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Asymptotics and Uniqueness of Solutions of the Elasticity System with the Mixed Dirichlet–Robin Boundary Conditions

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Abstract: We study properties of generalized solutions of the Dirichlet–Robin problem for an elasticity system in the exterior of a compact, as well as the asymptotic behavior of solutions of this mixed problem at infinity, with the condition that the energy integral with the weight $|x|^a$ is finite. Depending on the value of the parameter a, we have proved uniqueness (or non-uniqueness) theorems for the mixed Dirichlet–Robin problem, and also given exact formulas for the dimension of the space of solutions. The main method for studying the problem under consideration is the variational principle, which assumes the minimization of the corresponding functional in the class of admissible functions.

Keywords: asymptotics; elasticity system; Dirichlet–Robin boundary conditions; weighted dirichlet integral; sobolev spaces

MSC: 35J57; 35J50; 35B40

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1. Introduction

Dedicated to the blessed memory of my parents who went to heaven this year.

Let Ω be an unbounded domain in $\mathbb{R}^n, n \geq 2$, $\Omega = \mathbb{R}^n \setminus \overline{G}$ with the boundary $\partial\Omega \in C^1$, where G is a bounded simply connected domain (or a union of finitely many such domains) in $\mathbb{R}^n$, $\Omega \cup \partial\Omega = \overline{\Omega}$ is the closure of Ω, $x = (x_1, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $u = (u_1, \ldots, u_n)$.

In the domain Ω, we consider the linear system of elasticity theory

$$Lu \equiv (Lu)_i = \sum_{j,k,h=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kh}^{ij}\frac{\partial u_j}{\partial x_h}\right) = 0, \quad i = 1, \ldots, n. \tag{1}$$

Here and in what follows, we assume summation from 1 to n over repeated indices. We also assume that the coefficients are constant and the following conditions hold:

$$a_{kh}^{ij} = a_{hk}^{ji} = a_{ih}^{kj}, \quad \lambda_1|\xi|^2 \leq a_{kh}^{ij}\xi_k^i\xi_h^j \leq \lambda_2|\xi|^2,$$

where ξ is an arbitrary symmetric matrix, $\{\xi_k^i\}$, $\xi_k^i = \xi_i^k$, $|\xi|^2 = \xi_k^i\xi_k^i$, λ_1, λ_2 are positive constants.

We consider the following boundary-value problem for the system (1): find a vector-valued function u that satisfies (1) in Ω along with the homogeneous Dirichlet–Robin boundary conditions

$$u\big|_{\Gamma_1} = 0, \quad (\sigma(u) + \tau u)\big|_{\Gamma_2} = 0, \tag{2}$$

where $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $mes_{n-1}\Gamma_1 \neq 0$, $\sigma(u) = (\sigma_1(u), \ldots, \sigma_n(u))$, $\sigma_i(u) \equiv a_{kh}^{ij} \frac{\partial u_j(x)}{\partial x_h} \nu_k$, $i = 1, \ldots, n$, $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal vector to $\partial\Omega$, τ is an infinitely differentiable function on $\partial\Omega$ with uniformly bounded derivatives, and $\tau \geq 0$, $\tau \not\equiv 0$.

General boundary value problems for elliptic systems in domains with smooth boundaries were studied in [1–4]. Boundary value problems for the elasticity system in bounded domains are quite well studied. A presentation of the basic facts of this theory can be found in Fichera's monograph [5]. In [6–8], Kondratiev and Oleinik established generalizations of Korn's inequality and Hardy's inequality for bounded domains and a large class of unbounded domains, and applied these to investigate the main boundary value problems for the elasticity system, which were also considered in [9,10]. The paper [10] uses Korn's inequality and Hardy's inequality to study the uniqueness and stability of generalized solutions of mixed boundary value problems for the elasticity system in an unbounded domain provided that $E(u, \Omega)$ is finite.

In [11,12], shells of variable thickness are considered in three-dimensional Euclidean space around surfaces that have a limited principal curvature. Here the author derives the Korn interpolation inequality, the inequality also introduced in [13], and the second Korn inequality in domains in which no boundary or normalization conditions are imposed on the vector function u. The constants in the estimates are asymptotically optimal in terms of the thickness of the region. Note that this is the first paper that defines the asymptotic behavior of the optimal constant in the classical Korn second inequality for shells over the thickness of the domain in almost complete generality, and the inequality holds for almost all thin domains. In [14], the author extends the L^2 Korn interpolation inequality, as well as the second Korn inequalities, in thin domains, proved in [12], to the space L^p for any $1 < p < \infty$. Note the paper [15], in which the authors prove asymptotically sharp weighted Korn and Korn-type inequalities in thin domains with singular weights. The choice of weights is based on some factors; in particular, the spatial case arises when transforming Cartesian variables to polar change of variables in two dimensions.

In [16], a regularity result is proved for a system of linear elasticity theory with mixed boundary conditions on a curved polyhedral domain in weighted Sobolev spaces, for which the weight is determined by the distance to the set of edges. These results are then extended to other strongly elliptic systems and higher dimensions.

In [17,18] methods are proposed that allow one to construct the asymptotics of solutions of the Laplace and poly-Laplace equations in a neighborhood of singular points, which are zero and infinity, as well as the asymptotics of these equations on manifolds with singularities. In [19], asymptotics were constructed for the solution of the Laplace equation on manifolds with a beak-type singularity in a neighborhood of the singular point.

We also note the papers [20–22], in which the basic boundary value problems and problems with mixed boundary conditions for the biharmonic (polyharmonic) equation are studied. In particular, the existence and uniqueness of solutions in the ball were established, and necessary and sufficient conditions for the solvability of boundary value problems for the biharmonic (polyharmonic) equation, including those with a polynomial right-hand side, were obtained.

It is well known that if Ω is unbounded, then one must also characterize the behavior of a solution at infinity. This is usually done by requiring that the Dirichlet integral $D(u, \Omega)$ or the energy integral $E(u, \Omega)$ be finite, or a condition on the nature of the decay of the modulus of a solution as $|x| \to \infty$.

In this paper we study the properties of generalized solutions of the mixed Dirichlet–Robin problem for the elasticity system in an unbounded domain Ω with a finiteness condition of the weighted energy integral:

$$E_a(u, \Omega) \equiv \int_\Omega |x|^a \sum_{i,j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 dx < \infty, \quad a \in \mathbb{R}.$$

Imposing the same constraint on the behavior of the solution at infinity in various classes of unbounded domains, the author [23–37] studied the uniqueness (non–uniqueness) problem and found the dimensions of the spaces of solutions of boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation.

The main research method for constructing solutions to the mixed Dirichlet–Robin problem is the variational principle, which assumes the minimization of the corresponding functional in the class of admissible functions. Further, using Korn's and Hardy's-type inequalities [6–8], we obtain a criterion for the uniqueness (or non-uniqueness) of solutions to this problem in weighted spaces.

This article contains proofs of the results announced in [36].

Notation: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω.

We denote by $H^1(\Omega,\Gamma), \Gamma \subset \overline{\Omega}$ the Sobolev space of functions in Ω obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ with respect to the norm

$$||u; H^1(\Omega,\Gamma)|| = \left(\int_\Omega \sum_{|\alpha|\le 1} |\partial^\alpha u|^2 dx \right)^{1/2},$$

where $\partial^\alpha u = \partial^{|\alpha|} u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi–index, $\alpha_j \ge 0$ are integers, and $|\alpha| = \alpha_1 + \dots + \alpha_n$; if $\Gamma = \emptyset$, we denote $H^1(\Omega,\Gamma)$ by $H^1(\Omega)$.

$\overset{\circ}{H}{}^1(\Omega)$ is the space of functions in Ω obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u; H^1(\Omega)||$;

$\overset{\circ}{H}{}^1_{loc}(\Omega)$ is the space of functions in Ω obtained by the completion of $C_0^\infty(\Omega)$ with respect to the family of semi-norms

$$\|u; H^1(\Omega \cap B_0(R))\| = \left(\int\limits_{\Omega \cap B_0(R)} \sum_{|\alpha|\le 1} |\partial^\alpha u|^2 \, dx \right)^{1/2}$$

for all open balls $B_0(R) := \{x : |x| < R\}$ in $\mathbb{R}^n$ for which $\Omega \cap B_0(R) \ne \emptyset$.

We set $\partial^\alpha u = \partial^{|\alpha|} u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \ge 0$ are integers, and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Let

$$D(u,\Omega) = \int_\Omega |\nabla u|^2 \, dx, \quad E(u,\Omega) = \int_\Omega |\varepsilon(u)|^2 \, dx,$$

$$D_a(u,\Omega) = \int_\Omega |x|^a |\nabla u|^2 \, dx, \quad E_a(u,\Omega) = \int_\Omega |x|^a |\varepsilon(u)|^2 \, dx,$$

where

$$|\nabla u|^2 = \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} \right)^2, \quad |\varepsilon(u)|^2 = \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2,$$

$$\Omega_R = \Omega \cap \{x : |x| < R\}, \quad \partial\Omega_R = \partial\Omega \cup \{x : |x| = R\}.$$

By the cone K in $\mathbb{R}^n$ with vertex at x_0 we mean a domain such that if $x - x_0 \in K$, then $\lambda(x - x_0) \in K$ for all $\lambda > 0$. We assume that the origin $x_0 = 0$ lies outside $\overline{\Omega}$.

Let $\binom{n}{k}$ be the (n,k)-binomial coefficient, $\binom{n}{k} = 0$ for $k > n$.

2. Definitions and Auxiliary Statements

Definition 1. *A solution of the system* (1) *in* Ω *is a vector-valued function* $u \in H^1_{loc}(\Omega)$ *such that for any vector-valued function* $\varphi \in C_0^\infty(\Omega)$ *the following integral identity holds*

$$\int_\Omega a_{kh}^{ij} \frac{\partial u_j}{\partial x_h} \frac{\partial \varphi_i}{\partial x_k}\, dx = 0.$$

Before proceeding to the consideration of the boundary value problem (1), (2), we establish two auxiliary lemmas.

Lemma 1. *Let* u *be a solution of the system* (1) *in* Ω *such that* $E_a(u,\Omega) < \infty$*. Then*

$$u(x) = P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x),\ x \in \Omega, \tag{3}$$

where $P(x)$ *is a polynomial,* $\operatorname{ord} P(x) \le m = \max\{1, 1 - n/2 - a/2\}$*,* $\Gamma(x)$ *is the fundamental solution of the system* (1)*,* $C_\alpha = \text{const}$*,* $\beta_0 = 1 - n/2 + a/2$*,* $\beta \ge 0$ *is an integer, and the function* u^β *satisfies the estimate*

$$|\partial^\gamma u^\beta(x)| \le C_{\gamma\beta}(a,u)|x|^{1-n-\beta-|\gamma|}$$

for every multi-index γ*,* $C_{\gamma\beta} = \text{const}$.

Remark 1. *It is known [38], that there exists a fundamental solution* $\Gamma(x)$*, which for* $n > 2$ *has the following estimate*

$$|\partial^\alpha \Gamma(x)| \le C(\alpha)|x|^{2-n-|\alpha|}, \quad C(\alpha) = \text{const}.$$

For $n = 2$ *the fundamental solution has a representation* $\Gamma(x) = S(x)\ln|x| + T(x)$*, where* $S(x)$ *and* $T(x)$ *are square matrices of order 2 whose entries are homogeneous functions of order zero [39].*

Proof of Lemma 1. Consider the vector-valued function $v(x) = \theta_N(x)u(x)$, where $\theta_N(x) = \theta(|x|/N)$, $\theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$, $\theta(s) = 0$ for $s \le 1$, $\theta(s) = 1$ for $s \ge 2$, and also $N >> 1$ and $G \subset \{x : |x| < N\}$. We extend v to $\mathbb{R}^n$ by setting $v = 0$ on $G = \mathbb{R}^n \setminus \overline{\Omega}$. Then the vector-valued function v belongs to $C^\infty(\mathbb{R}^n)$ and satisfies the system

$$Lv = F_i, \quad i = 1, \dots, n,$$

where $F_i \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} F_i \subset \{x : |x| < 2N\}$. It is easy to see that $E_a(v, \mathbb{R}^n) < \infty$. If $a + n \ne 0$, then Korn's inequality ([7], § 3, inequality (1)) implies that $v(x) = w(x) + Ax$, where A is a constant skew-symmetric matrix and w satisfies $D_a(w,\Omega) < \infty$.

We can now use Theorem 1 of [40] since it is based on Lemma 2 of [40], which imposes no constraints on the sign of σ'. Hence the expansion

$$w(x) = P_0(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + w^\beta(x)$$

holds for each a, where $P_0(x)$ is a polynomial, $\operatorname{ord} P_0(x) \le \max\{1, 1 - n/2 - a/2\}$, $C_\alpha = \text{const}$, $\beta_0 = 1 - n/2 + a/2$ and

$$|\partial^\gamma w^\beta(x)| \le C_{\gamma\beta}|x|^{1-n-\beta-|\gamma|}, \qquad C_{\gamma\beta} = \text{const}.$$

Hence, by the definition of v, we obtain (3) with $P(x) = P_0(x) + Ax$.

Now let $a + n = 0$. Then for each $\delta > 0$,

$$E_{-n-\delta}(v, \mathbb{R}^n) \le E_{-n}(v, \mathbb{R}^n) < \infty.$$

By Korn's inequality ([7], § 3, inequality (1)), there exists a constant skew-symmetric matrix A such that

$$D_{-n-\delta}(v - Ax, \mathbb{R}^n) \le C E_{-n-\delta}(v, \mathbb{R}^n) < \infty,$$

where the constant C is independent of $v(x)$. Hence, using Theorem 1 of [40], we get

$$v(x) - Ax = P_0(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

where $P_0(x)$ is a polynomial, ord $P_0(x) \le 1$, $C_\alpha = \text{const}$, $\beta_0 = 1 - n/2 + a/2$ and

$$|\partial^\gamma v^\beta(x)| \le C_{\gamma\beta} |x|^{1-n-\beta-|\gamma|}, \qquad C_{\gamma\beta} = \text{const}\,.$$

Thus,

$$v(x) - Ax = P_0(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

which proves the Lemma for $a = -n$. □

Lemma 2. *Let u be a solution of the system (1) in Ω such that $E_a(u, \Omega) < \infty$ for some $a \ge 0$. Then for all $x \in \Omega$ equality (3) holds with u^β, satisfying an estimate similar to that in Lemma 1; in addition, $P(x) = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector.*

Proof. Let u be a solution of the system (1) in Ω. Then by Lemma 1, we have

$$u(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial, ord $P(x) \le 1$, and

$$R(x) = \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \qquad R(x) = O(|x|^{2-n}), \qquad n > 2.$$

Let us prove that $P(x) = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector. Obviously, if $E_a(u, \Omega) < \infty$ and $a \ge 0$, then $E(u, \Omega) < \infty$.

Assume that $n > 2$. It is easy to verify that $E(R(x), \Omega) < \infty$ for $n > 2$. Hence $E(P(x), \Omega) < \infty$ by the triangle inequality.

Let $P(x) = Ax + B$, that is, $P_i(x) = \sum_{j=1}^n a_{ij} x_j + b_i$. Then

$$E(P(x), \Omega) = \int_\Omega \sum_{i,j=1}^n (a_{ij} + a_{ji})^2 dx,$$

where the a_{ij} are the entries of A. The last integral converges if and only if $a_{ij} = -a_{ji}$, that is, A is a constant skew-symmetric matrix.

We consider now the case $n = 2$. It is known [39] that $\Gamma(x) = S(x) \ln |x| + T(x)$, where $S(x)$ and $T(x)$ are 2×2 matrices whose entries are homogeneous functions of order zero. Then $\nabla\Gamma(x) = O(|x|^{-1} \ln |x|)$, and therefore, $\nabla R(x) = O(|x|^{-1} \ln |x|)$. Assume that there exists k and l such that $a_{kl} + a_{lk} \ne 0$. Then

$$|\varepsilon_{kl}(u)| = |a_{kl} + a_{lk} + O(|x|^{-1} \ln |x|)| \ge \frac{1}{2} |a_{kl} + a_{lk}| \quad \text{for} \quad |x| >> 1.$$

Hence,

$$\infty > E(u, \Omega) = \int_\Omega \sum_{i,j=1}^n |\varepsilon_{ij}(u)|^2 \, dx \ge \int_\Omega |\varepsilon_{kl}(u)|^2 \, dx \ge \frac{1}{4} \int_{|x|>H} |a_{kl} + a_{lk}|^2 \, dx = \infty.$$

This contradiction demonstrates that $a_{kl} = -a_{lk}$ for all k and l, which completes the proof. $\square$

Definition 2. *A solution of the mixed Dirichlet–Robin problem* (1), (2) *is a vector-valued function* $u \in \overset{\circ}{H}{}^1_{loc}(\Omega, \Gamma_1)$, *such that for each vector-valued function* $\varphi \in \overset{\circ}{H}{}^1_{loc}(\Omega, \Gamma_1) \cap C_0^\infty(\mathbb{R}^n)$, *the following integral identity holds*

$$\int_\Omega a_{kh}^{ij} \frac{\partial u_j}{\partial x_h} \frac{\partial \varphi_i}{\partial x_k} \, dx + \int_{\Gamma_2} \tau \, u \, \varphi \, ds = 0. \tag{4}$$

Let $\mathrm{Ker}_0(L)$ be the space of generalized solutions of the mixed Dirichlet–Robin problem (1), (2), that have a finite energy integral, that is,

$$\mathrm{Ker}_0(L) = \{u : Lu = 0,\ u|_{\Gamma_1} = 0,\ (\sigma(u) + \tau u)|_{\Gamma_2} = 0,\ E(u, \Omega) < \infty\}.$$

We set by definition

$$\mathrm{Ker}_a(L) = \{u : Lu = 0,\ u|_{\Gamma_1} = 0,\ (\sigma(u) + \tau u)|_{\Gamma_2} = 0,\ E_a(u, \Omega) < \infty\}.$$

Let $\dim \mathrm{Ker}_0(L)$ and $\dim \mathrm{Ker}_a(L)$ be dimensions of $\mathrm{Ker}_0(L)$ and $\mathrm{Ker}_a(L)$, respectively. We shall calculate the values of $\dim \mathrm{Ker}_a(L)$ in their dependence on the parameter a.

3. Main Results

Theorem 1. *The mixed Dirichlet–Robin problem* (1), (2) *with the condition* $E(u, \Omega) < \infty$ *has* $n(n+1)/2$ *linearly independent solutions if* $n \geq 3$, *and at least one linearly independent solution if* $n = 2$.

Proof. Step 1. Let $n \geq 3$. For any constant skew-symmetric matrix A, we construct a generalized solution u_A of the Dirichlet–Robin problem for the system (1) in Ω with the boundary conditions

$$u_A|_{\Gamma_1} = Ax, \quad (\sigma(u_A) + \tau u_A)|_{\Gamma_2} = 0, \tag{5}$$

satisfying the conditions $E(u_A, \Omega) < \infty$, $D(u_A, \Omega) < \infty$, and

$$\int_\Omega |u_A|^2 |x|^{-2} \, dx < \infty. \tag{6}$$

Such a solution may be constructed by the variational method, minimizing the functional

$$\Phi(v) \equiv \int_\Omega a_{kh}^{ij} \frac{\partial v_j}{\partial x_h} \frac{\partial v_i}{\partial x_k} \, dx$$

in the class of admissible functions $\{v : v \in H^1(\Omega),\ \ v|_{\Gamma_1} = Ax, \quad v \text{ has compact support in } \overline{\Omega}\}$. The boundedness of the Dirichlet follows from Korn's inequality ([7]; §3, inequality (43)). Condition (6) is a consequence of Hardy's inequality [7].

Let $A_1, \ldots, A_p$, $p = (n^2 - n)/2$, be linearly independent constant skew-symmetric $n \times n$ –matrices. We consider the solutions $u_{A_1}, \ldots, u_{A_p}$.

Step 2. Now in the same way, for any constant vector $\vec{e} = \vec{e}_k \neq 0$,

$$e_k = (e_k^1, \ldots, e_k^n), \quad e_k^j = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k, j = 1, \ldots, n,$$

we construct a generalized solution u_{e_k} of the Dirichlet–Robin problem for the system (1) with the boundary conditions

$$u_{e_k}|_{\Gamma_1} = \vec{e}_k, \quad \big(\sigma(u_{e_k}) + \tau u_{e_k}\big)|_{\Gamma_2} = 0$$

and with conditions $E(u_{e_k}, \Omega) < \infty, \quad D(u_{e_k}, \Omega) < \infty,$

$$\int_\Omega |u_{e_k}|^2 |x|^{-2}\, dx < \infty. \tag{7}$$

Such a solution may be constructed by a variational method, minimizing the corresponding functional in the class of functions $\{v \in C^\infty(\overline{\Omega}), v_e|_{\Gamma_1} = \vec{e}, v$ has compact support in $\overline{\Omega}\}$. The boundedness of the Dirichlet follows from Korn's inequality ([7]; §3, inequality (43)). Condition (7) follows from Hardy's inequality ([7]; §3, inequality (27)).

Step 3. The solutions $u_{A_1} - A_1 x, \dots, u_{A_p} - A_p x$, $u_{e_1} - e_1, \dots, u_{e_n} - e_n$ are linearly independent. Indeed, if

$$\sum_{i=1}^{p} C_i (u_{A_i} - A_i x) + \sum_{i=1}^{n} c_i (u_{e_i} - e_i) = 0,$$

where C_i and c_i are constants, then $v \equiv \sum\limits_{i=1}^{p} C_i A_i x \equiv 0$, since

$$v = \sum_{i=1}^{p} C_i u_{A_i} + \sum_{i=1}^{n} c_i (u_{e_i} - e_i)$$

has a finite Dirichlet integral $D(v, \Omega) < \infty$. Therefore $C_i = 0$, $i = 1, \dots, p$. Hence

$$\sum_{i=1}^{n} c_i u_{e_i} = \sum_{i=1}^{n} c_i e_i = B$$

where B is a constant vector. Since the u_{e_i} satisfy (7), we have $B = 0$. The vectors e_i, $i = 1, \dots, n$ are linearly independent, and therefore $c_i = 0$, $i = 1, \dots, n$.

Thus, we have proved that the homogenous Dirichlet–Robin problem has at least $n(n+2)/2$ linearly independent generalized solutions.

Step 4. Let us now prove that any generalized solution u of the homogenous Dirichlet–Robin problem with the condition $E(u, \Omega) < \infty$ is a linear combination of the constructed solutions. According to Korn's inequality ([7]; §3, inequality (43)), there is skew-symmetric matrix A such that

$$D(u - Ax, \Omega) \le C E(u, \Omega), \quad C = \text{const}.$$

Let $A = \sum\limits_{i=1}^{p} m_i A_i$, $m_i = \text{const}$, $i = 1, \dots, p$. For the function $v = u + (u_A - Ax)$ we have $D(v, \Omega) < \infty$, since $D(u_{A_i}, \Omega) < \infty$. Hardy's inequality implies that

$$\int\limits_\Omega |v - B|^2 |x|^{-2}\, dx \le C D(v, \Omega) < \infty, \quad C = \text{const},$$

where B is a constant vector.

Let $B = \sum\limits_{i=1}^{n} M_i e_i$. We set $w = u + (u_A - Ax) + (u_B - B)$, where $u_B = \sum\limits_{i=1}^{n} M_i u_{e_i}$. It is easy to see that

$$w|_{\Gamma_1} = 0, \quad (\sigma(w) + \tau w)|_{\Gamma_2} = 0,$$

$$D(w, \Omega) < \infty, \quad \int\limits_\Omega |w|^2 |x|^{-2}\, dx < \infty.$$

Let us show that $w \equiv 0$. We substitute in the integral identity (4) for w the vector-valued function $\varphi = w\theta_N(x)$, where $\theta_N(x) = \theta(|x|/N)$, $\theta(s) = 1$ for $s \leq 1$, $\theta(s) = 0$ for $s \geq 2$, $\theta \in C^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, we get

$$\int_\Omega \mathcal{E}(w)\,\theta_N(x)\,dx + \int_{\Gamma_2} \tau|w|^2\theta_N(x)\,ds = -\int_\Omega a_{kh}^{ij}\frac{\partial w_j}{\partial x_h}\frac{\partial \theta_N(x)}{\partial x_k}\,w\,dx, \tag{8}$$

where $\mathcal{E}(w) \equiv a_{kh}^{ij}\frac{\partial w_j}{\partial x_h}\frac{\partial w_i}{\partial x_k}$.

We claim that the right-hand side of (8) approaches zero as $N \to \infty$. Indeed, the Cauchy–Schwartz inequality yields that

$$\left| -\int_\Omega a_{kh}^{ij}\frac{\partial w_j}{\partial x_h}\frac{\partial \theta_N(x)}{\partial x_k}\,w\,dx \right| \leq 2CJ_1(w)J_2(w),$$

where

$$J_1(w) \equiv \left(\int_{\{x:|x|>N\}} |\nabla w|^2\,dx\right)^{1/2}, \quad J_2(w) \equiv \left(\int_{\{x:N<|x|<2N\}} |w|^2|x|^{-2}\,dx\right)^{1/2}.$$

Since

$$\int_\Omega |w|^2|x|^{-2}\,dx < \infty, \quad D(w,\Omega) < \infty,$$

it follows that $J_2(w) \to 0$ and $J_1(w) \to 0$ as $N \to \infty$. Hence,

$$\int_\Omega \mathcal{E}(w)\,\theta_N(x)\,dx + \int_{\Gamma_2} \tau|w|^2\theta_N(x)\,ds \to 0 \quad \text{as} \quad N \to \infty.$$

Using the integral identity

$$\int_\Omega \mathcal{E}(w)dx + \int_{\Gamma_2} \tau|w|^2 ds = 0,$$

we find that if w is a solution of the homogeneous problem (1), (2), then $w = A_1x + B_1$. The set of points where $A_1x + B_1 = 0$ is a linear manifold of dimension less than $n-1$, since the rank of the matrix A_1 is ≥ 2 if $A_1 \not\equiv 0$. Consequently, $w = 0$. This conclusion follows from the fact that

$$\int_{\Gamma_2} \tau|w|^2 ds = 0,$$

and hence $w \equiv 0$ on a subset of $\partial\Omega$ of positive measure. This means that $u = -(u_A - Ax) - (u_B - B)$. The theorem is proved for $n \geq 3$.

Let now $n = 2$. For a nontrivial constant skew-symmetric matrix A, we construct a generalized solution u_A of the Dirichlet–Robin problem for the system (1) in Ω with the boundary conditions (5), minimizing the corresponding functional $\Phi(v)$ in the class of functions $\{v : v \in C^\infty(\overline{\Omega}),\ v_A|_{\Gamma_1} = Ax,\ v \text{ has a compact support in } \overline{\Omega}\}$. This solution satisfies $E(u_A,\Omega) < \infty$ and $D(u_A,\Omega) < \infty$. By Hardy's inequality [6] we get

$$\int_{|x|>N} |u_A|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty, \tag{9}$$

where $N >> 1$ is such that $G \subset \{x : |x| < N\}$.

We prove further that any generalized solution u of the homogeneous Dirichlet–Robin problem (1), (2) has the form $u = c_0(u_A - Ax)$, where $c_0 = \text{const}$, A is a skew-symmetric matrix, and $A \neq 0$.

We observe that $u_A - Ax \neq 0$, since $D(u_A - Ax) = \infty$. By Korn's inequality ([7]; §3, inequality (43)), there is a skew-symmetric matrix A_0 such that

$$D(u - A_0x,\Omega) \leq CE(u,\Omega), \quad C = \text{const}. \tag{10}$$

We set $w = u + (u_{A_0} - A_0 x)$, where $u_{A_0} = c_0 u_A$, if $A_0 = c_0 A$. It is easy to see that $D(w, \mathbb{R}^n) < \infty$, since (10) and $D(u_A, \Omega) < \infty$ by construction. Therefore, for w, by Hardy's inequality, the inequality of the form (9) holds.

Let us prove that $w = 0$. Substituting in the integral identity (4) for w the function $\varphi = w\theta_N(x)$, where $\theta_N(x) = \theta(\ln|x|/\ln N)$, $\theta(s) = 1$ for $s \le 1$, $\theta(s) = 0$ for $s \ge 2$, $\theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$. Further, as above, we obtain that $w \equiv 0$. This concludes the proof. □

Theorem 2. *If* $-n \le a < n-2$, $n \ge 2$, *then* $\dim \mathrm{Ker}_a(L) = n(n+1)/2$.

Proof. We first consider the case $0 \le a < n-2$, $n \ge 3$. Obviously, $\mathrm{Ker}_a(L) \subset \mathrm{Ker}_0(L)$ for $a \ge 0$.

We claim that $\mathrm{Ker}_0(L) \subset \mathrm{Ker}_a(L)$ if $a < n-2$. Indeed, let $u \in \mathrm{Ker}_0(L)$. By Lemma 2 we have equality (3):

$$u(x) = P(x) + R(x),$$

where $P(x) = Ax + B$, A is a constant skew-symmetric matrix, B is a constant vector, $R(x) = \sum\limits_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x)$.

It is easy to verify that $E_a(R(x), \Omega) < \infty$ and $E_a(P(x), \Omega) = 0$ for $0 \le a < n-2$. Hence $E_a(u, \Omega) < \infty$, that is, $u \in \mathrm{Ker}_a(L)$. Therefore $\mathrm{Ker}_0(L) \subset \mathrm{Ker}_a(L)$.

Thus, $\mathrm{Ker}_a(L) = \mathrm{Ker}_0(L)$ and $\dim \mathrm{Ker}_a(L) = \dim \mathrm{Ker}_0(L)$. Using Theorem 1 we have $\dim \mathrm{Ker}_a(L) = n(n+1)/2$.

Consider now the case when $-n \le a < 0$ and $n > 2$.

Let $u \in \mathrm{Ker}_a(L)$, where $-n \le a < 0$. By Korn's inequality ([7]; §3, inequality (43)), there is a constant skew-symmetric matrix A such that

$$D_a(u - Ax, \Omega) \le C E_a(u, \Omega),$$

where the constant C is independent of u.

For the function $v = u - Ax$ we have $D_a(v, \Omega) < \infty$ and $E_a(v, \Omega) < \infty$. Moreover, v is a solution of (1) in Ω. Hence, by Lemma 1, it has the form (3):

$$v = P(x) + R(x),$$

where $P(x)$ is a polynomial, and $R(x) = \sum\limits_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x)$.

Let us prove that $\mathrm{ord}\, P(x) = 0$. First, establish the inequality $D_a(P(x), \Omega) < \infty$. We have $D_a(v, \Omega) < \infty$, and it is easy to verify that $D_a(R(x), \Omega) < \infty$ for $-n \le a < 0$. Hence $D_a(P(x), \Omega) < \infty$ by the triangle inequality..

Let $\mathrm{ord}\, P(x) = k$, where $k \ne 0$. Then we have $|\nabla P(x)| \ge C|x|^{k-1}$ in the interior of some cone K. Hence,

$$\infty > D_a(P(x), \Omega) = \int_\Omega |x|^a |\nabla P(x)|^2 \, dx \ge C \int_{K \cap \{x : |x| > H\}} |x|^{a+2k-2+n} |x|^{-1} \, d|x|.$$

The last integral converges if and only if $a + 2k - 2 + n < 0$. Hence $k < 1$ and, therefore, $\mathrm{ord}\, P(x) = 0$ and $P(x) = B$, where B is a constant vector. Thus, $v(x) = B + R(x)$.

On the other hand, $v(x) = u(x) - Ax$. Hence we have $u = Ax + B + R(x)$, where A is a constant skew-symmetric matrix and B is a constant vector. It is easy to verify that $E(R(x), \Omega) < \infty$ and $E(Ax + B, \Omega) = 0$. Hence $E(u, \Omega) < \infty$, that is, $u \in \mathrm{Ker}_0(L)$. We obtain the embedding $\mathrm{Ker}_a(L) \subset \mathrm{Ker}_0(L)$. In addition, it is obvious that $\mathrm{Ker}_0(L) \subset \mathrm{Ker}_a(L)$ for $a < 0$.

Thus, $\mathrm{Ker}_a(L) = \mathrm{Ker}_0(L)$ and $\dim \mathrm{Ker}_a(L) = \dim \mathrm{Ker}_0(L)$. By Theorem 1 we obtain $\dim \mathrm{Ker}_0(L) = n(n+1)/2$. Hence $\dim \mathrm{Ker}_a(L) = n(n+1)/2$.

Step 1. Now let $n = 2$. For a non-trivial constant skew-symmetric matrix A, we construct a generalized solution $u_A(x)$ of the mixed Dirichlet–Robin problem for the system (1) in Ω with boundary conditions

$$u_A|_{\Gamma_1} = Ax, \quad (\sigma(u_A) + \tau u_A)|_{\Gamma_2} = 0,$$

by minimizing the functional $\Phi(v)$ in the class of admissible functions $\{v : v \in C^\infty(\overline{\Omega}),\, v|_{\Gamma_1} = Ax,\, v$ has compact support in $\overline{\Omega}\}$. The resulting solution satisfies $E(u_A, \Omega) < \infty$ and $D(u_A, \Omega) < \infty$. By Hardy's inequality [6] we obtain

$$\int_{|x|>N} |u_A|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty,$$

where $N >> 1$ and $G \subset \{x : |x| < N\}$.

Step 2. In this same way we obtain generalized solutions of the mixed Dirichlet–Robin for the system (1) in Ω with the boundary conditions

$$u_i|_{\Gamma_1} = \Gamma(x)\tilde{C}_i, \quad (\sigma(u_i) + \tau u_i)|_{\Gamma_2} = 0,\ i = 1,2,\ \tilde{C}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \tilde{C}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and with the properties $E(u_i, \Omega) < \infty$, $D(u_i, \Omega) < \infty$, and

$$\int_{|x|>N} |u_i|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty,\ i = 1,2.$$

The solutions $u_A - Ax$ and $u_i - \Gamma(x)\tilde{C}_i,\ i = 1,2$, are linearly independent. Indeed, if

$$(u_A - Ax)d_0 + \sum_{i=1}^{2} (u_i - \Gamma(x)\tilde{C}_i)d_i = 0,$$

for some constants d_0 and $d_i,\ i = 1,2$, then

$$v \equiv d_0 Ax + \Gamma(x)(d_1\tilde{C}_1 + d_2\tilde{C}_2) \equiv 0,$$

because $v = d_0 u_A + \sum\limits_{i=1}^{2} d_i u_i$ has a finite Dirichlet integral $D(v, \Omega) < \infty$ and

$$\int_{\Omega} |v|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty.$$

Thus, $Axd_0 + \Gamma(x)\vec{d} = 0$, where $\vec{d} = d_1\tilde{C}_1 + d_2\tilde{C}_2$.

Since $|Ax| \geq |x|$ and $|\Gamma(x)\vec{d}| \leq C \ln|x|$, it follows that $d_0 = 0$. Hence $\Gamma(x)\vec{d} = 0$, and applying the elasticity operator to this equation, we obtain

$$0 = L(\Gamma(x)\vec{d}) = L(\Gamma(x))\vec{d} = I\delta(x)\vec{d} = \begin{pmatrix} \delta(x) & 0 \\ 0 & \delta(x) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where I is the 2×2 unit matrix, $\delta(x)$ is the Dirac function. Hence it follows that $d_1 = d_2 = 0$. It is easy to verify that $E_a(u_A - Ax, \Omega) < \infty$ and $E_a(u_i - \Gamma(x)\tilde{C}_i, \Omega) < \infty,\ i = 1,2$, for $-2 \leq a < 0$.

Hence the Dirichlet–Robin problem (1), (2) has at least three linearly independent solutions satisfying $E_a(u, \Omega) < \infty$.

Step 3. We claim that each generalized solution u of the Dirichlet–Robin problemn (1), (2) with condition $E_a(u, \Omega) < \infty$ is a linear combination of the solutions constructed above. By Korn's inequality ([7]; §3, inequality (43)), there is a constant skew-symmetric matrix A_1 such that

$$D_a(u - A_1 x, \Omega) \leq C E_a(u, \Omega),$$

where the constant C is independent of u. For the function$v_1 = u - A_1 x$ we have $D_a(v_1, \Omega) < \infty$ and $E_a(v_1, \Omega) < \infty$. Since v_1 is a solution of the system (1) in Ω, it follows by Lemma 1 that

$$v_1(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial and $R(x) = \sum\limits_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x)$.

In a similar way to the above we can show that ord $P(x) = 0$ and $P(x) = B_1$, B_1 is a constant vector. Thus, $u = A_1 x + B_1 + \Gamma(x) C_0 + R_1(x)$, where

$$R_1(x) = \sum_{0<|\alpha|\leq\beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \quad R_1(x) = O(|x|^{-1} \ln |x|).$$

Let $A_1 = -CA$ and $C_0 = -(d_1 \tilde{C}_1 + d_2 \tilde{C}_2)$. We set

$$w = u - [C(u_A - Ax) + \sum_{i=1}^{2} d_i (u_i - \Gamma(x) \tilde{C}_i)].$$

Obviously, w is a solution of the system (1) in Ω, $w|_{\Gamma_1} = 0$, $(\sigma(w) + \tau w)|_{\Gamma_2} = 0$ and

$$w = B_1 + R_1(x) - C u_A - \sum_{i=1}^{2} d_i u_i.$$

It is easy to see that $D(w, \Omega) < \infty$ and $\int_\Omega |w|^2 |x|^{-2} |\ln |x||^{-2}\, dx < \infty$. Thus, $w(x)$ is a solution of the following problem (w):

$$\begin{cases} Lw = 0 \text{ in } \Omega, \\ w|_{\Gamma_1} = 0, \quad (\sigma(w) + \tau w)|_{\Gamma_2} = 0, \\ D(w, \Omega) < \infty, \quad \int_\Omega |w|^2 |x|^{-2} |\ln |x||^{-2}\, dx < \infty. \end{cases}$$

Let us prove that the solution $w(x)$ of problem (w) is unique, that is, $w(x) \equiv 0$, $x \in \Omega$. To this end, we write the integral identity (4) for the vector-valued function $\varphi = w\theta_N(x)$, where $\theta_N(x) = \theta\,(\ln |x| / \ln N)$, $\theta(s) = 1$ for $s \leq 1$, $\theta(s) = 0$ for $s \geq 2$, $\theta \in C^\infty(\mathbb{R}^n)$, $0 \leq \theta \leq 1$, we get

$$\int_\Omega \mathcal{E}(w) \theta_N(x)\, dx + \int_{\Gamma_2} \tau |w|^2 \theta_N(x)\, ds = -\int_\Omega a_{kh}^{ij} \frac{\partial w_j}{\partial x_h} \frac{\partial \theta_N(x)}{\partial x_k} w\, dx, \tag{11}$$

where $\mathcal{E}(w) \equiv a_{kh}^{ij} \frac{\partial w_j}{\partial x_h} \frac{\partial w_i}{\partial x_k}$.

We claim that the right-hand side of (11) approaches zero as $N \to \infty$. Indeed, the Cauchy–Schwartz inequality yields that

$$\left| -\int_\Omega a_{kh}^{ij} \frac{\partial w_j}{\partial x_h} \frac{\partial \theta_N(x)}{\partial x_k} w\, dx \right| \leq C \int\limits_{\Omega \cap \{x : N < |x| < N^2\}} |\nabla w| \frac{|w|}{|x| \ln N}\, dx \leq$$
$$\leq 2C J_1(w) J_2(w),$$

where

$$J_1(w) \equiv \left(\int_{\{x:|x|>N\}} |\nabla w|^2\, dx \right)^{1/2}, \quad J_2(w) \equiv \left(\int_{\{x:N<|x|<N^2\}} \frac{|w|^2}{|x|^2 |\ln N|^2}\, dx \right)^{1/2}.$$

Since
$$\int_{\Omega} |w|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty,\; D(w,\Omega) < \infty,$$
it follows that $J_2(w) \to 0$ and $J_1(w) \to 0$ as $N \to \infty$. Hence
$$\int_{\Omega} \mathcal{E}(w)\, \theta_N(x)\, dx + \int_{\Gamma_2} \tau |w|^2 \theta_N(x)\, ds \to 0 \quad \text{as} \quad N \to \infty.$$

Using the integral identity
$$\int_{\Omega} \mathcal{E}(w) dx + \int_{\Gamma_2} \tau |w|^2\, ds = 0,$$
we find that if w is a solution to the homogeneous problem (1), (2), then $w = A_2 x + B_2$. The set of all x such that $A_2 x + B_2 = 0$ is a linear manifold whose dimension is less than $n-1$, since the rank of the matrix A_2 is ≥ 2 if $A_2 \not\equiv 0$. Therefore, $w = 0$. The relation
$$\int_{\Gamma_2} \tau |w|^2\, ds = 0,$$
implies that $w \equiv 0$ on a set of positive measure on $\partial\Omega$, and therefore, $w(x) \equiv 0$, $x \in \Omega$. The theorem is proved. □

Theorem 3. *If* $n-2 \leq a < n,\; n \geq 2$, *then* $\dim \mathrm{Ker}_a(L) = n(n-1)/2$.

Proof. Step 1. Assume that $n \geq 3$. For each constant vector $\vec{e} = \vec{e}_k \neq 0$:
$$e_k = (e_k^1, \ldots, e_k^n), \quad e_k^j = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k, j = 1, \ldots, n,$$
we construct a generalized solution u_e of the Dirichlet–Robin problem for the system (1) with the boundary conditions
$$u_e|_{\Gamma_1} = \vec{e}, \quad (\sigma(u_e) + \tau u_e)|_{\Gamma_2} = 0 \tag{12}$$
and with the additional conditions $E(u_e, \Omega) < \infty, \quad D(u_e, \Omega) < \infty$, and
$$\int_{\Omega} |u_e|^2 |x|^{-2}\, dx < \infty. \tag{13}$$

Such a solution is constructed by the variational method. We minimize the corresponding functional over the class of admissible functions $\{v \in C^\infty(\overline{\Omega}),\, v_e\,|_{\Gamma_1} = \vec{e},\, v$ has compact support in $\overline{\Omega}\}$. The boundedness of the Dirichlet integral follows from Korn's inequality ([7]; §3, inequality (43)). Condition (13) follows from Hardy's inequality ([7]; §3, inequality (27)). By Lemma 2 the solution $u_e(x)$ takes the form
$$u_e(x) = P_e(x) + R_e(x), \tag{14}$$
where $P_e(x)$ is a polynomial, $P_e(x) = Ax + B$, with A being a constant skew-symmetric matrix and B being a constant vector, and
$$R_e(x) = \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C'_\alpha + u_e^\beta(x),\; R_e(x) \to 0,\; |x| \to \infty.$$

We claim that $P_e(x) \equiv 0$. Assume that $P_e(x) \not\equiv 0$. Then in the interior of a certain cone K we have $|P_e(x)| > C$ and
$$\infty > \int_{\Omega \cap K} |u_e|^2 |x|^{-2}\, dx > \frac{C^2}{2} \int_{K \cap \{x : |x| > H\}} |x|^{-2}\, dx = \infty.$$

This contradiction shows that $P_e(x) \equiv 0$. Thus,

$$u_e(x) = \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C'_\alpha + u_e^\beta(x). \tag{15}$$

Let us prove that

$$\int_\Omega a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \frac{\partial u_{ei}}{\partial x_k} dx = \int_{\partial\Omega} u_e\, a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \nu_k\, ds. \tag{16}$$

For a proof we consider a ball $Q_R = \{x : |x| < R\}$ with centre at the origin suck that $G \subset\subset Q_R$. Let $\Omega_R = \Omega \cap Q_R$, $\partial\Omega_R = \partial\Omega \cup \{x : |x| = R\}$. Then

$$\int_{\Omega_R} a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \frac{\partial u_{ei}}{\partial x_k}\, dx = \left(\int_{\partial\Omega} + \int_{|x|=R} \right) u_e\, a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \nu_k\, ds. \tag{17}$$

There exists a sequence of domains Ω_{R_k} such that $\Omega_{R_k} \subset \Omega_{R_{k+1}} \subset \cdots \subset \Omega_{R_n} \subset \mathbb{R}^n$ and $\cup_k \Omega_{R_k} = \Omega$.

We claim that the integrals in the right- and left-hand sides of (17) converge. Indeed, the Cauchy–Schwartz inequality yields that

$$\left| \int_{\Omega_R} a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \frac{\partial u_{ei}}{\partial x_k}\, dx \right| \le C \left[\int_{\Omega_R} \left| \frac{\partial u_{ej}}{\partial x_h} \right|^2 dx \right]^{1/2} \left[\int_{\Omega_R} \left| \frac{\partial u_{ei}}{\partial x_k} \right|^2 dx \right]^{1/2} < \infty,$$

because $\Omega_R \subset \Omega$ and $\int_\Omega |\nabla u_e|^2\, dx < \infty$.

We now claim that

$$\int_{|x|=R} u_e\, a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \nu_k\, ds \to 0 \quad \text{as} \quad |x| \to \infty.$$

In fact, by the Cauchy–Schwartz inequality and the estimates

$$|u_e| \le C|x|^{2-n}, \quad |\nabla u_e| \le C|x|^{1-n}$$

we obtain

$$\left| \int_{|x|=R} u_e\, a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \nu_k\, ds \right| \le \text{const} \int_{|x|=R} |u_e||\nabla u_e|\, ds$$

$$\le c \left[C \int_{|x|=R} |x|^{4-2n}\, ds \right]^{1/2} \left[C \int_{|x|=R} |x|^{2-2n}\, ds \right]^{1/2} = \text{const}\, R^{2-n} \to 0$$

as $R \to \infty$ for $n > 2$. The constants c and C are independent of R.

Letting R in (17) tend to infinity, we obtain the required Equation (16):

$$\int_\Omega \mathcal{E}(u_e)\, dx = \left(\int_{\Gamma_1} + \int_{\Gamma_2} \right) u_e\, \sigma(u_e)\, ds$$

and bearing in mind that $u_e\, |_{\Gamma_1} = \vec{e}, \quad (\sigma(u_e) + \tau u_e)|_{\Gamma_2} = 0$, we get

$$\int_\Omega \mathcal{E}(u_e)\, dx + \int_{\Gamma_2} \tau |u_e|^2\, ds = \vec{e} \int_{\Gamma_1} \sigma(u_e)\, ds, \tag{18}$$

where $\sigma(u_e) \equiv a_{kh}^{ij} \frac{\partial u_{ej}}{\partial x_h} \nu_k$.

We claim that the constant C'_0 is non-zero in (15). Indeed, if $C'_0 = 0$, then $|u_e| \le C|x|^{1-n}$ and $\sigma(u_e)| \le C|x|^{-n}$. Taking the scalar product of the system (1) and 1 and integrating over Ω_R, we obtain

$$\left(\int_{\partial\Omega} + \int_{|x|=R} \right) \sigma(u_e)\, ds = 0.$$

Since

$$\left|\int_{|x|=R} \sigma(u_e)\,ds\right| \le C\int_{|x|=R} |x|^{-n}\,ds = \text{const}\,R^{-1} \to 0 \quad \text{as} \quad R \to \infty,$$

it follows that

$$\int_{|x|=R} \sigma(u_e)\,ds \to 0 \quad \text{as} \quad R \to \infty.$$

Hence,

$$\int_{\partial\Omega} \sigma(u_e)\,ds = \left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)\sigma(u_e)\,ds \to 0 \quad \text{as} \quad R \to \infty,$$

and by (18) we obtain

$$\int_{\Omega} \mathcal{E}(u_e)\,dx + \int_{\Gamma_2} \tau|u_e|^2\,ds = -\vec{e}\cdot\int_{\Gamma_2} \sigma(u_e)\,ds.$$

Using the integral identity

$$\int_{\Omega} \mathcal{E}(u_e)dx + \int_{\Gamma_2} \tau|u_e|^2 ds = 0,$$

we get $\sigma(u_e) = 0$, since $\vec{e} \neq 0$. By [5], it follows that $u_e = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector.

On the other hand, Formula (15) with $\alpha = 0$ and $\beta = 0$ yields that

$$u_e(x) = C_0'\Gamma(x) + u_e^0(x).$$

Hence,

$$Ax + B = C_0'\Gamma(x) + u_e^0(x) \to 0 \quad \text{as} \quad |x| \to \infty,$$

that is, $Ax + B \to 0$, which is possible only if $A = 0$ and $B = 0$, that is, $u_e \equiv 0$.

However, $u_e\ |_{\Gamma_1} = \vec{e} \neq 0$ and $(\sigma(u_e) + \tau u_e)|_{\Gamma_2} = 0$. This contradiction shows that if $\vec{e} \neq 0$, then $C_0' \neq 0$.

Step 2. Let $\vec{e} \neq 0$ be an arbitrary vector in $\mathbb{R}^n$. We consider the solution u_e such that $u_e\ |_{\Gamma_1} = \vec{e}$, $(\sigma(u_e) + \tau u_e)|_{\Gamma_2} = 0$, and $u_e(x) = C_0'\Gamma(x) + u_e^0(x)$, where $\vec{C}_0' \neq 0$.

We can associate with each vector $\vec{e}$ in $\mathbb{R}^n$ the corresponding vector $\vec{C}_0'$ in $\mathbb{R}^n$, thus obtaining a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S : \vec{e} \to \vec{C}_0'$, where $\vec{e} \neq 0$, $\vec{C}_0' \neq 0$. It is easy to verify that the transformation S is linear and non-degenerate.

Let $\boldsymbol{e} = \{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$ be a basis in $\mathbb{R}^n$. For arbitrary linearly independent vectors $C_0' = \{C_{01}', \ldots, C_{0n}'\}$ there exists a unique linear transformation (matrix) S such that $\vec{C}_0' = S\vec{\boldsymbol{e}}$. Then

$$\vec{\boldsymbol{e}} = S^{-1}\vec{C}_0'. \tag{19}$$

Step 3. Consider now the elasticity system (1) in Ω with boundary conditions

$$u_A(x)\ |_{\Gamma_1} = Ax, \quad (\sigma(u_A) + \tau u_A)|_{\Gamma_2} = 0, \tag{20}$$

where A is a constant skew-symmetric matrix. For every such matrix A we construct a generalized solution of the system (1) with the boundary conditions (20) and the properties $E(u_A, \Omega) < \infty$, $D(u_A, \Omega) < \infty$,

$$\int_{\Omega} |u_A|^2|x|^{-2}\,dx < \infty. \tag{21}$$

Such a solution can be constructed using the variational method and minimizing the corresponding functional over the set of admissible functions $\{u : u \in H^1(\Omega),\ u\ |_{\Gamma_1} = Ax, \quad u$ has compact support in $\overline{\Omega}\}$. The Dirichlet integral is bounded by Korn's inequality ([7]; §3, inequality (43)). Condition (21) follows from Hardy's inequality [7]. By Lemma 2 we have

$$u_A(x) = P_A(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u_A^\beta(x). \tag{22}$$

As before, we can show that $P_A(x) \equiv 0$. Hence,

$$u_A(x) = \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u_A^\beta(x). \tag{23}$$

Step 4. Consider now the difference

$$v = (u_A - Ax) - (u_e - \vec{e}), \tag{24}$$

where $\vec{e} = S^{-1}\vec{C}_0$, and u_e and u_A are defined by (15) and (23) respectively. Obviously, v is a solution of (1) in Ω and $v\,|_{\Gamma_1} = 0$, $(\sigma(v) + \tau v)|_{\Gamma_2} = 0$.

We claim that $E_a(v, \Omega) < \infty$ for $n - 2 \le a < n$. Since $\vec{C}_0' = S\vec{e} = S(S^{-1}\vec{C}_0) = SS^{-1}\vec{C}_0 = \vec{C}_0$, it follows by (15) and (23) that

$$u_A - u_e = \sum_{0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha'' + u_0^\beta(x),$$

where $C_\alpha'' = C_\alpha - C_\alpha'$, $u_0^\beta = u_A^\beta - u_e^\beta$.

It is easy to verify that $E_a(\partial^\alpha \Gamma(x) C_\alpha'', \Omega) < \infty$ and $E_a(u_0^\beta, \Omega) < \infty$. Hence $E_a(u_A - u_e, \Omega) < \infty$. Note also that $E_a(Ax, \Omega) = 0$, $E_a(S^{-1}\vec{C}_0, \Omega) = 0$. Therefore $E_a(v, \Omega) < \infty$.

We now claim that $v \not\equiv 0$. For let $v \equiv 0$, that is, $u_A - Ax - u_{S^{-1}\vec{C}_0} + S^{-1}\vec{C}_0 \equiv 0$, where $u_A(x) \to 0$ and $u_{S^{-1}\vec{C}_0}(x) \to 0$ as $|x| \to \infty$. Then we obtain

$$|Ax| = |u_A - u_{S^{-1}\vec{C}_0} + S^{-1}\vec{C}_0| < \text{const}.$$

On the other hand, $|Ax| \to \infty$ as $|x| \to \infty$, $Ax \ne 0$. This contradiction shows that $v \not\equiv 0$.

Let us prove that if $A_1, \ldots, A_p$ is a basis in the space of skew-symmetric matrices, then $v_{A_1}, \ldots, v_{A_p}$ are linearly independent solutions, i.e., from the equality

$$\sum_{i=1}^p c_i v_{A_i} = 0, \quad c_i = \text{const},$$

follows that $c_i = 0$, $i = 1, \ldots, p$. Indeed, assume that

$$\sum_{i=1}^p c_i [u_{A_i} - u_{e_i} - A_i x + e_i] = 0,$$

where $c_i = \text{const}$, $i = 1, \ldots, p$, that is, let

$$\sum_{i=1}^p c_i A_i x = \sum_{i=1}^p c_i u_{A_i} - \sum_{i=1}^p c_i (u_{e_i} - e_i).$$

Then we set $W_1 \equiv \sum\limits_{i=1}^p c_i A_i x$, so that

$$W_1 = \sum_{i=1}^p c_i u_{A_i} - \sum_{i=1}^p c_i (u_{e_i} - e_i) \quad \text{and} \quad D_a(W_1, \Omega) < \infty.$$

To prove that $W_1 \equiv \sum_{i=1}^{p} c_i A_i x \equiv 0$, we put $T = \sum_{i=1}^{p} c_i A_i$, $Tx = \sum_{i=1}^{p} c_i A_i x$, where $T = ||t_{ij}||_{n\times n}$. Then

$$\int_\Omega |x|^a |\nabla T x|^2\, dx < \infty,$$

because $Tx = W_1$ and $\int_\Omega |x|^a |\nabla W_1|^2\, dx < \infty$. On the other hand,

$$\infty > \int_\Omega |x|^a |\nabla T x|^2\, dx = \int_\Omega |x|^a |t_{ij}|^2\, dx,$$

and the integral on the right-hand side is finite if and only if $t_{ij} = 0$, that is, $T = 0$ and $\sum_{i=1}^{p} c_i A_i = 0$, so that, $c_i = 0$, $i = 1, \dots, p$.

Thus, the mixed Dirichlet–Robin problem (1), (2) has at least $p = (n^2 - n)/2$ linearly independent generalized solutions.

Step 5. Let us show that each generalized solution $u(x)$ of the homogeneous problem (1), (2) such that $E_a(u, \Omega) < \infty$ is a linear combination of the solutions $v_{A_1}, \dots, v_{A_p}$, that is,

$$u = \sum_{i=1}^{p} c_i' v_{A_i}, \quad c_i' = \text{const}, \ i = 1, \dots, p.$$

By Lemma 2, the solution of the system (1) in Ω has a representation (3). Let us prove that there are $c_i' = \text{const}$, $i = 1, \dots, p$, such that the following equation holds for all $x \in \Omega$:

$$P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x) = \sum_{i=1}^{p} c_i' (u_{A_i} - u_{e_i} - A_i x + e_i).$$

Since $A_1, \dots, A_p$, $p = (n^2 - n)/2$, is a basis in the space of skew-symmetric matrices, there are $c_1, \dots, c_p$, such that $A = \sum_{i=1}^{p} c_i A_i$. We put

$$u_0 \equiv \sum_{i=1}^{p} c_i' (u_{A_i} - u_{e_i} - A_i x + e_i),$$

where $c_i' = -c_i$. Obviously, u_0 is a solution of (1) in Ω, $u_0 \ |_{\Gamma_1} = 0$, $(\sigma(u_0) + \tau u_0)|_{\Gamma_2} = 0$, and $E_a(u_0, \Omega) < \infty$ for $n - 2 \le a < n$.

Step 6. Consider now the difference $W = u - u_0$. By construction, $W \ |_{\Gamma_1} = 0$ and $(\sigma(W) + \tau W)|_{\Gamma_2} = 0$. It follows by the triangle inequality that $E_a(W, \Omega) < \infty$ for $n - 2 \le a < n$.

We claim that $W \equiv 0$ in Ω. Indeed, let $A = \sum_{i=1}^{p} c_i A_i$. Then

$$W(x) = b + Z(x),$$

where

$$b = B - \sum_{i=1}^{p} c_i' e_i, \quad Z(x) = \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x) - \sum_{i=1}^{p} c_i' [u_{A_i} - u_{e_i}],$$

that is, $Z = W - \vec{b}$ and $Z \ |_{\Gamma_1} = -\vec{b}$, $\quad (\sigma(Z) + \tau Z)|_{\Gamma_2} = 0$.

It is easy to see that $D(Z,\Omega) < \infty$ and $\int_\Omega |Z|^2|x|^{-2}\,dx < \infty$. Thus, we obtain problem (Z_b):

$$\begin{cases} LZ = 0 \quad \text{in} \quad \Omega, \\ Z\,|_{\Gamma_1} = -\vec{b}, \quad (\sigma(Z) + \tau Z)|_{\Gamma_2} = 0, \\ D(Z,\Omega) < \infty, \quad \displaystyle\int_\Omega |Z|^2|x|^{-2}\,dx < \infty. \end{cases}$$

By construction, we have problem (e):

$$\begin{cases} Lu_e = 0 \quad \text{in} \quad \Omega, \\ u_e\,|_{\Gamma_1} = \vec{e}, \quad (\sigma(u_e) + \tau u_e)|_{\Gamma_2} = 0, \\ D(u_e,\Omega) < \infty, \quad \displaystyle\int_\Omega |u_e|^2|x|^{-2}\,dx < \infty. \end{cases}$$

We shall now prove the uniqueness of a solution of problem (e). Let u_e' and u_e'' be solutions such that

$$u_e'\,|_{\Gamma_1} = \vec{e}, \quad (\sigma(u_e') + \tau u_e')\big|_{\Gamma_2} = 0, \quad u_e''\,|_{\Gamma_1} = \vec{e}, \quad \left(\sigma(u_e'') + \tau u_e''\right)\Big|_{\Gamma_2} = 0.$$

Then the function $u_0' = u_e' - u_e''$ satisfies

$$u_0'\,|_{\Gamma_1} = 0, \quad (\sigma(u_0') + \tau u_0')\big|_{\Gamma_2} = 0, \quad E(u_0',\Omega) < \infty, \quad D(u_0',\Omega) < \infty,$$

$$\int_\Omega |u_0'|^2|x|^{-2}\,dx < \infty.$$

We claim that $u_0' \equiv 0$ in Ω. Indeed, consider the integral identity (4) for u_0' and put $\varphi = u_0'\theta_N(x)$, where $\theta_N(x) = \theta(|x|/N)$, $\theta(s) = 1$ for $s \le 1$, $\theta(s) = 0$ for $s \ge 2$, $\theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$. We get

$$\int_\Omega \mathcal{E}(u_0')\theta_N(x)\,dx + \int_{\Gamma_2} \tau|u_0'|^2\theta_N(x)\,ds = -\int_\Omega a_{kh}^{ij} \frac{\partial u_{0j}'}{\partial x_h} \frac{\partial \theta_N(x)}{\partial x_k}\, u_0'\,dx, \tag{25}$$

where $\mathcal{E}(u_0') \equiv a_{kh}^{ij} \frac{\partial u_{0j}'}{\partial x_h} \frac{\partial u_{0i}'}{\partial x_k}$.

In the same way as in (11) (Theorem 2, case n = 2), we can show that the right-hand side of (25) tends to zero as $N \to \infty$. Hence,

$$\int_\Omega \mathcal{E}(u_0')\,\theta_N(x)\,dx + \int_{\Gamma_2} \tau|u_0'|^2\theta_N(x)\,ds \to 0 \quad \text{for} \quad N \to \infty.$$

Using the integral identity

$$\int_\Omega \mathcal{E}(u_0')\,dx + \int_{\Gamma_2} \tau|u_0'|^2 ds = 0,$$

we find that if u_0' is a solution to the homogeneous problem (1), (2), then $u_0' = A_0x + B_0$. The set of all x such that $A_0x + B_0 = 0$ is a linear manifold whose dimension is less than $n-1$, since the rank of the matrix A_0 is ≥ 2 if $A_0 \not\equiv 0$. Therefore, $u_0' = 0$. The relation

$$\int_{\Gamma_2} \tau|u_0'|^2 ds = 0,$$

implies that $u_0' \equiv 0$ on a set of positive measure on $\partial\Omega$, and therefore, $u_0' \equiv 0$, $x \in \Omega$. Thus, the solution to problem (e) is unique.

We now claim that

$$E_a(u_e,\Omega)=\int_\Omega |x|^a|\varepsilon(u_e)|^2\,dx=\infty \quad \text{for} \quad \vec{e}\neq 0. \tag{26}$$

First of all we show that if $C_0'\neq 0$ in (15), then

$$E_a(C_0'\Gamma(x),\Omega)=\int_\Omega |x|^a|\varepsilon(C_0'\Gamma(x))|^2\,dx=\infty.$$

By the properties of the fundamental solution of the elasticity system [38], if $\Gamma(x)=|x|^{2-n}U(x)$, where $U(x)$ is a homogeneous function of order zero, then $|x|^a|\varepsilon(C_0'\Gamma(x))|^2$ is a homogeneous function of order $a+2(1-n)$, that is,

$$0\not\equiv |x|^a|\varepsilon(C_0'\Gamma(x))|^2 \stackrel{def}{=} f(x)=C(C_0')|x|^{a+2-2n}f_0(x),$$

where $f_0(x)$ is a homogeneous function of order zero. We fix a point x_0 such that $f_0(x_0)\neq 0$. By continuity, $f_0(x)\neq 0$ in a neighborhood $Q_\delta(x_0)$ of x_0. We consider a cone K with vertex at the origin such that $Q_\delta(x_0)\subset K$. Then

$$E_a(C_0'\Gamma(x),\Omega)=\int_\Omega |x|^a|\varepsilon(C_0'\Gamma(x))|^2\,dx\geq C(C_0')\int_{K\cap\{x:|x|>H\}}|x|^{a+2-2n}\,dx=\infty$$

for $C_0'\neq 0$.

Applying the triangle inequality to the Formula (15) of the type $C_0'\Gamma(x)=u_e-R_{1e}(x)$, where

$$R_{1e}(x)=\sum_{0<|\alpha|\leq\beta}\partial^\alpha\Gamma(x)C_\alpha'+u_e^\beta(x),$$

we obtain

$$\infty=E_a(C_0'\Gamma(x),\Omega)\leq E_a(u_e,\Omega)+E_a(R_{1e}(x),\Omega).$$

It is easy to verify that $E_a(R_{1e}(x),\Omega)<\infty$. Hence $E_a(u_e,\Omega)=\infty$ for $\vec{e}\neq 0$, and we obtain the problem (e_a):

$$\begin{cases} Lu_e=0 \quad \text{in} \quad \Omega,\\ u_e\,|_{\Gamma_1}=\vec{e}, \quad (\sigma(u_e)+\tau u_e)|_{\Gamma_2}=0,\\ D(u_e,\Omega)<\infty, \ \int_\Omega |u_e|^2|x|^{-2}\,dx<\infty,\\ E_a(u_e,\Omega)=\infty, \quad \text{for} \quad \vec{e}\neq 0. \end{cases}$$

By Formula (26),

$$\int_\Omega |x|^a|\varepsilon(Z)|^2\,dx=\infty \quad \text{for} \quad \vec{b}\neq 0\ (\vec{e}\neq 0).$$

For the function $Z=W-\vec{b}$ we have

$$\infty=\int_\Omega |x|^a|\varepsilon(Z)|^2\,dx=\int_\Omega |x|^a|\varepsilon(W)|^2\,dx<\infty.$$

This contradiction shows that $\vec{b}=0$, that is, $\vec{e}=0$. Hence $W=Z$ is a solution of the following problem (z_0):

$$\begin{cases} LZ=0 \quad \text{in} \quad \Omega,\\ Z\,|_{\Gamma_1}=0, \quad (\sigma(Z)+\tau Z)|_{\Gamma_2}=0,\\ D(Z,\Omega)<\infty, \quad \int_\Omega |Z|^2|x|^{-2}\,dx<\infty. \end{cases}$$

By the unique solubility of the problem (e), we have $Z \equiv 0$ in Ω. Hence $W = 0$, and since $W|_{\Gamma_1} = 0$, $(\sigma(W) + \tau W)|_{\Gamma_2} = 0$, it follows that $W \equiv 0$ in Ω. This proves the theorem for $n \geq 3$.

The proof in the case $n = 2$ is carried out in a similar way. For a non-trivial constant skew-symmetric matrix A, we construct a generalized solution $u_A(x)$ of the mixed Dirichlet–Robin problem for the system (1) in Ω with the boundary conditions (20) by minimizing the corresponding functional over the class of admissible functions $\{u : u \in H^1(\Omega), \quad u|_{\Gamma_1} = Ax, \quad u$ has a compact support in $\overline{\Omega}\}$. This solution satisfies $E(u_A, \Omega) < \infty$ and $D(u_A, \Omega) < \infty$. By Hardy's inequality [6] we obtain

$$\int_{|x|>N} |u_A|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty,$$

where $N >> 1$ and $G \subset \{x : |x| < N\}$.

Let us prove that each generalized solution u of the problem (1), (2) satisfying the condition $E_a(u, \Omega) < \infty$ has the following form:

$$u = u_A - Ax,$$

where A is a constant skew-symmetric matrix. By Lemma 2, the solution of the system (1) has the form (22) with $P_A(x) = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector.

We claim that $A = 0$. For assuming that $A \neq 0$, we can write (22) in the following form:

$$u_A(x) = Ax + B + \Gamma(x)C_0 + R_1(x), \quad R_1(x) = \sum_{0<|\alpha|\leq\beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x).$$

By construction, $D(u_A, \Omega) < \infty$, that is,

$$\int_{\rho<|x|<R} |\nabla u_A|^2\, dx < C_1'$$

for each R. It is easy to see that

$$\int_\Omega |\nabla R_1(x)|^2\, dx < \infty.$$

Hence,

$$\int_{\rho<|x|<R} |\nabla R_1(x)|^2\, dx < C_2'$$

for any R. Since $\Gamma(x)$ is a fundamental solution of (1), $\Gamma(x) = S(x)\ln|x| + T(x)$, where $S(x)$ and $T(x)$ are (2×2)–matrices whose entries are homogeneous functions of order zero (see [39]), and so $|\nabla(\Gamma(x)C_0)| \leq C(C_0)|\ln|x|||x|^{-1}$. It follows that

$$\int_{\rho<|x|<R} |\nabla(\Gamma(x)C_0)|^2\, dx \leq C'(C_0) \int_{\rho<|x|<R} |x|^{-2}|\ln|x||^2\, dx \leq C'(C_0)(\ln R)^3.$$

By (22) and the triangle inequality, we have

$$\int_{|x|<R} |\nabla P_A(x)|^2\, dx \leq C_1' + C_2' + C'(C_0)(\ln R)^3.$$

On the other hand,

$$\nabla P_A(x) = A, \quad \int_{|x|<R} |\nabla P_A(x)|^2\, dx = \int_{|x|<R} |A|^2 |x| d|x| = C_3' R^2.$$

Hence $C_3' R^2 \leq C_1' + C_2' + C'(C_0)(\ln R)^3$ for each $R >> 1$. This contradiction shows that $A = 0$ and $P_A(x) = B$. Hence, $u_A = B + \Gamma(x)C_0 + R_1(x)$.

We now claim that in (22) the constant $C_0 = 0$. Assume that $C_0 \neq 0$. Then by the triangle inequality we obtain

$$\int_\Omega |\nabla(\Gamma(x)C_0)|^2\,dx < \infty.$$

On the other hand, $\Gamma(x)C_0 = (S(x)\ln|x| + T(x))C_0$. Hence in a certain cone K we have the inequality $|\nabla(\Gamma(x)C_0)|^2 \geq C(C_0)|x|^{-2}(\ln|x|)^2$. Consequently,

$$\infty > \int_\Omega |\nabla(\Gamma(x)C_0)|^2 dx \geq C(C_0)\int_{K\cap\{x:|x|>H\}} |\ln|x||^2|x|^{-2}\,dx = \infty.$$

his contradiction shows that $C_0 = 0$. Thus $u_A = B + R_1(x)$.

It is easy to verify that $E_a(u_A,\Omega) < \infty$ for $0 \leq a < 2$. Hence $E_a(u_A - Ax,\Omega) < \infty$ and $(u_A - Ax) \in \mathrm{Ker}_a(L)$, that is, the problem (1), (2), with condition $E_a(u,\Omega) < \infty$, $0 \leq a < 2$ has at least one non-zero solution, so that $\dim \mathrm{Ker}_a(L) \geq 1$.

On the other hand, $\mathrm{Ker}_a(L) \subset \mathrm{Ker}_0(L)$ for $a \geq 0$, and therefore, $\dim \mathrm{Ker}_a(L) \leq \dim \mathrm{Ker}_0(L)$. By Theorem 1, $\dim \mathrm{Ker}_0(L) = 1$. Thus, we have $\dim \mathrm{Ker}_a(L) = 1$ for $0 \leq a < 2$. The theorem is proved. □

Theorem 4. *If* $n \leq a < \infty$, $\quad n \geq 2$, *then* $\dim \mathrm{Ker}_a(L) = 0$.

Proof. Consider the case $n \geq 3$. Let $a = n$. We shall prove the theorem by contradiction. Assume that $\dim \mathrm{Ker}_a(L) > 0$. Then there is a u such that $u \in \mathrm{Ker}_a(L)$ and $u \not\equiv 0$. Since $a = n$, we have $u \in \mathrm{Ker}_n(L) \subset \mathrm{Ker}_{n-2}(L)$. Hence by Theorem 3 we obtain

$$u = u_A - Ax - u_e + \vec{e}, \tag{27}$$

where $\vec{e} = S^{-1}\vec{C}_0$ (see (19)) and $\vec{C}_0$ is defined by Formula (23). Substituting (15) and (23) in (27), we obtain

$$u = P(x) + (\vec{C}_0 - \vec{C}_0')\Gamma(x) + \sum_{0<|\alpha|\leq\beta} \partial^\alpha\Gamma(x)\tilde{C}_\alpha + u_0^\beta(x),$$

where $P(x) = -Ax + \vec{e}$, $\quad \tilde{C}_\alpha = C_\alpha - C_\alpha'$, $\quad u_0^\beta(x) = u_A^\beta(x) - u_e^\beta(x)$. Since $\vec{C}_0' = S\vec{e} = S(S^{-1}\vec{C}_0) = SS^{-1}\vec{C}_0 = \vec{C}_0$, it follows that $\vec{C}_0 - \vec{C}_0' = 0$. Hence,

$$u = P(x) + (\tilde{C}_1\nabla)\Gamma(x) + R_2(x), \tag{28}$$

where

$$R_2(x) = \sum_{1<|\alpha|\leq\beta} \partial^\alpha\Gamma(x)\tilde{C}_\alpha + u^\beta(x).$$

We claim that $\tilde{C}_1 \neq 0$ in (28). Indeed, we assume that $\tilde{C}_1 = 0$. Taking the scalar product of (1) with u and integrating over Ω_R, we obtain

$$\int_{\Omega_R} \mathcal{E}(u)\,dx = \left(\int_{\partial\Omega} + \int_{|x|=R}\right) u\,\sigma(u)\,ds. \tag{29}$$

We claim that

$$\int_{|x|=R} u\,\sigma(u)\,ds \to 0 \quad \text{for} \quad R \to \infty.$$

It is easy to verify that $|u| \leq C|x|$ and $|\sigma(u)| \leq C|x|^{-n-1}$ for $\tilde{C}_1 = 0$. Next, using the Cauchy–Schwartz inequality we obtain

$$\left|\int_{|x|=R} u\,\sigma(u)\,ds\right| \leq c\int_{|x|=R} |u||\sigma(u)|\,ds$$

$$\leq c \left[\int_{|x|=R} |x|^2 \, ds \right]^{1/2} \left[\int_{|x|=R} |x|^{(-n-1)2} \, ds \right]^{1/2} = cR^{-1} \to 0 \quad \text{as} \quad R \to \infty.$$

There exists a sequence of domains Ω_{R_k} such that $\Omega_{R_k} \subset \Omega_{R_{k+1}} \subset \cdots \subset \mathbb{R}^n$, $\cup_k \Omega_{R_k} = \Omega$. In equality (29) we pass to the limit as $R = R_k \to \infty$. By the Cauchy–Schwartz inequality,

$$\left| \int_{\Omega_R} \mathcal{E}(u) \, dx \right| \leq c \left[\int_{\Omega_R} \left| \frac{\partial u_j}{\partial x_h} \right|^2 \, dx \right]^{1/2} \left[\int_{\Omega_R} \left| \frac{\partial u_i}{\partial x_k} \right|^2 \, dx \right]^{1/2} < \infty,$$

because $\Omega_R \subset \Omega$ and $\int_\Omega |\nabla u|^2 \, dx < \infty$. Thus,

$$\int_\Omega \mathcal{E}(u) \, dx = \left(\int_{\Gamma_1} + \int_{\Gamma_2} \right) u \, \sigma(u) \, ds.$$

On the other hand, $u = u_A - Ax - (u_e - \vec{e})$ and $u \, |_{\Gamma_1} = 0, \quad (\sigma(u) + \tau u)|_{\Gamma_2} = 0$. Hence,

$$\int_{\Gamma_1} u \, \sigma(u) \, ds = 0 \quad \text{and} \quad \int_\Omega \mathcal{E}(u) \, dx + \int_{\Gamma_2} \tau |u|^2 \, ds = 0.$$

Using the integral identity

$$\int_\Omega \mathcal{E}(u) \, dx + \int_{\Gamma_2} \tau |u|^2 \, ds = 0,$$

we find that if u is a solution to the homogeneous problem (1), (2), then $u = A_0 x + B_0$. The set of all x such that $A_0 x + B_0 = 0$ is a linear manifold whose dimension is less than $n-1$, since the rank of the matrix A_0 is ≥ 2 if $A_0 \not\equiv 0$. Therefore, $u = 0$. The relation

$$\int_{\Gamma_2} \tau |u|^2 ds = 0,$$

implies that $u \equiv 0$ on a set of positive measure on $\partial\Omega$, and therefore, $u \equiv 0$.

This is a contradiction, since $u \in \mathrm{Ker}_a(L)$ and $u \not\equiv 0$. Thus, $\tilde{C}_1 \neq 0$ in (28).

By assumption, $E_a(u, \Omega) < \infty$. It is easy to verify that $E_a(R_2(x), \Omega) < \infty$ and $E_a(P(x), \Omega) = 0$. Now, by the triangle inequality we obtain

$$E_a((\tilde{C}_1 \nabla)\Gamma(x), \Omega) < \infty \quad \text{for} \quad a = n.$$

By the properties of the fundamental solution of the system (1) (see [38]) we have $\Gamma(x) = |x|^{2-n} U(x)$, where $U(x)$ is a homogeneous function of order zero. Hence $|x|^n |\varepsilon((\tilde{C}_1 \nabla)\Gamma(x))|^2$ is a homogeneous function of order $(-n)$, that is,

$$0 \not\equiv |x|^n |\varepsilon((\tilde{C}_1 \nabla)\Gamma(x))|^2 \overset{def}{\equiv} f(x) = |x|^{-n} f_0(x),$$

where $f_0(x)$ is a homogeneous function of order zero. We fix a point x_0 such that $f_0(x_0) \neq 0$. By continuity, $f_0(x) \neq 0$ in a neighborhood $Q_\delta(x_0)$ of x_0. We consider a cone K with vertex at the origin such that $Q_\delta(x_0) \subset K$. Then

$$\infty > \int_\Omega |x|^n |\varepsilon((\tilde{C}_1 \nabla)\Gamma(x))|^2 \, dx \geq C(\tilde{C}_1) \int_{\{x:|x|>H\}} |x|^{-n} \, dx = \infty.$$

This contradiction shows that $u \equiv 0$. This completes the proof for $n \geq 3$.

Consider now the case $n = 2$. It sufficient to show that $\dim \mathrm{Ker}_a(L) = 0$ for $a = 2$. Assume that $\dim \mathrm{Ker}_a(L) > 0$, that is, there exists u such that $u \in \mathrm{Ker}_a(L)$ and $u \not\equiv 0$. Since $a = 2$, it follows that $u \in \mathrm{Ker}_a(L) \subset \mathrm{Ker}_0(L)$. Hence by Theorem 3 we obtain

$$u = u_A - Ax$$

and for u_A Lemma 2 yields a representation (22), that is,

$$u_A(x) = P_A(x) + R(x),$$

where $P_A(x) = Ax + B$, A is a constant skew-symmetric matrix and B is a constant vector. Substituting the expansion of $u_A(x)$ in the representation of $u(x)$, we obtain

$$u = B + \Gamma(x)C_0 + R_1(x), \quad R_1(x) = \sum_{0<|\alpha|\le\beta} \partial^\alpha \Gamma(x) C_\alpha + u_A^\beta(x).$$

We prove that $C_0 = 0$ by contradiction. Indeed, assume that $C_0 \neq 0$. Then we have $E_a(R(x),\Omega)=E_a(u,\Omega)<\infty$.

On the other hand, $\Gamma(x) = S(x)\ln|x| + T(x)$, where $S(x)$ and $T(x)$ are (2×2)– matrices whose entries are homogeneous functions of order zero (see [39]). Hence,

$$|\varepsilon(R(x))|^2 \ge C|x|^{-2}(\ln|x|)^2$$

in some cone K, and therefore

$$\infty > E_a(R(x),\Omega) = \int_\Omega |x|^a |\varepsilon(R(x))|^2\,dx$$

$$\ge C(C_0)\int_{K\cap\{x:|x|>H\}} |x|^{a-2}(\ln|x|)^2\,dx = \infty$$

for $a \ge 2$. This contradiction shows that $C_0 = 0$. Thus,

$$u = B + R_1(x), \quad R_1(x) = O(|x|^{-1}\ln|x|).$$

Taking the scalar product of (1) and u and integrating over Ω_R, we obtain

$$\int_{\Omega_R} \mathcal{E}(u)\,dx = \left(\int_{\partial\Omega} + \int_{|x|=R}\right) u\,\sigma(u)\,ds.$$

In view of the boundary conditions $u\ |_{\Gamma_1}= 0$, $(\sigma(u)+\tau u)|_{\Gamma_2} = 0$, and $\int_{\Gamma_1} u\,\sigma(u)\,ds = 0$, we have

$$\int_\Omega \mathcal{E}(u)\,dx + \int_{\Gamma_2} \tau|u|^2\,ds = \int_{|x|=R} u\,\sigma(u)\,ds. \tag{30}$$

Since $|u| \le C$ and $|\sigma(u)| \le C|x|^{-2}\ln|x|$, it follows that

$$\left|\int_{|x|=R} u\,\sigma(u)\,ds\right| \le \int_{|x|=R} |u||\sigma(u)|\,ds$$

$$\le C\int_{|x|=R} |x|^{-2}\ln|x|\,ds = CR^{-1}\ln R \to 0 \quad \text{as} \quad R\to\infty.$$

Passing to the limit as $R \to \infty$ in equality (30), we obtain

$$\int_\Omega \mathcal{E}(u)\,dx + \int_{\Gamma_2} \tau|u|^2\,ds = 0.$$

Using the obtained integral identity, we conclude that if u is a solution to the homogeneous problem (1), (2), then $u = A_1 x + B_1$, where A_1 is a constant skew-symmetric matrix, B_1 is a constant vector. Hence,

$$A_1 x + B_1 = B + R_1(x),$$

where $R_1(x) \to 0$ as $|x| \to \infty$. Thus, we have $A_1 x + (B_1 - B) \to 0$, which is possible only if $A_1 = 0$ and $B_1 = B$. In view of boundary conditions (2), $B = 0$ and $u \equiv 0$, so that $\dim \operatorname{Ker}_a(L) = 0$ for $n = 2$ and $2 \le a < \infty$. The theorem is proved. $\square$

Theorem 5. *The mixed Dirichlet–Robin problem* (1), (2) *with the condition* $E_a(u, \Omega) < \infty$ *has* $k(r,n)$ *linearly independent solutions for* $-2r - n \le a < -2r - n + 2$, *that is,*

$$\dim \operatorname{Ker}_a(L) = k(r,n),$$

where $r > 0$ *and*

$$k(r,n) = \begin{cases} n\left(\binom{r+n-1}{n-1} + \binom{r+n-2}{n-1}\right), & \textit{if} \quad n \ge 3, \\ 4r + 2, & \textit{if} \quad n = 2. \end{cases}$$

Here $\binom{r}{s}$ *is binomial coefficient from* r *to* s, $\binom{r}{s} = 0$ *if* $s > r$.

Proof. Assume that $n > 2$. To prove the theorem, we need to determine the number of linearly independent polynomial solutions of a system (1), the degree of which does not exceed the fixed number.

Let $P = (P_1, \dots, P_n)$ be a polynomial solution of the system (1) of degree r. Then the degree of the polynomial P_i does not exceed r, and P can be represented in the following form:

$$P = \sum_{s=0}^{r} P^{(s)}, \qquad (P)$$

where $P^{(s)} = (P_1^{(s)}, \dots, P_n^{(s)})$ is a homogeneous polynomial of degree s, satisfying the system (1) (see [38]).

The space of polynomials in $\mathbb{R}^n$ of degree at most r has dimension $(r+n)!/r!n!$ (see [41]). Hence the dimension of the space of vector-valued polynomials in $\mathbb{R}^n$ of degree at most r is equal to

$$\frac{n(r+n)!}{r!n!} = \frac{(r+n)!}{r!(n-1)!}.$$

Polynomials of this kind solving the elasticity system form a space of dimension

$$\frac{(r+n)!}{r!(n-1)!} - \frac{(r+n-2)!}{(r-2)!(n-1)!} = n\left(\frac{(r+n)!}{r!n!} - \frac{(r+n-2)!}{(r-2)!n!}\right),$$

because each equation of the elasticity system is equivalent to the vanishing of some polynomial of degree $(r-2)$.

We denote by $k(r,n)$ the number of linearly independent polynomial solutions of (1) whose degree is at most r, and let $l(r,n)$ be the number of linearly independent homogeneous polynomials of degree r that are solutions of (1). Using representation (P) we obtain

$$k(r,n) = \sum_{s=0}^{r} l(s,n),$$

where

$$l(s,n) = n\left(\binom{s+n-2}{n-2} + \binom{s+n-3}{n-2}\right) \text{ for } s \ge 1, \quad l(0,n) = n.$$

We now prove the following statements are true:

(i) The Dirichlet–Robin problem (1), (2) with the condition $E_a(u,\Omega) < \infty$ has $k(r,n)$ linearly independent solutions for $-2r-n \leq a < -2r-n+2$;

(ii) Each system of $k(r,n)+1$ solutions is linearly dependent.

(i) Let $w_1,\ldots,w_k$ be a basis in the space of polynomial solutions of (1) whose degrees do not exceed r. Since $\operatorname{ord} w_i \leq r$, it follows that $E_a(w_i,\Omega) < \infty$ for $-2r-n \leq a < -2r-n+2$. For each w_i, $i=1,\ldots,k$ we consider the solution v_i of the system (1) such that $v_i\,|_{\Gamma_1} = w_i$, $\quad (\sigma(v_i)+\tau v_i)|_{\Gamma_2} = 0$ and

$$D(v_i,\Omega) < \infty, \quad \int_\Omega |v_i|^2|x|^{-2}\,dx < \infty.$$

Such a solution we can construct by the variational method, minimizing the corresponding functional over the class of admissible functions $\{v : v \in H^1(\Omega), v\,|_{\Gamma_1} = w,\ v \text{ has compact support in } \overline{\Omega}\}$.

Consider next the difference: $z_i = w_i - v_i$. We have $Lz_i = 0$ in Ω, $z_i\,|_{\Gamma_1} = 0$, $(\sigma(z_i)+\tau z_i)|_{\Gamma_2} = 0$ and $E_a(z_i,\Omega) < \infty$.

Let us prove that z_i, $i=1,\ldots,k$, are linearly independent. Indeed, if

$$Z \equiv \sum_{i=1}^{k} c_i z_i = 0,\ c_i = \text{const}, \quad \text{then} \quad \sum_{i=1}^{k} c_i(w_i - v_i) = 0,$$

that is,

$$W \equiv \sum_{i=1}^{k} c_i w_i = \sum_{i=1}^{k} c_i v_i = V.$$

Hence, $|W|^2 = |V|^2$, $|\nabla W|^2 = |\nabla V|^2$ and

$$\begin{gathered} D(W,\Omega) = D(V,\Omega) < \infty, \\ \int_\Omega |W|^2|x|^{-2}\,dx = \int_\Omega |V|^2|x|^{-2}\,dx < \infty. \end{gathered} \tag{31}$$

By Lemma 1, the solution V of the system (1) in Ω has the following form:

$$V(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial, and

$$R(x) = \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + V^\beta(x), \quad R(x) = O(|x|^{2-n}).$$

It is easy to verify that

$$D(R(x),\Omega) < \infty, \quad \int_\Omega |R(x)|^2|x|^{-2}\,dx < \infty \quad \text{for} \quad n > 2.$$

By the triangle inequality,

$$D(P(x),\Omega) < \infty, \quad \int_\Omega |P(x)|^2|x|^{-2}\,dx < \infty.$$

We claim that $P(x) \equiv 0$. Indeed, assume that $\operatorname{ord} P(x) = r$. Then in the interior of a certain cone K we have $|\nabla P(x)| \geq C|x|^{r-1}$. Hence,

$$\infty > D(P(x),\Omega) \geq C \int_{K\cap\{x:|x|>H\}} |x|^{(r-1)2}\,dx = C\int_{|x|>H} |x|^{2r-2+n}|x|^{-1}\,d|x|.$$

This integral converges only when $r < 0$. Therefore, $P(x) \equiv 0$.

Thus, $V(x) = R(x)$, where $R(x) \to 0$ as $|x| \to \infty$, that is, $V(x) \to 0$ as $|x| \to \infty$. Hence,

$$\sum_{i=1}^{k} c_i w_i \equiv W = V \to 0 \quad \text{as} \quad |x| \to \infty,$$

and by the estimates (31) we obtain $\sum\limits_{i=1}^{k} c_i w_i \equiv 0$.

Since w_i is a basis in the space of polynomial solutions of (1) whose degrees do not exceed r, it follows that $c_i = 0, \quad i = 1, \ldots, k$. Hence the problem has at least $k(r,n)$ linearly independent solutions.

(ii) Let us prove that each solution u of the system (1) with boundary conditions $u \mid_{\Gamma_1} = 0$, $(\sigma(u) + \tau u)|_{\Gamma_2} = 0$ and $E_a(u, \Omega) < \infty$ can be represented as a linear combination of the solutions z_i, $i = 1, \ldots, k$, $z_i = w_i - v_i$. By Lemma 1, every solution of the system (1) in Ω may be written as

$$u(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial of degree $\operatorname{ord} P(x) \leq m = [1 - n/2 - a/2]$,

$$R(x) = \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x).$$

Since $-2r - n \leq a < -2r - n + 2$, it follows that $-n/2 - a/2 \leq r < 1 - n/2 - a/2$ and, therefore, $r = [1 - n/2 - a/2] = m$. Hence $\operatorname{ord} P(x) \leq r$.

We claim that $P(x)$ is a solution of the system (1). Indeed,

$$0 = Lu = LP(x) + LR(x), \quad \text{where} \quad LR(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Since $LP(x)$ is a polynomial and $LP(x) = -LR(x) \to 0$ as $|x| \to \infty$, it follows that $LP(x) \equiv 0$, that $P(x)$ is a polynomial solution of the system (1). Hence it is represented as a linear combination of the functions w_i, $i = 1, \ldots, k$:

$$P(x) = \sum_{i=1}^{k} c_i w_i.$$

We claim that $u = \sum\limits_{i=1}^{k} c_i z_i$. We set

$$u_0 = u - \sum_{i=1}^{k} c_i z_i.$$

By our construction of the solutions, after elementary transformations we obtain

$$u_0 = R(x) + \sum_{i=1}^{k} c_i v_i.$$

Let us prove that $u_0 \equiv 0$. Indeed, u_0 is a solution, that is, $Lu_0 = 0$ in Ω, $u_0 \mid_{\Gamma_1} = 0$, and $(\sigma(u_0) + \tau u_0)|_{\Gamma_2} = 0$. By the construction of the solutions v_i we have

$$D(v_i, \Omega) < \infty, \quad \int_\Omega |v_i|^2 |x|^{-2} \, dx < \infty, \; i = 1, \ldots, k.$$

Moreover, it is easy to verify that

$$D(R(x), \Omega) < \infty, \quad \int_\Omega |R(x)|^2 |x|^{-2} \, dx < \infty.$$

Hence,

$$D(u_0,\Omega) < \infty, \quad \int_\Omega |u_0|^2|x|^{-2}\,dx < \infty.$$

Since $u_e(x)$ is a unique solution of problem (e) in Theorem 3, it follows that $u_0 \equiv 0$. This proves the theorem for $n > 2$.

The proof for $n = 2$ is similar. We claim that

(i) The Dirichlet–Robin problem (1), (2) along with the condition $E_a(u,\Omega) < \infty$ has $k(r,2)$ linearly independent solutions for $-2r-2 \le a < -2r$;

(ii) Each system of $k(r,2)+1$ solutions is linearly dependent.

(i) Let $w_1,\ldots,w_k$ be a basis in the space of polynomial solutions of (1) whose degrees do not exceed r. The vector-valued functions $u_1 = (\mathrm{const},0)$ and $u_2 = (0,\mathrm{const})$ are linearly independent polynomial solutions of (1). Therefore we can assume without loss of generality that $w_1 \equiv (1,0)$, $w_2 \equiv (0,1)$. The condition $\operatorname{ord} w_i \le r$ shows that $|w_i| \le c|x|^r$ and, therefore, $E_a(w_i,\Omega) < \infty$ for $-2r-2 \le a < -2r$.

For each w_i, $i = 3,\ldots,k$ we consider a solution v_i of the system (1) such that $v_i \;|_{\Gamma_1} = w_i$, $\quad (\sigma(v_i)+\tau v_i)|_{\Gamma_2} = 0$, $\quad D(v_i,\Omega) < \infty$ and, by Hardy's inequality [6], we have

$$\int_{|x|>N} |v_i|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty,$$

where $N >> 1$ such that $G \subset \{x : |x| < N\}$.

Such a solution may be constructed by the variational method, by minimizing the corresponding functional over the class of admissible functions. In the same way, we can construct solutions of (1) with boundary conditions $v_i \;|_{\Gamma_1} = \Gamma(x)\tilde{c}_i$, $\quad (\sigma(v_i)+\tau v_i)|_{\Gamma_2} = 0$, $\quad i = 1,2$, where $\tilde{c}_1 = (1,0)$ and $\tilde{c}_2 = (0,1)$ such that $E(v_i,\Omega) < \infty$ and $D(v_i,\Omega) < \infty$. By Hardy's inequality [6] we obtain

$$\int_{|x|>N} |v_i|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty,$$

where $N >> 1$ and $G \subset \{x : |x| < N\}$.

Let $z_i = w_i - v_i$, $i = 3,\ldots,k,k+1,k+2$, where $w_{k+1} = \Gamma(x)\tilde{c}_1$ and $w_{k+2} = \Gamma(x)\tilde{c}_2$. Then $Lz_i = 0$ in Ω, $z_i \;|_{\Gamma_1} = 0$, $\quad (\sigma(z_i)+\tau z_i)|_{\Gamma_2} = 0$, and

$$E_a(z_i,\Omega) \le CD_a(z_i,\Omega) < \infty.$$

We claim that z_i, $i = 3,\ldots,k+2$ are linearly independent. Indeed, if

$$Z \equiv \sum_{i=3}^{k+2} c_i' z_i = 0,\ c_i' = \mathrm{const}, \quad \text{then} \quad \sum_{i=3}^{k+2} c_i'(w_i - v_i) = 0,$$

that is,

$$W \equiv \sum_{i=3}^{k+2} c_i' w_i = \sum_{i=3}^{k+2} c_i' v_i \equiv V.$$

Hence $|W|^2 = |V|^2$, $|\nabla W|^2 = |\nabla V|^2$, and

$$D(W,\Omega) = D(V,\Omega) < \infty,$$
$$\int_{|x|>N} |W|^2|x|^{-2}|\ln|x||^2\,dx = \int_{|x|>N} |V|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty.$$

By Lemma 1, the solution V of the system (1) in Ω has a representation

$$V(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial and $R(x) = \sum_{\beta_0<|\alpha|\le\beta} \partial^\alpha\Gamma(x)C_\alpha + v^\beta(x)$.

We prove that $\operatorname{ord} P(x) = 0$ by contradiction. Indeed, assume that $\operatorname{ord} P(x) = k$, where $k \neq 0$. Then in the interior of a certain cone K we have $|P(x)| \geq C|x|^k$. In addition, $R(x) \sim \ln|x|$. Hence,

$$|V(x)| = |P(x) + R(x)| \geq |P(x)| - |R(x)| \geq C|x|^k - C\ln|x| \geq \frac{C}{2}|x|^k$$

for $|x| >> 1$. This yields the inequalities

$$\infty > \int_{|x|>N} |V|^2|x|^{-2}|\ln|x||^{-2}\,dx \geq \frac{C}{2}\int_{|x|>N} |x|^{2k-2}|\ln|x||^{-2}\,dx.$$

The resulting integral diverges for $k \geq 1$. Hence $\operatorname{ord} P(x) = 0$ and $P(x) = C = \text{const}$. Thus,

$$V = C + \Gamma(x)C_0 + R_1(x), \quad R_1(x) = \sum_{0<|\alpha|\leq\beta} \partial^\alpha\Gamma(x)C_\alpha + v^\beta(x).$$

We now claim that $C_0 = 0$. Indeed, if $C_0 \neq 0$, then

$$|V| = |C + \Gamma(x)C_0 + R_1(x)| \geq \text{const}(C_0)|\ln|x|| - C - |R_1(x)| \geq \frac{\text{const}(C_0)}{2}\ln|x|,$$

because $|R_1(x)| \ll 1$ for $|x| >> 1$. By Hardy's inequality [6],

$$\infty > \int_{|x|>N} |V|^2|x|^{-2}|\ln|x||^{-2}\,dx$$

$$\geq \frac{\text{const}(C_0)}{2}\int_{|x|>N} |\ln|x||^2|x|^{-2}|\ln|x||^{-2}\,dx = \infty.$$

This contradiction shows that $C_0 = 0$. Hence,

$$C + R_1(x) = \sum_{i=3}^{k+2} c_i' w_i,$$

and, therefore,

$$\sum_{i=3}^{k+2} c_i' w_i - C = R_1(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{32}$$

Let $C = -(c_1', c_2')$, then by our choice of the function w_1 and w_2, we obtain

$$-C = c_1' w_1 + c_2' w_2.$$

Hence, by (32),

$$\sum_{i=1}^{k+2} c_i' w_i = 0, \quad \text{that is,} \quad \sum_{i=1}^{k} c_i' w_i = -c_{k+1}' w_{k+1} - c_{k+2}' w_{k+2}.$$

Since $w_{k+1} = \Gamma(x)\tilde{c}_1$ and $w_{k+2} = \Gamma(x)\tilde{c}_2$, it follows that

$$\sum_{i=1}^{k} c_i' w_i = -c_{k+1}'\Gamma(x)\tilde{c}_1 - c_{k+2}'\Gamma(x)\tilde{c}_2. \tag{33}$$

The left-hand side of (33) is a polynomial while its right-hand side has logarithmic growth as $|x| \to \infty$, therefore they both vanish, and $\sum\limits_{i=1}^{k} c_i' w_i = 0$.

Since w_i, $i = 1,\dots,k$, form a basis in the space of polynomial solutions, it follows that $c'_i = 0$, $i = 1,\dots,k$, $\Gamma(x)(c'_{k+1}\tilde{c}_1 + c'_{k+2}\tilde{c}_2) = 0$, and in view of the linear independence of the vectors $\tilde{c}_1$ and $\tilde{c}_2$, we obtain $c'_{k+1} = c'_{k+2} = 0$.

Hence the z_i, $i = 3,\dots,k+2$, are linearly independent solutions, that is, the Dirichlet–Robin problem (1), (2) supplemented with the condition $E_a(u,\Omega) < \infty$ has at least, $k(r,2)$ linearly independent solutions.

(ii) Let us prove that each solution u of (1) with the boundary conditions (2) and $E_a(u,\Omega) < \infty$ for $-2r-2 \le a < -2r$ may be represented as a linear combination of the sulutions z_i, $i = 3,\dots,k+2$, $z_i = w_i - v_i$.

By Lemma 1, a sloution u of (1) in Ω has the form

$$u = P(x) + \Gamma(x)C_0 + R_1(x),$$

where $P(x)$ is a polynomial of degree $\operatorname{ord} P(x) \le m = [-a/2]$ and $R_1(x) = \sum\limits_{0<|\alpha|\le\beta} \partial^\alpha\Gamma(x)C_\alpha + u^\beta(x)$.

Since $-2r-2 \le a < -2r$, we have $-1-a/2 \le r < -a/2$ and, therefore, $r = [-a/2] = m$. Hence $\operatorname{ord} P(x) \le r$.

We claim that $P(x)$ is a solution of (1). Indeed, we have

$$0 = Lu(x) = LP(x) + LR(x),$$

where $LR(x) \to 0$ as $|x| \to \infty$.

Since $LP(x)$ is a polynomial and $LP(x) = -LR(x) \to 0$ as $|x| \to \infty$, it follows that $LP(x) \equiv 0$, that is, $P(x)$ is a polynomial solution of (1). Hence it is a finite linear combination of the functions w_i, $i = 1,\dots,k$:

$$P(x) = \sum_{i=1}^{k} d_i w_i, \quad d_i = \text{const}.$$

Let $C_0 = d_{k+1}\tilde{c}_1 + d_{k+2}\tilde{c}_2$. To show that $u = \sum\limits_{i=3}^{k+2} d_i z_i$, we put

$$u_0 = u - \sum_{i=3}^{k+2} d_i z_i.$$

After elementary transformations we obtain

$$u_0 = P(x) + \Gamma(x)C_0 + R_1(x) - \sum_{i=3}^{k+2} d_i(w_i - v_i$$

$$= \Psi + R_1(x) + \sum_{i=3}^{k+2} d_i v_i,$$

where $\Psi = \sum\limits_{i=1}^{2} d_i w_i$.

We claim that $u_0 \equiv 0$. Indeed, we have $Lu_0 = 0$ in Ω, $u_0 \,|_{\Gamma_1} = 0$, $(\sigma(u_0) + \tau u_0)|_{\Gamma_2} = 0$. By the construction of the solutions v_i we get

$$D(v_i,\Omega) < \infty, \quad \int_{|x|>N} |v_i|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty, \quad i = \overline{3,k+2}.$$

It is easy to verify that $D(R_1(x),\Omega) < \infty$, $D(\Psi,\Omega) < \infty$ and

$$\int_{|x|>N} |R_1(x)|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty, \quad \int_{|x|>N} |\Psi|^2|x|^{-2}|\ln|x||^{-2}\,dx < \infty.$$

Hence it follows by the triangle inequality that $D(u_0, \Omega) < \infty$ and

$$\int_{|x|>N} |u_0|^2 |x|^{-2} |\ln|x||^{-2}\, dx < \infty.$$

Using the unique solubility of problem (w), we now see from Theorem 3 that $u_0 \equiv 0$. Hence,

$$u = \sum_{i=3}^{k+2} d_i(w_i - v_i).$$

Thus the problem (1), (2) has $(k+2) - 2 = k = k(r, 2)$ linearly independent solutions. The proof is complete. □

4. Conclusions

The problem of studying boundary value problems for the system of elasticity theory began to be dealt with at the beginning of the 20th century. One of the first papers initiating the systematic investigation of these problems was Fredholm's classical paper [42], in which the first boundary value problem for the linear elasticity system in the case of an isotropic homogeneous body was studied by the method of integral equations. The second boundary value problem for the elasticity system in the case of a bounded domain was studied by Korn [43], who was the first to establish inequalities between the Dirichlet integral $D(u, \Omega)$ of the solution and the energy $E(u, \Omega)$ of the system, which are now known as Korn's inequalities. Friedrichs's paper [44] played a major role in the analysis of the mathematical aspects of the stationary elasticity theory. In that paper Korn's inequalities are proved and the first and the second boundary value problems of the elasticity theory are analyzed in a bounded domain by the variational method. Here we also note Fichera's monograph [5], who used Korn's inequalities and functional methods to study various boundary value problems for the elasticity system. For a wide class of unbounded domains Kondratiev and Oleinik [6–8] established generalizations of Korn's and Hardy's inequalities and used them for the analysis of the main boundary value problems for the elasticity system. In particular, they investigated the existence, uniqueness and stability of solutions of boundary value problems with a finite energy integral.

This article considers the boundary value problem for the elasticity system in the exterior of a compact set with the mixed boundary conditions: the Dirichlet condition on one part of the boundary and the Robin condition on the other; and also with the condition of boundedness of the energy integral $E_a(u, \Omega)$ with the weight $|x|^a$, which characterizes the behavior of the solution of this problem at infinity. Depending on the value of the parameter a, for each interval, we determine the dimension of the kernel of the operator of the theory of elasticity. The main research method for constructing solutions to the mixed Dirichlet–Robin problem is the variational principle, which assumes the minimization of the corresponding functional in the class of admissible functions. Further, using Korn's and Hardy's-type inequalities, we obtain a criterion for the uniqueness (or non-uniqueness) of solutions to this problem in weighted spaces. These results find their practical application in the field of shell theory, mechanics of deformable solids, as well as in the study of some problems in the theory of scattering, optics, applied and astrophysics.

Note that a new inequality called the Korn's interpolation inequality (since it interpolates between the first and second Korn's inequalities) was applied to study shells. An asymptotically exact version of the interpolation estimate was proved by Harutyunyan (see [12], and other papers) for practically any thin domains and any vector field.

Further, this theory has found its development in many papers in the field of mathematical physics and applied mathematics; some of them are given in the bibliography.

5. Application

As an application, we note the book [45], in which astronomical optics and the elasticity theory give a very complete and comprehensive description of what is known in this field. After extensive introduction to optics and elasticity, this book discusses a multimode deformable mirror of variable curvature, as well as in-depth active optics, its theory, and fields of application.

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