{p_1} dy \right)^{\frac{p}{p_1}} dx \right)^{\frac{1}{p}} < \infty.$$

By using the general abstract result above, the existence, uniqueness and uniformly $L^p(\tilde{\Omega})$ estimates for solution of the problem (1f)–(1h) is obtained.

Moreover, we choose $E = L^{p_1}(0, b)$ and A to be degenerated differential operator in $L^{p_1}(0, b)$ defined by

$$D(A) = \left\{ u \in W_{\gamma}^{[2], p_1}(0, 1), \sum_{i=0}^{v_k} \alpha_{ki} u^{[i]}(0) + \beta_{ki} u^{[i]}(b) = 0, k = 1, 2 \right\},$$

$$A(x)u = b_1(x, y)u^{[2]} + b_2(x, y)u^{[1]}, x \in \mathbb{R}_+^n, y \in (0, b), v_k \in \{0, 1\}, \quad (1i)$$

where $u^{[i]} = \left(y^{\gamma} \frac{d}{dy}\right)^{\gamma} u$ for $0 \leq \gamma < \frac{1}{2}$, $b_1 = b_1(x, y)$ is a continuous, $b_2 = b_2(x, y)$ is a bounded function on $y \in [0, 1]$ for a.e. $x \in \mathbb{R}_+^n$, α_{ki}, β_{ki} are complex numbers and $W_{\gamma}^{[2], p_1}(0, b)$ is a weighted Sobolev space defined by

$$W_{\gamma}^{[2], p_1}(0, b) = \{ u : u \in L^{p_1}(0, b), u^{[2]} \in L^{p_1}(0, b),$$

$$\|u\|_{W_{\gamma}^{[2], p_1}} = \|u\|_{L^{p_1}} + \|u^{[2]}\|_{L^{p_1}} < \infty.$$

Then, we obtain the existence, uniqueness and uniformly $L^p(\tilde{\Omega})$ estimates for solutions of the following mixed problem for degenerate N-S equation

$$\frac{\partial u}{\partial t} - B_{\varepsilon}u + (u \cdot B_{1\varepsilon})u + B_{1\varepsilon}\varphi + \left(b_1 \frac{\partial^2 u}{\partial y^2} + b_2 \frac{\partial^1 u}{\partial y} \right) = f(x, y, t), \quad (1j)$$

$$B_{2\varepsilon}u = 0, x \in \mathbb{R}_+^n, y \in (0, b), t \in (0, T), u = u(x, y, t),$$

$$\sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0, t) = 0, v_k \in \{0, 1, \dots, 2m-1\}, k = 1, 2, \dots, m,$$

$$L_k u = \sum_{i=0}^{v_k} \alpha_{ki} u^{[i]}(x, 0, t) + \beta_{ki} u^{[i]}(x, b, t) = 0, k = 1, 2, \quad (1k)$$

$$u(x, y, 0) = a(x, y). \quad (1l)$$

Let E be a Banach space and $L^p(\Omega; E)$ denotes the space of strongly measurable E -valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega; E)} = \left(\int_{\Omega} \|f(x)\|_E^p dx \right)^{\frac{1}{p}}, 1 \leq p < \infty.$$

The Banach space E is called an *UMD-space* if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$ is bounded in $L^p(\mathbb{R}, E)$, $p \in (1, \infty)$ (see. e.g., [19], § 4). *UMD* spaces include e.g., L^p, l^p spaces and Lorentz spaces $L_{pq}, p, q \in (1, \infty)$.

Let E_1 and E_2 be two Banach spaces. Let $B(E_1, E_2)$ denote the space of all bounded linear operators from E_1 to E_2 . For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\psi = \{\lambda \in \mathbb{C}, |\arg \lambda| \leq \psi\} \cup \{0\}, \quad 0 \leq \psi < \pi.$$

A linear operator A is said to be positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A + \lambda I)^{-1}\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$ for any $\lambda \in S_\psi$ where I is the identity operator in E (see e.g., [1], §1.15.1). The positive operator A is said to be R -positive in a Banach space E if the set $\{\lambda(A + \lambda)^{-1} : \lambda \in S_\psi\}$ is R -bounded (see [19], §4). The operator $A(s)$ is said to be positive in E uniformly with respect to parameter s with bound $M > 0$ if $D(A(s))$ is independent on s , $D(A(s))$ is dense in E and $\|(A(s) + \lambda)^{-1}\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S_\psi$, where the constant M does not depend on s and λ .

Assume E_0 and E are two Banach spaces and E_0 is continuously and densely included into E . Here, Ω is a measurable set in $\mathbb{R}^n$ and m is a positive integer. Let $W^{m,p}(\Omega; E_0, E)$ denote the space of all functions $u \in L^p(\Omega; E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega; E_0, E)} = \|u\|_{L^p(\Omega; E_0)} + \sum_{k=1}^n \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega; E)} < \infty.$$

Let $H^{s,p}(\mathbb{R}^n; E)$, $-\infty < s < \infty$ denotes the E -valued fractional Sobolev space of order s that is defined as:

$$H^{s,p}(\mathbb{R}^n; E) = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n; E)$$

with the norm

$$\|u\|_{H^{s,p}} = \left\| (I - \Delta)^{\frac{s}{2}} u \right\|_{L^p(\mathbb{R}^n; E)}.$$

It clear that $H^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E)$. It is known that if E is a UMD space, then $H^{m,p}(\mathbb{R}^n; E) = W^{m,p}(\mathbb{R}^n; E)$ for positive integer m (see e.g., [25], §15). $H^{s,p}(\mathbb{R}^n; E_0, E)$ denote the Fractional Sobolev-Lions type space i.e.,

$$H^{s,p}(\mathbb{R}^n; E_0, E) = \{u \in H^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0)\},$$

$$\|u\|_{H^{s,p}(\mathbb{R}^n; E_0, E)} = \|u\|_{L^p(\mathbb{R}^n; E_0)} + \|u\|_{H^{s,p}(\mathbb{R}^n; E)} < \infty\}.$$

2. Regularity Properties of Solutions for System of ADEs with Parameters

In this section, we will derive the maximal regularity properties of the BVP for system of ADE with small parameters in half-space

$$B_\varepsilon u + (A + \lambda)u = f(x), \quad x \in \mathbb{R}_+^n, \quad (2a)$$

$$L_k u = \sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0) = \eta_k, \quad v_k \in \{0, 1, \dots, 2m-1\}, \quad k = 1, 2, \dots, m, \quad (2b)$$

$$B_\varepsilon u = \sum_{k=1}^n (-1)^m \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}}, \quad \sigma_i = \frac{1}{2m} \left(i + \frac{1}{q} \right), \quad q \in (1, \infty),$$

α_{ki} are complex numbers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, ε_k are small positive parameters, m is a positive integer with $m \geq 1$ and A is a linear operator in a Banach space E . Here, $u = u_\varepsilon(x) =$

$(u_1(x), u_2(x), \dots, u_n(x)), u_k(x) = u_{k\varepsilon}(x), \varphi = \varphi(x, t)$ are represent the E -valued unknown velocity and pressure like functions, respectively and $f = (f_1(x), f_2(x), \dots, f_n(x))$.

Let $X_q = X_q(E) = (L^q(\mathbb{R}_+^n; E))^n$ denotes the class of E -valued system of function $f = (f_1(x), f_2(x), \dots, f_n(x))$ with norm

$$\|f\|_{X_q} = \left(\sum_{i=1}^n \|f_i\|_{L^q(\mathbb{R}_+^n; E)}^q \right)^{\frac{1}{q}}, \quad q \in (1, \infty).$$

and let

$$X_q^m(A) = (W^{m,q}(\mathbb{R}_+^n; E(A), E))^n$$

with norm

$$\|u\|_{X_q^m(A)} = \sum_{i=1}^n \|u_i\|_{W^{m,q}(\mathbb{R}_+^n; E(A), E)}.$$

Let

$$\theta_k = \frac{\frac{1}{q} + \nu_k}{2m}, \quad W^{k,m,q}(E) = \left(W^{m\theta_k,q}(\mathbb{R}^{n-1}, (E(A), E)_{\theta_k,q}, E) \right)^n.$$

By reasoning as in ([9], Theorem 2) we have

Theorem 1. Let E be a UMD space and A be an R -positive operator in E . Assume m is a nonnegative number, $q \in (1, \infty)$, $\alpha_\nu \neq 0$, $0 < \varepsilon_k \leq 1$. Then for all $f \in X$, $\lambda \in S_\psi$ with sufficiently large $|\lambda| > 0$ problem (2a) and (2b) has a unique solution u that belongs to $X_q^m(A)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^n \sum_{i=0}^{2m} \varepsilon_k^{\frac{i}{2m}} |\lambda|^{1-\frac{i}{2m}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \|Au\|_{X_q} \leq C \left(\|f\|_{X_q} + \sum_{k=1}^m \|\eta_k\|_{W^{k,2m,q}(E)} \right) \quad (2c)$$

Consider the operator Q_ε generated by problem (2a) and (2b), i.e.,

$$D(Q_\varepsilon) = \left\{ u \in X_q^{2m}(A), L_k u = 0 \right\}, \quad Q_\varepsilon u = B_\varepsilon u + Au. \quad (2d)$$

From Theorem 1 we obtain the following results:

Result 1. Suppose the all conditions of Theorem 1 are satisfied. Then, there exists a resolvent $(Q_\varepsilon + \lambda)^{-1}$ for $\lambda \in S_\psi$ satisfying the following uniform estimate

$$\sum_{k=1}^n \sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} \varepsilon_k^{\frac{i}{2m}} \left\| \frac{\partial^i}{\partial x_k^i} (Q_\varepsilon + \lambda)^{-1} \right\|_{B(X_q)} \leq C.$$

It is clear that the solution (2a) and (2b) depend on parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, i.e., $u = u_\varepsilon(x)$. In view of the Theorem 1, we derive the properties of solutions (2a) and (2b).

From Theorem 1 we obtain:

Result 2. For $\lambda \in S_\psi$ there exists a resolvent $(Q_\varepsilon + \lambda)^{-1}$ of the operator Q_ε satisfying the following uniform estimate

$$\sum_{k=1}^n \sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} \varepsilon_k^{\frac{i}{2m}} \left\| \frac{\partial^i}{\partial x_k^i} (Q_\varepsilon + \lambda)^{-1} \right\|_{B(X_q)} \leq C. \quad (2e)$$

3. The Stokes System with Small Parameters

In this section, we derive the maximal regularity properties of the stationary abstract Stokes problem

$$B_\varepsilon u + (u, B_{1\varepsilon}) u + B_{1\varepsilon} \varphi + Au = f(x), \quad B_{2\varepsilon} u = 0, \quad (3a)$$

$$L_k u = \sum_{i=0}^{v_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i} (x', 0) = 0, \quad v_k \in \{0, 1, \dots, 2m-1\}, \quad k = 1, 2, \dots, m,$$

where

$$B_\varepsilon u = \sum_{k=1}^n (-1)^m \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}}, \quad \sigma_i = \frac{1}{2m} \left(i + \frac{1}{q} \right), \quad q \in (1, \infty),$$

$$B_{1\varepsilon} \varphi = \left(\varepsilon_1^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_1^m}, \varepsilon_2^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_2^m}, \dots, \varepsilon_n^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_n^m} \right), \quad B_{2\varepsilon} u = - \sum_{k=1}^n \varepsilon_k^{\frac{1}{2}} \frac{\partial^m u_k}{\partial x_k^m},$$

α_{ki} are complex numbers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, ε_k are small positive parameters, m is a positive integer with $m \geq 1$ and A is a linear operator in a Banach space E . Here, $u = u_\varepsilon(x) = (u_1(x), u_2(x), \dots, u_n(x))$, $u_k(x) = u_{k\varepsilon}(x)$, $\varphi = \varphi(x, t)$ are represent the E -valued unknown velocity and pressure like functions, respectively and $f = (f_1(x), f_2(x), \dots, f_n(x))$.

Here and hereafter E^* will denoted the conjugate of E and $(\cdot, \cdot)$ (resp. $\langle \cdot, \cdot \rangle$) denotes the duality pairing of functions on $\mathbb{R}_+^n$ (resp. $\mathbb{R}^{n-1}$) and $X_{q'} = X_{q'}(E^*) = \left(L^{q'}(\mathbb{R}_+^n; E^*) \right)^n$ will denote the dual space of X_q , where $q^{-1} + (q')^{-1} = 1$. Let $X_{\sigma q} = L_\sigma^q(\mathbb{R}_+^n; E)$ denote the E -valued solenoidal space. Let A be a positive operator in E . Let $Y^{2,q}(A) = W^{m,q}(\mathbb{R}_+^n; E(A), E)$. The spaces $(W^{m,q}(\mathbb{R}_+^n; E))^n$, $(Y^{m,q}(A))^n$ will be denoted by X_q^m and $X_q^m(A)$, respectively. Let

$$Bu = \sum_{k=1}^n (-1)^m \frac{\partial^{2m} u}{\partial x_k^{2m}}, \quad B_1 u = - \left(\frac{\partial^m u_1}{\partial x_1^m}, \frac{\partial^m u_2}{\partial x_2^m}, \dots, \frac{\partial^m u_n}{\partial x_n^m} \right),$$

$$B_2 u = - \sum_{k=1}^n \frac{\partial^m u_k}{\partial x_k^m}.$$

Consider the space

$$Y_q(A, B_1) = \{u \in X_q(E(A)), \quad \operatorname{div} B_1 u \in L^q(\mathbb{R}_+^n; E),$$

$$\|u\|_{Y_q(A, B_1)} = \left(\|u\|_{X_q(E(A))}^q + \|\operatorname{div} B_1 u\|_{L^q(\mathbb{R}_+^n; E)}^q \right)^{\frac{1}{q}} \}.$$

$Y_q(A, B_1)$ becomes a Banach space with this norm. Consider the problem

$$Bu + (A + \lambda) u = f(x), \quad x \in \mathbb{R}_+^n, \quad (3b)$$

$$L_k u = \sum_{i=0}^{v_k} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i} (x', 0) = 0, \quad v_k \in \{0, 1, \dots, m-1\}, \quad k = 1, 2, \dots, m.$$

By using Theorem 1, we obtain the following

Corollary 1. *Let E be a UMD space and A be an R -positive operator in E . Assume m is a nonnegative number, $q \in (1, \infty)$. Then for all $f \in X_q^{-m}$ problem (3b) has a unique solution $u \in X_q^m(A)$ and the following estimate holds*

$$\|u\|_{X_q^m(A)} \leq C \|f\|_{X_q^{-m}}.$$

It is known that (see e.g., [11,12]) vector field $u \in (L^q(\mathbb{R}_+^n))^n$ has a Helmholtz decomposition. In the following theorem we generalize this result for E -valued function space X_q .

By reasoning as in [11,12] and ([22], Theorem 3.1) we have decomposition result via operator $\tilde{B}$ generated by problem (3b).

Theorem 2. Let E be an UMD space and $q \in (1, \infty)$. Assume there exists a constant $C_0 > 0$ such that

$$\|u\|_{(E(A), E)_{\frac{1}{q}, q}} \geq C_0 \|\operatorname{div} B_1 u\|_E \text{ for } u \in X_q^m(A), t \in [0, T], x \in \mathbb{R}_+^n. \quad (3c)$$

Then $u \in X_q$ has a Helmholtz decomposition i.e., there exists a bounded linear projection operator P_q from X_q onto $X_{q\sigma}$ with null space

$$N(P_q) = \left\{ B_1 \varphi \in X_q : \varphi \in L_{loc}^q(\mathbb{R}_+^n; E) \right\}.$$

In particular, all $u \in X_q$ has a unique decomposition $u = u_0 + B_1 \varphi$ with $u_0 = P_q u \in X_{q\sigma}$ so that

$$\|B_1 \varphi\|_{X_q} + \|u_0\|_{X_q} \leq C \|u\|_{X_q}.$$

For proving the Theorem 2 we need the following lemma:

By reasoning as in ([12], Lemma 2) we get:

Lemma 1. $C_0^\infty(\mathbb{R}_+^n; E)$ is dense in $Y_q(A, B_1)$.

Here, $\langle, \rangle$ and $(,)$ denotes the duality pairing of abstract functions defined in $\mathbb{R}^{n-1}$ and $\mathbb{R}_+^n$, respectively. From [22] we have

Proposition 1. There exists a unique bounded linear operator $u \rightarrow \frac{\partial^i u}{\partial x_n^i}(x', 0)$, $i = 0, 1, \dots, m-1$ from $X_q^m(A)$, $q \in (1, \infty)$ onto

$$\left(W^{i, m, q}(E) \right)^* = \left(W^{m, q}(\mathbb{R}^{n-1}, (E(A^*), E^*)_{\theta_i, 1/q'}, E^*) \right)^n$$

such that

$$\langle u(x', 0), v(x', 0) \rangle = (B_2 u, v) + (u, B_1 v), v \in W^{m, q'}(\mathbb{R}_+^n, E(A), E)$$

and the following estimate holds

$$\left\| \frac{\partial^i u}{\partial x_n^i}(x', 0) \right\|_{W^{i, m, q}(E)} \leq C \left(\|u\|_{X_q} + \|B_2 u\|_{L^q(\mathbb{R}_+^n; E)} \right), \quad (3d)$$

where

$$\theta_i = \frac{i + \frac{1}{q}}{m}, \frac{1}{q} + \frac{1}{q'} = 1.$$

Proof. For $u \in Y_q(A, B_1)$ consider the linear form

$$T_u(v_i) = (B_2 u, \Phi) + (u, B_1 \Phi), \Phi \in X_{q'}^{2m-1}(A), \frac{\partial^i}{\partial x_n^i} \Phi(x', 0) = v_i. \quad (3e)$$

□

By virtue of trace theorem in $W^{s, q}(0, a; E(A), E)$, the interpolation of intersection and dual spaces (see e.g., ([22], §1.8.2, 1.12.1, 1.11.2)) and by localization argument we obtain that the operator

$v \rightarrow \frac{\partial^i}{\partial x_n^i} v(x', 0)$ is a bounded linear and surjective from $X_q^m(A)$ onto $W^{i,m,q}(E)$. Hence, we can find for each $v_i \in W^{i,m,q}(E)$ an element $\Phi \in X_q^{2m-1}(A)$ so that

$$\frac{\partial^i}{\partial x_n^i} \Phi(x', 0) = v_i, \quad \|\Phi\|_{X_q^{2m-1}(A)} \leq C \|v_i\|_{W^{i,q}(E)}.$$

Therefore, from (3e) we get

$$\begin{aligned} |T_u(v_i)| &\leq \left(\|u\|_{X_q} + \|B_2 u\|_{L^q(\mathbb{R}_+^n; E)} \right) \|\Phi\|_{X_q^{2m-1}(A)} \\ &\leq C \left(\|u\|_{X_q} + \|B_2 u\|_{L^q(\mathbb{R}_+^n; E)} \right) \|v_i\|_{W^{i,q}(E)}. \end{aligned}$$

This implies the existence of an element

$$\frac{\partial^i}{\partial x_n^i} u(x', 0) \in \left(W^{i,m,q}(E) \right)^* = \left(W^{m_i,q} \left(\mathbb{R}^{n-1}, (E(A^*), E^*)_{\theta_i, 1/q', E^*} \right) \right)^n$$

such that

$$\left\langle \frac{\partial^i}{\partial x_n^i} u(x', 0), v_i \right\rangle = T_u(v_i) \quad \text{for } v_i \in W^{i,q}(E)$$

and

$$\left\| \frac{\partial^i}{\partial x_n^i} u(x', 0) \right\|_{(W^{i,q}(E))^*} \leq C \left(\|u\|_{X_q} + \|B_2 u\|_{L^q(\mathbb{R}_+^n; E)} \right),$$

where

$$m_i = 1 - (2m - 1) \theta_i.$$

Thus, we have proved the existence of the operators $u \rightarrow \frac{\partial^i}{\partial x_n^i} u(x', 0)$. The uniqueness follows from Lemma 1.

Now we are going to construct the projection operator P_q . Let $f \in X_q$ and $f = (f_1(x), f_2(x), \dots, f_n(x))$. Consider the boundary value problem

$$Bu + Au = B_2 f(x), \quad x \in \mathbb{R}_+^n, \quad (3f)$$

$$L_k u = \sum_{i=0}^{\nu_k} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0) = 0, \quad k = 1, 2, \dots, m.$$

Since $B_2 f(x) \in X_q^{-m}$, in view of Corollary 1, then for all $f \in X_q$ problem (3f) has a unique solution $u_1 \in X_q^m(A)$ and the following estimate holds

$$\|u_1\|_{X_q^m(A)} \leq C \|B_2 f\|_{X_q^{-m}}.$$

Thus we have

$$\|u_1\|_{X_q^m(A)} \leq C \|f\|_{X_q}, \quad f - B_1 u \in X_q. \quad (3g)$$

Now consider the problem

$$Bu + Au = 0, \quad L_k u = v_k - L_k u_1, \quad x \in \mathbb{R}_+^n, \quad k = 1, 2, \dots, m. \quad (3h)$$

By Theorem 1, we obtain that for all $v_k \in W^{\theta_k, m, q}(E)$ problem (3h) has a unique solution $u_2 \in X_q^m(A)$ and the following estimate holds

$$\|u_2\|_{X_q^m(A)} \leq C \sum_{k=1}^n \|v_k - L_k u\|_{W_{k,m,q}(E)},$$

where $\theta_k = \frac{\nu_k + \frac{1}{q}}{m}$. For any $f \in X_q$, we take the solution of (3f), then that of (3h) and put $u = u_1 + u_2$. We define

$$P_q u = u - B_1 u.$$

Then by reasoning as in [12,16] we have

Lemma 2. Let E be an UDM space and $q \in (1, \infty)$. Then, $P_q X_q$ is a closed subspace of X_q .

Lemma 3. Let E be an UMD space and $q \in (1, \infty)$. Then, the operator P_q is a linear bounded operator in X_q and $P_q f = f$ if $B_1 f(x) = 0$.

Lemma 4. Assume E is an UMD space, A is an R -positive operator in E and $q \in (1, \infty)$. Then the conjugate of P_q is defined as $P_q^* = P_{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and this operator is bounded linear in $(L^{q'}(\mathbb{R}_+^n; E^*))^n$.

Let

$$W_q = \left\{ \nabla \varphi : \varphi \in W^{1,q}(\mathbb{R}_+^n; E) \right\}, \quad (P_q X_q)^\perp =$$

$$\left\{ f \in (L^{q'}(\mathbb{R}_+^n; E^*))^n, \langle f, v \rangle = 0 \text{ for any } v \in P_q X_q \right\}.$$

From Lemmas 3 and 4 we obtain

Lemma 5. Assume E is an UMD space and $q \in (1, \infty)$. Then

$$(P_q X_q)^\perp = W_{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Lemma 6. Assume E is an UMD space and $q \in (1, \infty)$. Then

$$X_{q\sigma}^\perp = W_{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Now we are ready to prove the Theorem 2.

Proof of Theorem 2. From Lemmas 5 and 6 we get that $X_{q\sigma} = (P_q X_q)^\perp$. Then, by construction of P_q we have

$$X_q = X_{q\sigma} \oplus W_q.$$

By Lemmas 2 and 3, we obtain the estimate (3a). Moreover, by Lemma 5, W_q is a close subspace of X_q . Then, it is known that the dual space of quotient space X_q/W_q is $W_q^\perp$. By first assertion we have $X_q/W_q = X_{q\sigma}$. $\square$

Theorem 3. Let E be an UMD space, A is an R -positive operator in E , $q \in (1, \infty)$. Then, problem (3a) and (3b) has a unique solution $u \in X_q^{2m}(A)$ for $f \in X_q$, $\varphi \in W^{m,q}(\mathbb{R}_+^n; E)$, $\lambda \in S_\psi$ and the following coercive uniform estimate holds

$$\sum_{k=1}^n \sum_{i=0}^{2m} \varepsilon_k^{\frac{i}{2m}} |\lambda|^{1-\frac{i}{2m}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \|Au\|_{X_q} + \|B_{1\varepsilon} \varphi\|_{X_q} \leq C \|f\|_{X_q} \quad (3i)$$

with $C = C(q, A)$ independent of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \lambda$ and f .

Proof. By applying the operator P_q to problem (2a) and (2b) we get the Stokes problem (3a) and (3b). It is clear to see that

$$D(O_{\varepsilon q}) = D(Q_\varepsilon) \cap X_{\sigma q},$$

where $O_{\varepsilon q}$ is the abstract Stokes operator generated by problem (3a) and (3b) and Q_ε is an abstract elliptic operator in X_q defined by (2e). $\square$

Then Theorem 2 we obtain the assertion.

Result 3. From the Theorem 3 we get that $Q_{\varepsilon q}$ is a positive operator in X_q and also generates a bounded holomorphic semigroup $S_\varepsilon(t) = \exp(-Q_{\varepsilon q}t)$ for $t > 0$. In a similar way as in [11] we show

Proposition 2. The following estimate holds

$$\|Q_{\varepsilon q}^\alpha S_\varepsilon(t)\| \leq Ct^{-\alpha},$$

uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ for $\alpha \geq 0$ and $t > 0$.

Proof. From Theorem 3 we obtain that the operator $O_{\varepsilon q}$ is uniformly positive in X_q , i.e., for $\lambda \in S_{\psi, \chi}$, $0 < \psi < \pi$ the following estimate holds

$$\|(Q_{\varepsilon q} + \lambda)^{-1}\| \leq M|\lambda|^{-1},$$

where the constant M is independent of λ and ε . Then, by using Danford integral and operator calculus (see e.g., in [10]) we obtain the assertion.

Now we can prove the main result of this section $\square$

Theorem 4. Let $0 < \varepsilon_k \leq 1$. Then, for $f \in L^p(0, T; X_q) = B(p, q)$ and $a \in (X_q^{2m}(A), X_q)_{\frac{1}{p}, p}$, $p, q \in (1, \infty)$ there is a unique solution $(u, B_{1\varepsilon}\varphi)$ of the problem (1d) and the following uniform estimate holds

$$\begin{aligned} & \left\| \frac{\partial u}{\partial t} \right\|_{B(p, q)} + \sum_{k=1}^n \left\| \varepsilon_k \frac{\partial^2 u}{\partial x_k^2} \right\|_{B(p, q)} + \|Au\|_{B(p, q)} + \|B_{1\varepsilon}\varphi\|_{L^{p, q}} \leq \\ & C \left(\|f\|_{B(p, q)} + \|a\|_{(X_q^{2m}(A), X_q)_{\frac{1}{p}, p}} \right) \end{aligned} \quad (3j)$$

with $C = C(T, p, q)$ independent of f and ε .

Proof. The problem (1d) can be expressed as the following abstract parabolic problem

$$\frac{du}{dt} + Q_{\varepsilon q}u = f(t), \quad u(0) = a. \quad (3k)$$

By Proposition 2, operator $O_{\varepsilon q}$ is uniform positive and generates holomorphic semigroup in X_q . Moreover, by using ([9], Theorem 3) we get that the operator $Q_{\varepsilon q}$ is R -positive in X_q uniformly with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. Since E is a UMD space, in a similar way as in ([20], Theorem 4.2) we obtain that for all $f \in L^p(0, T; X_q)$ and $a \in (X_q^{2m}(A), E)_{\frac{1}{p}, p}$ there is a unique solution $u \in W^{1, p}(0, T, D(O_{\varepsilon q}), E)$ of the problem (4b) so that the following uniform estimate holds

$$\left\| \frac{du}{dt} \right\|_{L^p(0, T; X_q)} + \|Q_{\varepsilon q}u\|_{L^p(0, T; X_q)} \leq \quad (3l)$$

$$C \left(\|f\|_{L^p(0,T;X_q)} + \|a\|_{(X_q^{2m}(A),E)_{\frac{1}{p},p}} \right).$$

From the estimates (3k) and (3l) we obtain the assertion. $\square$

Result 4. It should be noted that if $\varepsilon_1 = \varepsilon_2 = \dots \varepsilon_n = 1$ we obtain maximal regularity properties of abstract Stokes problem without any parameters in principal part.

Remark 2. There are a lot of positive operators in concrete Banach spaces. Therefore, putting in (1d) concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A , by virtue of Theorem 3 and 4 we can obtain the maximal regularity properties of different class of stationary and instationary Stokes problems, respectively, which occur in numerous physics and engineering problems.

4. Existence and Uniqueness for N–S Equation with Parameters

In this section, we study the N–S problem (1a)–(1c) in X_q . The problem (1a)–(1c) can be expressed as

$$\frac{du}{dt} + Q_{eq}u = Fu + P_q f, u(0) = 0, t > 0, Fu = -P_q(u, B_{1\varepsilon})u. \quad (4a)$$

We consider the Equation (4a) in integral form

$$u(t) = S_\varepsilon(t)a + \int_0^t S_\varepsilon(t-s) [Fu(s) + P_q f(s)] ds, t > 0. \quad (4b)$$

For proving the main result we need the following lemma which is obtained from ([11], Theorem 2).

Lemma 7. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. For any $0 \leq \alpha \leq 1$ the domain $D(Q_{eq}^\alpha)$ is the complex interpolation space $[X_q, D(Q_{eq})]_\alpha$, [1], §1).

Lemma 8. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. For each $k = 1, 2, \dots, n$ the operator $u \rightarrow Q_{eq}^{-\frac{1}{2}} P_q \left(\frac{\partial^m}{\partial x_k^m} \right) u$ extends uniquely to a uniformly bounded linear operator from $L^q(\mathbb{R}_+^n; E)$ to X_q .

Proof. Since Q_{eq} is a positive operator, it has a fractional powers O_{eq}^α . From the Lemma 7, It follows that the domain $D(Q_{eq}^\alpha)$ is continuously embedded in $(H_q^{2\alpha}(\mathbb{R}_+^n; E(A), E))^n \cap X_{q\sigma}$ for any $\alpha > 0$. Then by using the duality argument and due to uniform positivity of $O_{eq}^{\frac{1}{2}}$ we obtain the following uniform estimate

$$\left\| Q_{eq}^{-\frac{1}{2}} P_q \left(\frac{\partial^m}{\partial x_k^m} \right) u \right\|_{L^q(\mathbb{R}_+^n; E)} \leq C \|u\|_{X_q}. \quad (4c)$$

$\square$

By reasoning as in [10] we obtain the following

Lemma 9. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $0 \leq \delta < \frac{1}{2} + \frac{n}{2} \left(1 - \frac{1}{q} \right)$. Then the following uniform estimate holds

$$\left\| Q_{eq}^{-\delta} P_q(u, B_{1\varepsilon})v \right\|_q \leq M \left\| O_{eq}^\theta u \right\|_q \left\| O_{eq}^\sigma u \right\|_q$$

provided that $\theta > 0, \sigma > 0, \sigma + \delta > \frac{1}{2}$ and

$$\theta + \sigma + \delta > \frac{n}{2q} + \frac{1}{2}.$$

Proof. Assume that $0 < \nu < \frac{n}{2} \left(1 - \frac{1}{q}\right)$. Since $D(Q_\varepsilon^\alpha)$ is continuously embedded in $\left(H_q^{2\alpha}(\mathbb{R}_+^n; E(A), E)\right)^n \cap X_{q\sigma}$ and $L^{q'}(\mathbb{R}_+^n; E) \cap X_{q'}$ is the same as $X_{s'}$, by Sobolev embedding theorem we obtain that the operators

$$Q_{\varepsilon, q'}^{-\nu} : X_{q'} \rightarrow D(Q_{\varepsilon, q'}^\nu) \rightarrow X_{s'}$$

is bounded, where

$$\frac{1}{s'} = \frac{1}{q'} - \frac{2\nu}{n}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

By duality argument then, we get that the operator $u \rightarrow Q_{\varepsilon, q}^{-\nu}$ is bounded from X_s to X_q , where

$$\frac{1}{s} = 1 - \frac{1}{s'} = \frac{1}{q} + \frac{2\nu}{n}.$$

Consider first the case $\delta > \frac{1}{2}$. Since $P(u, B_{1\varepsilon})v$ is bilinear in u, v , it suffices to prove the estimate on a dense subspace. Therefore assume that u and v are smooth. Since $\operatorname{div} B_{1\varepsilon}u = 0$, we get

$$(u, B_{1\varepsilon})v = \sum_{k=1}^n \varepsilon_k^{\frac{1}{2}} \frac{\partial^m}{\partial x_k^m} (u_k v).$$

Taking $\nu = \delta - \frac{1}{2}$ and using the uniform boundedness of $Q_{\varepsilon, q}^{-\nu}$, from X_s to X_q and Lemma 8 for all $\varepsilon > 0$ we obtain the uniform estimate

$$\left\| Q_{\varepsilon q}^{-\delta} P(u, B_{1\varepsilon})v \right\|_q = \left\| \varepsilon_k Q_{\varepsilon, q}^{\frac{1}{2}-\nu} \sum_{k=1}^n P_q B_{1\varepsilon} (u_k v) \right\|_q \leq \|u\|_s \|v\|_s.$$

By assumption we can take r and η such that

$$\frac{1}{r} \geq \frac{1}{q} - \frac{2\theta}{n}, \quad \frac{1}{\eta} \geq \frac{1}{q} - \frac{2\sigma}{n}, \quad \frac{1}{r} + \frac{1}{\eta} = \frac{1}{s}, \quad r > 1, \quad \eta < \infty.$$

Since $D(Q_{\varepsilon, q}^\alpha)$ is continuously embedded in $\left(H_q^{2\alpha}(\mathbb{R}_+^n; E(A), E)\right)^n \cap X_{q\sigma}$, then by Sobolev embedding we get

$$\| |u| |v| \|_s \leq \|u\|_r \|v\|_\eta \leq M \left\| Q_{\varepsilon, q}^\theta u \right\|_r \left\| Q_{\varepsilon, q}^\sigma v \right\|_\eta,$$

i.e., we have the required result for $\delta > \frac{1}{2}$. In particular, we get

$$\left\| Q_{\varepsilon q}^{-\frac{1}{2}} P_q(u, B_{1\varepsilon})v \right\|_q \leq M \left\| Q_{\varepsilon, q}^\theta u \right\|_r \left\| Q_{\varepsilon, q}^\sigma v \right\|_\eta, \quad \theta + \sigma \geq \frac{n}{2q}, \quad \sigma > 0.$$

Similarly we obtain

$$\left\| P_q(u, B_{1\varepsilon})v \right\|_q \leq C \|u\|_r \|v\|_\eta \leq C \left\| Q_{\varepsilon, q}^\theta u \right\|_r \left\| Q_{\varepsilon, q}^{\beta+\frac{1}{2}} v \right\|_\eta$$

for $\frac{1}{r} + \frac{1}{\eta} = \frac{1}{q}$ and $\delta = 0$. The above two estimates show that the map $v \rightarrow P_q(u, B_{1\varepsilon})v$ is a uniform bounded operator from $D(Q_{\varepsilon q}^\beta)$ to $D(Q_{\varepsilon q}^{-\frac{1}{2}})$ and from $D(Q_{\varepsilon q}^{\beta+\frac{1}{2}})$ to X_q . By using the Lemma 7 and the interpolation theory of Banach spaces for $0 \leq \delta \leq \frac{1}{2}$ we obtain the uniform estimate

$$\|P_q(u, B_{1\varepsilon})v\|_q \leq C \|Q_{\varepsilon,q}^\theta u\|_r \|Q_{\varepsilon,q}^\sigma v\|_\eta.$$

□

By using Lemma 9 and iteration argument, by reasoning as in Fujita and Kato [13] we obtain the following.

Theorem 5. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $\gamma < 1$ be a real number and $\delta \geq 0$ such that

$$\frac{n}{2q} - \frac{1}{2} \leq \gamma, \quad -\gamma < \delta < 1 - |\gamma|.$$

Suppose that $a \in D(Q_{\varepsilon q}^\gamma)$, and that $\|Q_{\varepsilon q}^{-\delta} P f(t)\|$ is continuous on $(0, T)$ and satisfies

$$\|Q_{\varepsilon q}^{-\delta} P f(t)\| = o(t^{\gamma+\delta-1}) \text{ as } t \rightarrow 0.$$

Then there is $T_* \in (0, T)$ independent of ε and local solution of (4a) u such that $u \in C([0, T_*]; D(Q_{\varepsilon q}^\gamma))$, $u(0) = a$,

$u \in C((0, T_*]; D(Q_{\varepsilon q}^\alpha))$ for some $T_* > 0$, $\|Q_{\varepsilon q}^\alpha u(t)\| = o(t^{\gamma-\alpha})$ uniformly in ε as $t \rightarrow 0$ for all α with $\gamma < \alpha < 1 - \delta$. Moreover, the solution of (4a) is unique if $u \in C((0, T_*]; D(Q_{\varepsilon q}^\beta))$ and $\|Q_{\varepsilon q}^\alpha u(t)\| = o(t^{\gamma-\beta})$ uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ as $t \rightarrow 0$ for $\beta > |\gamma|$.

Proof. We introduce the following iteration scheme

$$u_0(t) = S_\varepsilon(t)a + \int_0^t S_\varepsilon(t-s)P f(s)ds,$$

$$u_{m+1}(t) = u_0(t) + \int_0^t S_\varepsilon(t-s)F u_m(s)ds, m \geq 0.$$

By estimating the term $u_0(t)$ in (4c) and by using the Lemma 9 for $\gamma \leq \alpha < 1 - \delta$ we get the uniform estimate

$$\|Q_{\varepsilon q}^\alpha u_0(t)\| \leq \|Q_{\varepsilon q}^\alpha S_\varepsilon(t)a\| + \int_0^t \|Q_{\varepsilon q}^{\alpha+\delta} S_\varepsilon(t-s)\| \|Q_{\varepsilon q}^{-\delta} P f(s)\| ds \leq$$

$$\|Q_{\varepsilon q}^\alpha S_\varepsilon(t)a\| + C_{\alpha+\delta} \int_0^t |(t-s)|^{-(\alpha+\delta)} \|Q_{\varepsilon q}^{-\delta} P f(s)\| ds \leq M_\alpha t^{\gamma-\alpha}$$

with

$$M_\alpha = \sup_{0 < t \leq T, \varepsilon > 0} t^{\alpha-\gamma} \|Q_{\varepsilon q}^{\alpha+\delta} S_\varepsilon(t)a\| + C_{\alpha+\delta} N B(1 - \delta - \alpha, \gamma + \alpha),$$

where $N = \sup_{0 < t \leq T} t^{1-\gamma-\delta} \|Q_{\varepsilon q}^{-\delta} P f(t)\|$ and $B(\beta, \eta)$ is the beta function. Here we suppose $\gamma + \delta > 0$. By induction assume that $u_m(t)$ satisfies the following

$$\|Q_{\varepsilon q}^{\alpha} u_m(t)\| \leq M_{\alpha m} t^{\gamma-\alpha}, \quad \gamma \leq \alpha < 1 - \delta. \quad (4d)$$

We shall estimate $Q_{\varepsilon q}^{\alpha} u_{m+1}(t)$ by using (4b). To estimate the term $\|Q_{\varepsilon q}^{-\delta} F u_m(s)\|$ we suppose

$$\theta + \sigma + \delta = 1 + \gamma, \quad \gamma < \theta < 1 - \delta, \quad \gamma < \sigma < 1 - \delta,$$

$$\theta > 0, \quad \sigma > 0, \quad \delta + \sigma > \frac{1}{2},$$

so that the numbers θ, σ, δ satisfy the assumptions of Lemma 9. Using Lemma 9 and (4d), we get

$$\|Q_{\varepsilon q}^{-\delta} F u_m(s)\| \leq C M_{\theta m} M_{\sigma m} s^{\gamma+\delta-1}.$$

Therefore, we obtain

$$\begin{aligned} \|Q_{\varepsilon q}^{\alpha} u_m(t)\| &\leq M_{\alpha} t^{\gamma-\alpha} + M_{\alpha+\delta} \int_0^t \|(t-s)\|^{-(\alpha+\delta)} \|Q_{\varepsilon q}^{-\delta} F u_m(s)\| ds \leq \\ &M_{\alpha m+1} t^{\gamma-\alpha} \end{aligned}$$

with

$$M_{\alpha m+1} = M_{\alpha} + M_{\alpha+\delta} M B(1 - \delta - \alpha, \gamma + \delta) M_{\theta m} M_{\sigma m}.$$

We get the uniform estimate. So, the remaining part of proof is obtained the same as in ([10], Theorem 2.3). $\square$

By reasoning as in [13] we obtain

Lemma 10. *Let the operator A_{ε} be uniform positive in a Banach space E and α be a positive number with $0 < \alpha < 1$. Then, the following uniform inequality holds*

$$\|A_{\varepsilon}^{\alpha} (e^{-A_{\varepsilon} t} - I) u\|_E \leq \frac{t^{\alpha}}{\alpha} \|A_{\varepsilon}^{\alpha} u\|_E$$

for all $u \in E$.

Proposition 3. *Let E be a space satisfying a multiplier condition, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let u be the solution given by Theorem 5 Then $Q_{\varepsilon q}^{\alpha} u$ for $\gamma < \alpha < 1 - \delta$ is uniform Hölder continuous on every interval $[\eta, T_*]$, $0 < \eta < T_*$ for all parameters $\varepsilon_k > 0$.*

Proof. It suffices to prove the Hölder continuity of $Q_{\varepsilon}^{\alpha} v$, where

$$v(t) = \int_0^t S_{\varepsilon}(t-s) [F u(s) + P f(s)] ds.$$

Using the Lemma 10 we get the uniform estimate

$$\|(e^{-h Q_{\varepsilon q}} - I) Q_{\varepsilon q}^{-\alpha}\|_{B(E)} \leq \frac{h^{\alpha}}{\alpha}, \quad h > 0.$$

Then by reasoning as in ([10], Proposition 2.4) we obtain the assertion. $\square$

Theorem 6. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Assume $P_q f : (0, T_*] \rightarrow X_q$ is Hölder continuous on each subinterval $[\eta, T_*]$. Then, the solution of (4b) given by Theorem 5 satisfies Equation (4a) for all $\varepsilon_k > 0$. Moreover, $u \in D(O_{\varepsilon q})$ for $t \in (0, T_*)$.

Proof. It suffices to show Hölder continuity of $Fu(t)$ on each interval $[\eta, T_*]$. It is clear to see that $u(\eta) \in X_q$ and

$$u(t) = S_\varepsilon(t)u(\eta) + \int_0^t S_\varepsilon(t-s)[Fu(s) + P_q f(s)]ds, \quad t \in [\eta, T_*].$$

Since $P_q f$ is continuous on $[\eta, T_*]$ we get

$$\|P_q f(t)\| = o(t - \eta)^{-\alpha}, \quad t \rightarrow \eta, \alpha > 0.$$

The uniqueness of $u(t)$ ensured by Theorem 5, implies the following uniform estimates

$$C([\eta, T_*]; D(Q_{\varepsilon q}^\nu)) \cap C([\eta, T_*]; D(Q_{\varepsilon q}^\alpha)),$$

$$Q_{\varepsilon q}^\alpha \|u(t)\| = o(t - \eta)^{\nu - \alpha}, \quad t \rightarrow \eta, \nu < \alpha < 1,$$

where $\nu = \max\{\gamma, 0\}$. So, by Proposition 4, $Q_{\varepsilon q}^\alpha u(t)$ is continuous on every subinterval $[\eta, T_*]$. Since we can choose θ, σ so that

$$\theta + \sigma = 1 + \nu, \nu < \theta < 1, \max\left\{\gamma, \frac{1}{2}\right\} < \sigma < 1.$$

□

Lemma 8 implies that $Fu(t)$ is Hölder continuous on every interval $[\eta, T_*]$.

5. Regularity Properties

The purposes of this section is to show that the solutions of (1a) are smooth if the data are smooth. For simplicity, we assume $P_q f = 0$. The proof when $P_q f \neq 0$ is the same. Consider first all of the Stokes problem (3d) and (3e).

By reasoning as in ([13], Lemma 2.14) we obtain

Lemma 11. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $f \in C^\mu([0, T]; X_q)$ for some $\mu \in (0, 1)$. Then for every $\eta \in (0, \mu)$ we have

$$v(t) = \int_0^t S_\varepsilon(t-s)f(s)ds \in C^\eta((0, T]; D(Q_{\varepsilon q})) \cap C^{1+\eta}((0, T]; X_q).$$

In a similar way as Lemmas 2, 5 and 6 in [10] we obtain, respectively:

Lemma 12. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. For $u, v \in W^{m,q}(\mathbb{R}_+^n; E(A), E)$, $q \in (1, \infty)$ the following hold:

(1)

$$P_q u \in W^{m,q}(\mathbb{R}_+^n; E(A), E) \cap X_{q\sigma}, \quad \|P_q u\|_{W^{m,q}(\mathbb{R}_+^n; E)} \leq C_{m,q} \|u\|_{W^{m,q}(\mathbb{R}_+^n; E)};$$

(2) for $m > \frac{n}{q}$ there exists a constant $C_{m,q}$ such that

$$\|P_q(u, B_{1\varepsilon})v\|_{W^{m,q}(\mathbb{R}_+^n; E)} \leq C_{m,q} \|u\|_{W^{m,q}(\mathbb{R}_+^n; E)} \|v\|_{W^{m,q}(\mathbb{R}_+^n; E)};$$

(3) when $q > n$ we have

$$\|P_q(u, B_{1\varepsilon})v\|_{L^q(\mathbb{R}_+^n; E)} \leq C_q \|u\|_{W^{1,q}(\mathbb{R}_+^n; E)} \|v\|_{W^{1,q}(\mathbb{R}_+^n; E)}.$$

Lemma 13. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $u = u_\varepsilon(t)$ be solution of (4b) for $P_q f = 0$, then $u \in C^\mu((0, T]; D(Q_{\varepsilon q}))$ and $\frac{du}{dt} \in C^\mu((0, T]; X_q)$ for $\mu \in (0, \frac{1}{2})$. Moreover,

$$Fu \in C^\mu((0, T]; W^{1,q}(\mathbb{R}_+^n; E(A), E)).$$

Lemma 14. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $u = u_\varepsilon(t)$ be solution of (4b) for $P_q f = 0$, then $u \in C^\mu((0, T]; D(Q_{\varepsilon q}^{\frac{1}{2}}))$ for $\mu \in (0, \frac{1}{2})$.

Now, by reasoning as in ([10], Proposition 3.5) we can state the following

Proposition 4. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let E be Banach algebra, $q > n$ and $a \in X_q$. Suppose that the solution $u = u_\varepsilon(t)$ of (4b) for $P_q f = 0$ given by Theorem 5 exists on $[0, T]$. Then $u \in C^\infty(\mathbb{R}_+^n \times [0, T]; E)$.

Proof. The solution $u = u_\varepsilon(t)$ of (4b) for $P_q f = 0$ given by Theorem 5 is expressed as

$$u(t) = S_\varepsilon(t)a + \int_0^t S_\varepsilon(t-s)Fu(s)ds, \quad t > 0, \quad (5a)$$

where $Fu = -P_q(u, \nabla)u$. From (5a) we get

$$Q_{\varepsilon q}^{\frac{1}{2}}u(t) = S_\varepsilon(t-\eta)Q_{\varepsilon q}^{\frac{1}{2}}u(\eta) + \int_\eta^t Q_{\varepsilon q}S_\varepsilon(t-s)Q_{\varepsilon q}^{-\frac{1}{2}}Fu(s)ds, \quad t > 0 =$$

$$S_\varepsilon(t-\eta)Q_{\varepsilon q}^{\frac{1}{2}}u(\eta) + v(t), \quad v(t) = v_\varepsilon(t) = \int_\eta^t Q_{\varepsilon q}S_\varepsilon(t-s)Q_{\varepsilon q}^{-\frac{1}{2}}Fu(s)ds.$$

Since $S_\varepsilon(t-\eta)Q_{\varepsilon q}^{\frac{1}{2}}u(\eta) \in C^\infty((\delta, T]; X_q)$ and $0 < \eta < T$, we will examine only $v(t)$. Integrating by parts, we obtain

$$v(t) = \int_\eta^t \frac{d}{ds}S_\varepsilon(t-s)Q_{\varepsilon q}^{-\frac{1}{2}}Fu(s)ds = \varepsilon Q_{\varepsilon q}^{-\frac{1}{2}}Fu(t) - \quad (5b)$$

$$S_\varepsilon(t-\eta)Q_{\varepsilon q}^{\frac{1}{2}}Fu(\delta) - \int_\eta^t S_\varepsilon(t-s)Q_{\varepsilon q}^{-\frac{1}{2}}\frac{d}{ds}(Fu)(s)ds.$$

Moreover, since $u(s) \in D(Q_{\varepsilon q})$ for all $\varepsilon_k > 0, 0 < s \leq T$, we have

$$(Fu)(s) = -\sum_{k=1}^n P_q\left(\varepsilon_k^{\frac{1}{2}}\frac{\partial^m}{\partial x_k^m}\right)[u_k(s)u(s)],$$

where

$$u(s) = (u_1(s), u_2(s), \dots, u_n(s)), u_k = u_{k\varepsilon}.$$

Hence, by Lemma 7 we get the following uniform estimate

$$\begin{aligned} \left\| Q_{\varepsilon q}^{-\frac{1}{2}} \frac{d}{ds} F u \right\|_{X_q} &= \left\| \sum_{k=1}^n Q_{\varepsilon q}^{-\frac{1}{2}} P_q \left(\varepsilon_k^{\frac{1}{2}} \frac{\partial^m}{\partial x_k^m} \right) \left[\frac{du_{k\varepsilon}}{ds} u_\varepsilon + u_{k\varepsilon} \frac{du_\varepsilon}{ds} \right] \right\|_{X_q} \\ &\leq C \|u_\varepsilon\|_{L^\infty(\mathbb{R}_+^n; E)} \left\| \frac{du_\varepsilon}{ds} \right\|_{X_q} \leq C \left\| Q_{\varepsilon q}^{\frac{1}{2}} u_\varepsilon \right\|_{X_q} \left\| \frac{du_\varepsilon}{ds} \right\|_{X_q}. \end{aligned}$$

This estimates together with Lemma 13 shows that

$$Q_{\varepsilon q}^{-\frac{1}{2}} \frac{d}{ds} F u \in C^\mu((0, T]; X_q).$$

Lemmas 11 and 12 now imply that

$$\frac{dv}{dt} \in C^\mu((0, T]; X_q).$$

Since $D\left(Q_{\varepsilon q}^{\frac{1}{2}}\right) \subset \left(W^{1,q}(\mathbb{R}_+^n; E(A), E)\right)^n$, Corollary 5.1, Lemmas 5.3, 5.4 and the identity $u(t) = Q_{\varepsilon q}^{\frac{1}{2}}\left(Fu - \frac{du}{dt}\right)$ imply

$$u \in C^\mu\left((0, T]; \left(W^{3,q}(\mathbb{R}_+^n; E(A), E)\right)^n\right).$$

Then the proof will be completed as in ([10], Proposition 3.5) by using the induction. $\square$

Now we can state the main result of this section

Theorem 7. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let E be Banach algebra and $a \in X_q$. Suppose that the solution $u = u_\varepsilon(t)$ of (4b) for $P_q F = 0$ given by Theorem 5 exists on $[0, T]$. Then $u \in C^\infty(\mathbb{R}_+^n \times [0, T_*]; E)$.

Proof. For $q > n$ the assertion is obtained from the Proposition 4. Let us show that the assertion is valid for $1 < q \leq n$. Indeed, the solution $u = u_\varepsilon(t)$ of (5b) for $P_q F = 0$ given by Theorem 5 satisfies the Equation (5a) on every subinterval $[\eta, T_*]$, $0 < \eta < T$. Theorem 6 shows that $u_\varepsilon(\eta) \in D(Q_{\varepsilon q})$. Since $0 \leq \frac{n}{2q} - \frac{1}{2} \leq \gamma < 1$, we have $D(Q_{\varepsilon q}^\gamma) \subset X_n$ so that $D(Q_{\varepsilon q}) \subset X_s$ for some $s > n$. By (4b) this means that we may assume $q > n$ and $a \in X_q$. $\square$

6. Existence of Global Solutions

In this section, we prove the existence and estimate of a global solution of the problem (1a)–(1c). The proofs of these theorems are based on the theory of holomorphic semigroups and fractional powers of generators. We assume for simplicity that $f = 0$, although it is not difficult to include nonzero f under appropriate conditions. The main result is the following

Theorem 8. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$ and $a \in X_q$. There exists a $T > 0$ and a unique solution $u = u_\varepsilon$ of (1a)–(1c) so that $t^{(1-\frac{n}{q})/2} u \in C([0, T]; X_q)$ for $n \leq q \leq \infty$ and $t^{(1-\frac{n}{2q})} B_{1\varepsilon} u \in C([0, T]; X_q)$ for $n \leq q < \infty$. Moreover, the following estimates hold

$$\sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{q})/2} u_\varepsilon \right\|_{X_q} \leq C, \quad \sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \right\|_{X_q} \leq C.$$

Proof. The solution $u = u_\varepsilon(t)$ of (4b) for $Pf = 0$ given by Theorem 5 is expressed as

$$u(t) = u_0(t) + G_\varepsilon u(t), \quad (6a)$$

where,

$$u_0(t) = S_\varepsilon(t)a, \quad G_\varepsilon u(t) = \int_0^t S_\varepsilon(t-s)Fu(s)ds, \quad t > 0.$$

By applying the generalized Minkovskii inequality and by Proposition 1 we can see that

$$\|S_\varepsilon(t)u\|_{X_p} \leq C\varepsilon_k^{\frac{n}{2}(1+\frac{1}{p})} t^{-\frac{n}{2}(1-\frac{1}{p})} \|u\|_{X_p}, \quad k = 1, 2, \dots, n.$$

By using the above estimate we get

$$\|S_\varepsilon(t)u\|_{X_q} \leq C\varepsilon_k^{\frac{n}{2}(2+\frac{1}{q}-\frac{1}{p})} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u\|_{X_p}, \quad (6b)$$

$$\|B_{1\varepsilon}S_\varepsilon(t)u\|_{X_q} \leq C\varepsilon_k^{\frac{n}{2}(\frac{3}{2}-\frac{1}{p}-\frac{1}{q})} t^{-(1+\frac{n}{2}(\frac{1}{p}-\frac{1}{q}))} \|u\|_{X_p} \text{ for } 1 < p \leq q < \infty. \quad (6c)$$

Moreover, by using (6a), (6b) and by applying the Hölder inequality we get

$$\|F(u, v)\|_{X_q} \leq C \|u\|_{L^r} \|\nabla v\|_{L^s}, \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{s}. \quad (6d)$$

Then in view of (6a) and (6d) we obtain the following uniform estimate

$$\|G_\varepsilon u\|_{(L^{m/\gamma})^n} \leq C \int_0^t (t-s)^{-(\alpha+\beta-\gamma)/2} \|u(s)\|_{m/\alpha} \|B_{1\varepsilon}u(s)\|_{m/\beta} ds, \quad (6e)$$

$$\|B_{1\varepsilon}G_\varepsilon u\|_{(L^{m/\gamma})^n} \leq C \int_0^t (t-s)^{-(1+\alpha+\beta-\gamma)/2} \|u(s)\|_{m/\alpha} \|B_{1\varepsilon}u(s)\|_{m/\beta} ds, \quad (6f)$$

where

$$\alpha, \beta, \gamma > 0, \quad \gamma \leq \alpha + \beta < n.$$

Then solving the Equation (6a) by successive approximation, starting with $u_0 = S_\varepsilon(t)a$ we get

$$u_{k+1} = u_0 + G_\varepsilon u_k, \quad u_k = u_{k\varepsilon}(t), \quad k = 0, 1, 2, \dots \quad (6g)$$

First by reasoning as in ([1], Theorem 1) and by using (6c)–(6e) we show by induction that $u_k = u_{k\varepsilon}$ exists, moreover,

$$t^{(1-\delta)/2}u_{k\varepsilon} \in C([0, \infty); (L^{n/\delta}(\mathbb{R}_+^n; E))^n),$$

$$t^{1/2}B_{1\varepsilon}u_{k\varepsilon} \in C([0, \infty); (L^n(\mathbb{R}_+^n; E))^n)$$

and for $\delta \in (0, 1)$ the following uniform estimates hold

$$\sup_{t, \varepsilon_k} \|t^{(1-\delta)/2}u_{k\varepsilon}\|_{L^{n/\delta}} \leq M_k, \quad \sup_{t, \varepsilon_k} \|t^{1/2}B_{1\varepsilon}u_{k\varepsilon}\|_{L^q} \leq M'_k. \quad (6h)$$

By applying (6c)–(6e) for $q = n$ and $p = \frac{n}{\delta}$ we have

$$M_0 = M'_0 = C \|a\|_{(L^n(\mathbb{R}_+^n; E))^n}, \quad (6i)$$

where C is a positive constant. From (6e) and (6g) for $n \leq p < \infty$ we obtain

$$\|u_{\varepsilon k+1}\|_{X_p} \leq \|u_{\varepsilon 0}\|_{X_p} \leq CM_k M'_k \int_0^t (t-s)^{-(1+\delta-n/q)/2} s^{-(1-\delta/2)} ds \leq Mt^{-(1-n/q)/2}.$$

It follows that $u_{\varepsilon k}(t)$ converges to a limit function u_ε uniformly with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, moreover, $u_\varepsilon \in C([0, T]; L^n(\mathbb{R}_+^n; E)^n)$ for $p = n$ and u_ε satisfies (6a) for $n < p < \infty$. $\square$

Theorem 9. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. There is a $\mu > 0$ such that if $\|a\|_{X_q} < \mu$, then there exists a global solution u_ε of (1a) – (1c), so that $t^{(1-\frac{n}{q})/2} u_\varepsilon \in C([0, \infty); X_q)$ for $n \leq q \leq \infty$, $t^{(1-\frac{n}{q})/2} u_\varepsilon$ and $t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \in C([0, \infty); X_q)$ for $n \leq q < \infty$. Moreover, the following uniform estimates hold

$$\sup_{t, \varepsilon_k} \left\| t^{(1-\frac{n}{q})/2} u_\varepsilon \right\|_{X_q} \leq C, \quad \sup_{t, \varepsilon_k} \left\| t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \right\|_{X_q} \leq C. \quad (6j)$$

Proof. It is clear to see from proof of Theorem 6.1 that M_k and M'_k are bounded by a constant M if $M_0 \leq \lambda$. By (7i) this is true if $\|a\|_{X_q}$ is sufficiently small. In this case, as in [15] we prove that the sequences $t^{(1-\delta)/2} u_{\varepsilon k}$, $t^{1/2} B_{1\varepsilon} u_{\varepsilon k}$ are bounded on $(0, \infty)$ uniformly in k and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ i.e.,

$$\sup_{t, \varepsilon_k} \left\| t^{(1-\delta)/2} u_{\varepsilon k} \right\|_{(L^{n/\delta})^n} \leq M_1, \quad \sup_{t, \varepsilon_k} \left\| t^{1/2} B_{1\varepsilon} u_{\varepsilon k} \right\|_{X_q} \leq M_2. \quad (6k)$$

Then (6k) is obtained from (6j). $\square$

Remark 3. Let E be a UMD space, A an R -positive operator in E , $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Theorem 9 shows that L^p norms of $u_\varepsilon(t)$ decay as $t \rightarrow \infty$ for $p > q$ uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$.

For $p = q$ we obtain the following result.

Theorem 10. Let the all conditions of Theorem 9 hold. Then $\|u_\varepsilon(t)\|_{X_p} \rightarrow 0$ uniformly in ε as $t \rightarrow \infty$. More precisely, we have

$$\|u_\varepsilon(t) - u_{0\varepsilon}(t)\|_{X_p} = O\left(t^{-\frac{\delta}{2}}\right) \text{ as } t \rightarrow \infty,$$

where, $u_{0\varepsilon}(t) = S_\varepsilon(t)a$ and $\delta < \min\left\{1, n - \frac{n}{q}, \frac{n}{q} - 1\right\}$.

7. The Wentzell–Robin Type Mixed Problem for Navier–Stokes Equations

Consider the N–S problem (1f)–(1h). Let

$$\tilde{\Omega} = \mathbb{R}_+^n \times (0, 1), \quad X_{\mathbf{p}} = (L^{\mathbf{p}}(\tilde{\Omega}))^n, \quad X_{\mathbf{p}}^{2m,2} = W^{2m,2,\mathbf{p}}((\tilde{\Omega}))^n.$$

$$B(\mathbf{p}, q) = L^q(0, T; X_{\mathbf{p}}), \quad \mathbf{p} = (p_1, p), \quad p_1, p, q \in (1, \infty)$$

denotes the product of Lebesgue spaces with corresponding mixed norm and $(X_{\mathbf{p}}^{2m,2}, X_{\mathbf{p}})_{\frac{1}{p}, p}$ denote real interpolation space between $Y_{\mathbf{p}}^{2m,2}$ and $X_{\mathbf{p}}$.

Theorem 11. Suppose a is positive, b is a real-valued functions on $\tilde{\Omega}$. Moreover, $a(x, \cdot) \in C[0, 1]$ for all $x \in \mathbb{R}_+^n$, $a(\cdot, y) \in C_b(\mathbb{R}_+^n)$ for all $y \in [0, 1]$ and $b(x, y)$ is a bounded function on $\tilde{\Omega}$ with

$$\exp \left(- \int_{\frac{1}{2}}^x b(x, y) a^{-1}(x, y) dy \right) \in L_1(0, 1), \text{ for } x \in \mathbb{R}_+^n.$$

Let $0 < \varepsilon_k \leq 1$ and $a \in (X_{\mathbf{p}}^{2m,2}, X_{\mathbf{p}})_{\frac{1}{p}, p}$ for $p, p_1 \in (1, \infty)$. Then, there exists a $T > 0$ and a unique solution $u = u_\varepsilon$ of (1f)–(1h) so that $t^{(1-\frac{n}{q})/2} u \in C([0, T]; B(\mathbf{p}, q))$ for $n \leq p \leq \infty$ and $t^{(1-\frac{n}{2q})} B_{1\varepsilon} u \in C([0, T]; B(\mathbf{p}, q))$ for $n \leq p < \infty$. Moreover, the following uniform estimates hold

$$\sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{q})/2} u_\varepsilon \right\|_{B(\mathbf{p}, q)} \leq C, \quad \sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \right\|_{B(\mathbf{p}, q)} \leq C.$$

Proof. Let $E = L^{p_1}(0, 1)$. It is known [19] that $L^{p_1}(0, 1)$ is an UMD space for $p_1 \in (1, \infty)$. Consider the operator A defined by

$$D(A) = W^{2,p_1}(\tilde{\Omega}; B_j u = 0), \quad Au = a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y}.$$

Therefore, the problem (1f)–(1h) can be rewritten in the form of (1a)–(1c), where $u(x) = u(x, \cdot)$, $f(x) = f(x, \cdot)$ are functions with values in $E = L^{p_1}(0, 1)$. From [23,24] we get that the operator A generates analytic semigroup in $L^{p_1}(0, 1)$. Moreover, we obtain that the operator A is R -positive in L^{p_1} . Then from Theorem 8 we obtain the assertion. $\square$

8. The Mixed Problem for Degenerate Navier–Stokes Equation

Consider the N–S problem (1j)–(1l). Let

$$\tilde{\Omega} = \mathbb{R}_+^n \times (0, b), \quad X_{\mathbf{p}} = (L^p(\tilde{\Omega}))^n, \quad X_{\mathbf{p}}^{2m,2} = W^{2m,[2],\mathbf{p}}((\tilde{\Omega}))^n.$$

The main aim of this section is to prove the following result:

Theorem 12. Let $0 < \varepsilon_k \leq 1$ and $a \in (X_{\mathbf{p}}^{2m,[2]}, X_{\mathbf{p}})_{\frac{1}{p}, p}$ for $p, p_1 \in (1, \infty)$. Suppose $b_1(\cdot, y) \in C(\mathbb{R}_+^n)$ for all $y \in [0, b]$, $b_1(x, \cdot) \in C([0, b])$ for all $x \in \mathbb{R}_+^n$, $b_1(\cdot, \cdot) \in C[0, 1]$ and $b_2(x, y)$ is a bounded function on $\tilde{\Omega}$. Moreover, assume $0 \leq \gamma < \frac{1}{2}$, $\alpha_{10}\beta_{20} - \alpha_{20}\beta_{10} \neq 0$, $\alpha_{20}\beta_{11} + \alpha_{21}\beta_{10} - \alpha_{10}\beta_{21} - \alpha_{11}\beta_{20} \neq 0$, $\alpha_{11}\beta_{21} + \alpha_{21}\beta_{11} \neq 0$, $\alpha_{11}\beta_{21} - \alpha_{11}\alpha_{21} \neq 0$, $\alpha_{11} \neq \beta_{11}$ for $v_k = 1$ and $|\alpha_{k0}| + |\beta_{k0}| > 0$, $\alpha_{10}\alpha_{20} + \beta_{10}\beta_{20} \neq 0$ for $v_k = 0$. Then, there exists a $T > 0$ and a unique solution $u = u_\varepsilon$ of (1j)–(1l) so that $t^{(1-\frac{n}{q})/2} u \in C([0, T]; B(\mathbf{p}, q))$ for $n \leq p \leq \infty$ and $t^{(1-\frac{n}{2q})} B_{1\varepsilon} u \in C([0, T]; B(\mathbf{p}, q))$ for $n \leq p < \infty$. Moreover, the following uniform estimates hold

$$\sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{q})/2} u_\varepsilon \right\|_{B(\mathbf{p}, q)} \leq C, \quad \sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{n}{2q})} B_{1\varepsilon} u_\varepsilon \right\|_{B(\mathbf{p}, q)} \leq C.$$

Proof. Let $E = L^{p_1}(0, b)$. It is known [19] that $L^{p_1}(0, b)$ is an UMD space for $p_1 \in (1, \infty)$. Consider the operator A defined by

$$D(A) = W^{[2],p_1}(\Omega; L_k u = 0), \quad Au = b_1 \frac{\partial^{[2]} u}{\partial y^2} + b_2 \frac{\partial^{[1]} u}{\partial y}.$$

Therefore, the problem (1j)–(1l) can be rewritten in the form of (1a)–(1c), where $u(x) = u(x, \cdot)$, $f(x) = f(x, \cdot)$ are functions with values in $E = L^{p_1}(0, b)$. From [7] we get that the operator A generates analytic semigroup in $L^{p_1}(0, b)$. Moreover, we obtain that the operator A is R -positive in L^{p_1} . Then from Theorem 8 we obtain the assertion. $\square$

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