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Article

Homogenization of a 2D Tidal Dynamics Equation

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Abstract: This work deals with the homogenization of two dimensions' tidal equations. We study the asymptotic behavior of the sequence of the solutions using the sigma-convergence method. We establish the convergence of the sequence of solutions towards the solution of an equivalent problem of the same type.

Keywords: homogenization; tiidal equation; sigma convergence

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1. Introduction

Ocean tides have been investigated by many authors, starting from [1,2]. The last decades have been marked by rapid progress in both theoretical and experimental studies of ocean tidal phenomena. Today, experimental and theoretical knowledge of ocean tides are used in order to address important problems in Oceanography, Atmospheric Sciences, Geophysics, as well as in Electronics and Telecommunications. It is important to point out that Laplace [3] was the first author to make the first major input in the theoretical formulation for tides of water on a rotating globe. Indeed, he formulated a system of partial differential equations that relate the horizontal flow to the surface height of the ocean. The existence and uniqueness of solutions of the deterministic tidal equation while using the classical compactness method have been proven in [2,4]. In our work, we consider the deterministic counterpart of a model of tidal dynamics that was studied by Manna et al. [5] and originally proposed by Marchuk and Kagan [2], where they considered the model of tidal dynamics derived by taking the shallow water model on a rotating sphere, with the latter being a slight generalization of the one considered earlier by Laplace.

Our objective is to carry out the homogenization of (2)–(5) under a suitable structural assumption on the coefficients of the operator that is involved in (2). These assumptions cover several physical behaviors, such as the periodicity, the almost periodicity, and much more. In order to achieve our goal, we shall use the concept of sigma-convergence [6], which is roughly a formulation of the well-known two-scale convergence method [7–13] in the context of algebras with mean value [6,14–16]. Therefore, our study falls within the framework of homogenization beyond the periodic setting, but including the periodic study as a special case.

The outline of the paper is as follows. The statement of the model problem, together with the derivation of appropriate uniform estimates, are the objectives of Section 2. Section 3 deals with the fundamentals of the sigma-convergence method. The homogenization process is performed in Section 4, while, in Section 5, we provide some applications of the main homogenization result.

2. Setting of the Problem and Uniform Estimates

2.1. Statement of the Problem

The tidal dynamics system that was developed by Manna et al. [5] for suitably normalized velocity $\mathbf{u}$ and tide height z reads as

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) + g\nabla z = f \text{ in } Q = \Omega \times (0,T) \\ \dfrac{\partial z}{\partial t} + \operatorname{div}(h\mathbf{u}) = 0 \text{ in } Q \\ \mathbf{u} = 0 \text{ on } \partial\Omega \times (0,T) \\ \mathbf{u}(x,0) = \mathbf{u}^0(x) \text{ and } z(x,0) = z^0(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where Ω is an open bounded subset, where $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{pmatrix} -\alpha \,\triangle & -\eta \\ \eta & -\alpha \,\triangle \end{pmatrix} \text{ and } \mathbf{B}\,(u) = \gamma \left|\mathbf{u} + \omega^0\right| \left(\mathbf{u} + \omega^0\right),$$

α and η (the Coriolis parameter) being positive constants, ω^0 a given function and $\gamma(x) = r/h(x)$.

In our work, we ignore the Coriolis parameter ($\eta = 0$), so that $\mathbf{A}(\mathbf{u}) = -\alpha\Delta\mathbf{u}$. However, instead of the Laplace operator, we rather consider a general linear elliptic operator of order 2 in divergence form, leading to the investigation of the limiting behavior (when $0 < \varepsilon \to 0$) of the generalized sequence $(\mathbf{u}_\varepsilon, z_\varepsilon)_\varepsilon$ of solution to the system (2)–(5), below

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial t} - \operatorname{div}\left(A_0^\varepsilon \nabla \mathbf{u}_\varepsilon\right) + \mathbf{B}\left(\mathbf{u}_\varepsilon\right) + g\nabla z_\varepsilon = \mathbf{f} \text{ in } Q \tag{2}$$

$$\frac{\partial z_\varepsilon}{\partial t} + \operatorname{div}\left(h\mathbf{u}_\varepsilon\right) = 0 \text{ in } Q \tag{3}$$

$$\mathbf{u}_\varepsilon = 0 \text{ on } \partial\Omega \times (0,T) \tag{4}$$

$$\mathbf{u}_\varepsilon\,(x,0) = \mathbf{u}^0\,(x) \text{ and } z_\varepsilon\,(x,0) = z^0\,(x) \text{ in } \Omega, \tag{5}$$

where Ω is a Lipschitz bounded domain of $\mathbb{R}^2$ and T is a positive real number. Here, $\mathbf{u}_\varepsilon$ and z_ε stand, respectively, for the total transport 2-D vector (the vertical integral of the velocity) and the deviation of the free surface with respect to the ocean bottom. We have chosen the Dirichlet boundary condition in order to simplify the presentation. It is worth noticing that other boundary conditions can be considered, such as the Robin one: the difficulties may be only of technical types, with the method being the same.

In (2)–(5), ∇ (resp. div) is the gradient (resp. divergence) operator in Ω and the functions A_0^ε, h, $\mathbf{u}^0$, z^0, and $\mathbf{B}$ are constrained, as follows:

(A1) The function A_0^ε is defined by $A_0^\varepsilon\,(x) = A_0(x, x/\varepsilon)$ $(x \in \Omega)$, where $A_0 \in \mathcal{C}(\overline{\Omega}, L^\infty(\mathbb{R}^2_y))^{2\times 2}$ is a symmetric matrix with

$$A_0(x,y)\xi \cdot \xi \geq \alpha\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,\ x \in \overline{\Omega} \text{ and a.e. } y \in \mathbb{R}^2, \tag{6}$$

where $\alpha > 0$ is a given constant independent of x, y and ξ.

(A2) The operator $\mathbf{B}$ is defined on $L^4(\Omega)^2$ by $\mathbf{B}\,(\mathbf{v}) = \gamma\left|\mathbf{v} + \omega^0\right|\left(\mathbf{v} + \omega^0\right)$ $(\mathbf{v} \in L^4\,(\Omega)^2)$, where $\omega^0 \in L^2(0,T; H_0^1(\Omega)^2)$ is a given function and $\gamma(x) = r/h(x)$ (for a fixed real number $r > 0$), h being a continuously differentiable function satisfying

$$\min_{x\in\Omega} h(x) = \beta > 0,\ \max_{x\in\Omega} h(x) = \mu \text{ and } \max_{x\in\Omega} |\nabla h(x)| \leq M,$$

where $M > 0$ is a given constant, which is equal to zero at a constant ocean depth. The functions $\mathbf{u}^0$, z^0 and $\mathbf{f}$ are such that $\mathbf{u}^0 \in L^2(\Omega)^2$, $z^0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^2)$, and g is the gravitational constant.

(A3) We assume further that, for all $x \in \overline{\Omega}$, the matrix-function $A_0(x,\cdot)$ has its entries in $B^2_{\mathcal{A}}(\mathbb{R}^2)$, where $\mathcal{A}$ is an algebra with mean value on in $\mathbb{R}^2$, while $B^2_{\mathcal{A}}(\mathbb{R}^2)$ stands for the generalized Besicovitch space that is associated to $\mathcal{A}$.

Remark 1. *The operator* $\mathbf{B}$ *continuously sends* $L^4(\Omega)^2$ *into* $L^2(\Omega)^2$ *with the following properties (see ([5], Lemma 3.3)): for* $\mathbf{u}, \mathbf{v} \in L^4(\Omega)^2$, *we have*

$$(\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v}) \geq 0; \tag{7}$$

$$\|\mathbf{B}(\mathbf{u})\|_{L^2(\Omega)^2} \leq \|\gamma\|_\infty \|\mathbf{u}\|_{L^4(\Omega)^2}; \tag{8}$$

$$\|\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v})\|_{L^2(\Omega)^2} \leq \|\gamma\|_\infty \left(\|\mathbf{u}\|_{L^4(\Omega)^2} + \|\mathbf{v}\|_{L^4(\Omega)^2}\right) \|\mathbf{u} - \mathbf{v}\|_{L^4(\Omega)^2}. \tag{9}$$

The assumption (**A3**), which depends on the algebra with a mean value $\mathcal{A}$, is crucial in the homogenization process. It shows how the microstructures are distributed in the medium Ω and, therefore, allows for us to pass to the limit.

Before dealing with the well-posedness of (2)–(5), we first need to define the concept of solutions that we will deal with.

Definition 1. *Let* $\mathbf{u}^0 \in L^2(\Omega)^2$, $z^0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^2)$, $\omega^0 \in L^2(0,T;H^1_0(\Omega)^2)$ *and* $0 < T < \infty$. *The couple* $(\mathbf{u}_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ *is a weak solution to the problem (2)–(5) if*

$$\begin{aligned}
\mathbf{u}_\varepsilon &\in \mathcal{C}\left(0,T;L^2(\Omega)^2\right) \cap L^2\left(0,T;H^1_0(\Omega)^2\right); \\
\frac{\partial \mathbf{u}_\varepsilon}{\partial t} &\in L^2\left(0,T;H^{-1}(\Omega)^2\right); \\
z_\varepsilon &\in L^\infty\left(0,T;L^2(\Omega)\right),\ \frac{\partial z_\varepsilon}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right);
\end{aligned}$$

and for all $\varphi \in L^2(0,T;H^1_0(\Omega)^2)$ *and* $\psi \in L^2(0,T;L^2(\Omega))$, *we have*

$$\begin{aligned}
&\int_0^T \left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t}, \varphi\right) dt + \int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi dxdt + \int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi dxdt + \int_Q g \nabla z_\varepsilon \varphi dxdt \\
&= \int_0^T (\mathbf{f}(t), \varphi(t)) dt
\end{aligned} \tag{10}$$

and

$$\int_0^T \left(\frac{\partial z_\varepsilon}{\partial t}, \psi\right) dt + \int_0^T (\mathrm{div}(h\mathbf{u}_\varepsilon), \psi)\, dt = 0. \tag{11}$$

In the above definition, $(\cdot,\cdot)$ stands for the duality pairings between any Hilbert space X and its topological dual X'. We also recall that the operator $\mathrm{div}\left(A_0^\varepsilon \nabla \mathbf{u}_\varepsilon\right)$ acts in a diagonal way, which is, for $\mathbf{v} = (v_1, v_2) \in H^1_0(\Omega)^2$, we have

$$\begin{aligned}
(\mathrm{div}\,(A_0^\varepsilon \nabla \mathbf{u}_\varepsilon), \mathbf{v}) &= -\int_\Omega A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{v} dx \\
&\equiv -\sum_{i=1}^2 \int_\Omega A_0^\varepsilon \nabla u_\varepsilon^i \cdot \nabla v_i dx
\end{aligned}$$

where $\mathbf{u}_\varepsilon = (u_\varepsilon^i)_{1\leq i\leq 2}$. This being so, the following existence and uniqueness result holds.

Theorem 1. *Assume that* (**A1**)–(**A2**) *are satisfied. Subsequently, there exists (for each* $\varepsilon > 0$*) a unique weak solution* $(\mathbf{u}_\varepsilon, z_\varepsilon)$ *to the problem* (2)–(5) *in the sense of Definition* 1.

Proof. We note that, in the problem stated in [5], if we replace the Laplace operator by $-\operatorname{div}(A_0^\varepsilon \nabla \mathbf{u}_\varepsilon)$ and we neglect therein the Coriolis parameter, then the proof follows exactly the lines of that of ([5], Propositions 3.6 and 3.7). □

2.2. A Priori Estimates

The following result will be useful in deriving the uniform estimates for $(\mathbf{u}_\varepsilon, z_\varepsilon)$

Lemma 1 ([5], Lemma 3.1)**.** *For any real-valued smooth function* φ *compactly supported in* $\mathbb{R}^2$*, we have*

$$\|\varphi\|_{L^4(\Omega)}^4 \leq 2\,\|\varphi\|_{L^2(\Omega)}^2\,\|\nabla\varphi\|_{L^2(\Omega)}^2. \tag{12}$$

The following lemma provides us with the a priori estimates.

Lemma 2. *Under assumptions* (**A1**)–(**A2**)*, the weak solution* $(\mathbf{u}_\varepsilon, z_\varepsilon)$ *of problem* (2)–(5) *in the sense that Definition* 1 *satisfies the following estimates*

$$\sup_{0\leq t\leq T}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2} \leq C; \tag{13}$$

$$\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{H_0^1(\Omega)^2}^2\,dt \leq C; \tag{14}$$

$$\left\|\frac{\partial \mathbf{u}_\varepsilon}{\partial t}\right\|_{L^2\left(0,T;H^{-1}(\Omega)^2\right)} \leq C; \tag{15}$$

$$\sup_{0\leq t\leq T}\|z_\varepsilon(t)\|_{L^2(\Omega)} \leq C; \tag{16}$$

$$\left\|\frac{\partial z_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \leq C \tag{17}$$

where the positive constant C *is independent of* ε*.*

Proof. We first deal with Equation (2). In the variational form of (2), we choose the test function $\mathbf{u}_\varepsilon(t)$ associated to (4) to obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}^2 + \left(A_0^\varepsilon\nabla\mathbf{u}_\varepsilon(t), \nabla\mathbf{u}_\varepsilon(t)\right) + \left(B(\mathbf{u}_\varepsilon(t)), \mathbf{u}_\varepsilon(t)\right) \\ &\quad + \left(g\nabla z_\varepsilon(t), \mathbf{u}_\varepsilon(t)\right) = \left(\mathbf{f}(t), \mathbf{u}_\varepsilon(t)\right). \end{aligned} \tag{18}$$

By the divergence theorem, we have

$$\left(g\nabla z_\varepsilon(t), \mathbf{u}_\varepsilon(t)\right) = -\left(g z_\varepsilon(t), \operatorname{div}(\mathbf{u}_\varepsilon(t))\right). \tag{19}$$

Applying Young's inequality in the form

$$ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2 \tag{20}$$

to (19) (with $\delta = \frac{2g}{\alpha}$), we obtain

$$
\begin{aligned}
\left| g\left(\nabla z_{\varepsilon}\left(t\right), \mathbf{u}_{\varepsilon}\left(t\right)\right)\right| &= \left| -g\left(z_{\varepsilon}\left(t\right), \operatorname{div}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)\right)\right| \\
&\leq \frac{g}{2}\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \frac{\alpha}{2g}\left\|\operatorname{div}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)\right\|^{2}_{L^{2}(\Omega)}\right) \\
&\leq \frac{g}{2}\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \frac{\alpha}{2g}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}\right).
\end{aligned}
\tag{21}
$$

In (7) if we take $\mathbf{u} = \mathbf{u}_{\varepsilon}$ and $\mathbf{v} = 0$ to get $\left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right) - \gamma\left|\omega^{0}\right|^{2}, \mathbf{u}_{\varepsilon}\left(t\right)\right) \geq 0$, which yields

$$
\begin{aligned}
\left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right), \mathbf{u}_{\varepsilon}\left(t\right)\right) &= \left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right) - \gamma\left|\omega^{0}\right|^{2}, \mathbf{u}_{\varepsilon}\left(t\right)\right) + \left(\gamma\left|\omega^{0}\left(t\right)\right|^{2}, \mathbf{u}_{\varepsilon}\left(t\right)\right) \\
&\geq \left(\gamma\left|\omega^{0}\left(t\right)\right|^{2}, \mathbf{u}_{\varepsilon}\left(t\right)\right) \\
&\geq -\frac{r}{\beta}\left\|\omega^{0}\left(t\right)\right\|^{2}_{L^{4}(\Omega)^{2}}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|_{L^{2}(\Omega)^{2}} \\
&\geq -\frac{r}{2\beta}\left(\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}} + \left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right).
\end{aligned}
\tag{22}
$$

Using again (20), but this time with $\delta = 1$, we get

$$
\left(\mathbf{f}(t), \mathbf{u}_{\varepsilon}\left(t\right)\right) \leq \frac{1}{2}\left(\left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} + \left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right).
\tag{23}
$$

Putting together (6), (21)–(23), we derive, from (18), the following

$$
\begin{aligned}
&\frac{d}{dt}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} + 2\alpha\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}} \\
&\leq \left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \frac{r}{\beta}\left(\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}} + \left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right) \\
&+ g\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \frac{\alpha}{2g}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}\right) + \left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} \\
&= \left(1 + \frac{r}{\beta}\right)\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} + \frac{2g^{2}}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \frac{r}{\beta}\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}} \\
&+ \frac{\alpha}{2}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}} + \left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}.
\end{aligned}
\tag{24}
$$

Integrating (24) with respect to t, we obtain

$$
\begin{aligned}
&\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} + 2\alpha\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}} ds \\
&\leq \left(1 + \frac{r}{\beta}\right)\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{L^{2}(\Omega)^{2}} ds + \frac{2g^{2}}{\alpha}\int_{0}^{t}\left\|z_{\varepsilon}\left(s\right)\right\|^{2}_{L^{2}(\Omega)} ds + \frac{r}{\beta}\int_{0}^{t}\left\|\omega^{0}\left(s\right)\right\|^{4}_{L^{4}(\Omega)^{2}} ds \\
&+ \frac{\alpha}{2}\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}} ds + \int_{0}^{t}\left\|\mathbf{f}\left(s\right)\right\|^{2}_{L^{2}(\Omega)^{2}} ds + \left\|\mathbf{u}^{0}\left(t\right)\right\|_{L^{2}(\Omega)^{2}}.
\end{aligned}
\tag{25}
$$

Next. dealing with (3), which we multiply by $z_{\varepsilon}(t)$ and then integrate the resulting equality over Ω, we get

$$
\frac{1}{2}\frac{d}{dt}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)} + \left(\operatorname{div}\left(h\mathbf{u}_{\varepsilon}\left(t\right)\right), z_{\varepsilon}\left(t\right)\right) = 0.
\tag{26}
$$

However,

$$
\begin{aligned}
|(\mathrm{div}\,(h\mathbf{u}_\varepsilon(t)), z_\varepsilon(t))| &= |(h\,\mathrm{div}\,\mathbf{u}_\varepsilon(t), z_\varepsilon(t)) + (\mathbf{u}_\varepsilon(t)\cdot\nabla h, z_\varepsilon(t))| \\
&\leq |(h\,\mathrm{div}\,\mathbf{u}_\varepsilon(t), z_\varepsilon(t))| + |(\mathbf{u}_\varepsilon(t)\cdot\nabla h, z_\varepsilon(t))| \\
&\leq \|h\|_\infty \|\mathbf{u}_\varepsilon(t)\|_{H_0^1(\Omega)^2} \|z_\varepsilon(t)\|_{L^2(\Omega)} + M\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2}\|z_\varepsilon(t)\|_{L^2(\Omega)} \\
&\leq \frac{\mu}{2}\left(\frac{\alpha}{2\mu}\|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(\Omega)^2} + \frac{2\mu}{\alpha}\|z_\varepsilon(t)\|^2_{L^2(\Omega)}\right) \\
&+ \frac{M}{2}\left(\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)}\right).
\end{aligned}
\tag{27}
$$

Taking (27) into account and integrating (26) in t gives

$$
\begin{aligned}
\|z_\varepsilon(t)\|^2_{L^2(\Omega)} \leq M\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2}ds + \left(\frac{2\mu^2}{\alpha}+M\right)\int_0^t \|z_\varepsilon(s)\|^2_{L^2(\Omega)}ds \\
+ \frac{\alpha}{2}\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2}ds + \left\|z^0\right\|_{L^2(\Omega)}.
\end{aligned}
\tag{28}
$$

Summing up inequalities (25) and (28), gives readily

$$
\begin{aligned}
&\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2}ds \\
&\leq \lambda_1\int_0^t \left(\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(s)\|^2_{L^2(\Omega)}\right)ds + \frac{r}{\beta}\int_0^t \left\|\omega^0(s)\right\|^4_{L^4(\Omega)^2}ds + \lambda_2,
\end{aligned}
$$

where

$$
\lambda_1 = \max\left(1+M+\frac{r}{\beta}, \frac{2\mu^2}{\alpha}+M+\frac{2g^2}{\alpha}\right)
$$

and

$$
\lambda_2 = \int_0^T \|\mathbf{f}(s)\|^2_{L^2(\Omega)^2}ds + \left\|\mathbf{u}^0\right\|^2_{L^2(\Omega)^2} + \left\|z^0\right\|^2_{L^2(\Omega)}.
$$

Now, appealing to inequality (12) (in Lemma 1) and owing to the fact that $\omega^0 \in L^2(0,T;H_0^1(\Omega)^2)$, we have

$$
\begin{aligned}
\left\|\omega^0(s)\right\|^4_{L^4(\Omega)} &\leq C\left\|\omega^0(s)\right\|^2_{L^4(\Omega)}\left\|\nabla\omega^0(s)\right\|^2_{L^2(\Omega)} \\
&\leq C\left\|\omega^0(s)\right\|^4_{H_0^1(\Omega)} \quad \text{for a.e. } s\in(0,T),
\end{aligned}
$$

so that

$$
\left\|\omega^0(s)\right\|_{L^2(0,T;L^4(\Omega)^2)} \leq C\left\|\omega^0(s)\right\|_{L^2(0,T;H_0^1(\Omega)^2)} \leq C.
$$

We are, therefore, led to

$$
\begin{aligned}
&\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2}ds \\
&\leq C + \lambda_1\int_0^t (\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(s)\|^2_{L^2(\Omega)})ds.
\end{aligned}
$$

Applying the Gronwall inequality leads to

$$
\sup_{0\leq t\leq T}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2} \leq C,\ \sup_{0\leq t\leq T}\|z_\varepsilon(t)\|_{L^2(\Omega)} \leq C,\ \int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(\Omega)^2}dt \leq C.
\tag{29}
$$

From (10), we obtain, for all $\varphi \in L^2(0,T;H_0^1(\Omega)^2)$,

$$\left|\left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t},\varphi\right)\right| \leq C\left\|\mathbf{u}_\varepsilon\right\|_{L^2(0,T;H_0^1(\Omega)^2)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)} + \left\|\mathbf{B}\left(\mathbf{u}_\varepsilon\right)\right\|_{L^2(Q)^2}\left\|\varphi\right\|_{L^2(Q)^2} + C\left\|z_\varepsilon\right\|_{L^2(Q)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)} + \left\|\mathbf{f}\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)}.$$

Next, while using the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\|\mathbf{B}(\mathbf{u}_\varepsilon)\|_{L^2(Q)} \leq C\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^4(\Omega)^2)} \leq C\|\mathbf{u}_\varepsilon\|_{L^2(0,T;H_0^1(\Omega)^2)}.$$

Therefore, we infer, from (29), that

$$\left|\left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t},\varphi\right)\right| \leq C\|\varphi\|_{L^2(0,T;H_0^1(\Omega)^2)},$$

from which

$$\left\|\frac{\partial \mathbf{u}_\varepsilon}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega)^2)} \leq C.$$

We follow the same way of reasoning to see that

$$\left\|\frac{\partial z_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

This concludes the proof. □

3. Fundamentals of the Sigma-Convergence Method

Here, we are concerned with the main features of the sigma-convergence method, which we define. The reader may find more details in [14,15].

We first recall that by an algebra with mean value $\mathcal{A}$ on $\mathbb{R}^d$ is meant any Banach algebra consisting of bounded uniformly contiuous functions on $\mathbb{R}^d$, satisfying

(i) $\mathcal{A}$ contains the constants;
(ii) $u(\cdot + a) \in \mathcal{A}$ for any $u \in \mathcal{A}$ and $a \in \mathbb{R}^d$; and,
(iii) for any $u \in \mathcal{A}$, the limit $M(u) = \lim_{R\to\infty} ⨍_{B_R} u(y)dy$ exists and is called the *mean value* of u.

In (iii), above, $⨍_{B_R}$ is the integral mean over the open ball B_R centered at 0 and of radius R: $⨍_{B_R} = |B_R|^{-1}\int_{B_R}$.

For obvious purposes, we define the generalized Besicovitch space $B^p_{\mathcal{A}}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ associated to a given algebra with mean value $\mathcal{A}$, as the completion with respect to the seminorm $\|\cdot\|_p$ defined on $\mathcal{A}$ by $\|u\|_p = (M(|u|^p))^{1/p}$ $(u \in \mathcal{A})$. It is worth noticing that $\|\cdot\|_p$ is well defined, since $|u|^p \in \mathcal{A}$ for any $p > 0$ and $u \in \mathcal{A}$. We may also define the Banach counterpart $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$ of $B^p_{\mathcal{A}}(\mathbb{R}^d)$ by cutting with the kernel of the seminorm $\|\cdot\|_p$: $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d) = B^p_{\mathcal{A}}(\mathbb{R}^d)/\mathcal{N}$ with $\mathcal{N} = \{u \in B^p_{\mathcal{A}}(\mathbb{R}^d) : \|u\|_p = 0\}$.

In the current work, we assume that all of the algebras with a mean value are ergodic (see [6,17] for the definition). We also need a further space, say $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$, which is defined, as follows:

$$B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d) = \{u \in W^{1,p}_{loc}(\mathbb{R}^d) : \nabla_y u \in B^p_{\mathcal{A}}(\mathbb{R}^d)^d \text{ and } M(\nabla_y u) = 0\}.$$

We identify two elements of $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ by their gradients, which is, $u = v$ in $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ if $\left\|\nabla_y(u-v)\right\|_p = 0$. Equipped with the gradient norm $\|u\|_{\#,p} = \left\|\nabla_y u\right\|_{p'}$, $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ is a Banach space ([18], Theorem 3.12).

We are now able to define the concept of sigma-convergence.

Definition 2. *A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q))$ $(1 \leq p < \infty)$ is said to:*

(i) weakly Σ-converge in $L^p(Q)$ to $u_0 \in L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$ as if $\varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x,t) f\left(x,t,\frac{x}{\varepsilon}\right) dxdt \to \int_Q M(u_0(x,t,\cdot)f(x,t,\cdot))dxdt \tag{30}$$

for every $f \in L^{p'}(Q;\mathcal{A})$, $\frac{1}{p}+\frac{1}{p'}=1$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ;

(ii) strongly Σ-converge in $L^p(Q)$ to $u_0 \in L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$ if (30) holds and further

$$\|u_\varepsilon\|_{L^p(Q)} \to \|u_0\|_{L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))}. \tag{31}$$

We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-strong Σ.

Remark 2. *(1) We can prove that the weak Σ-convergence in $L^p(Q)$ implies the weak convergence in $L^p(Q)$. (2) The convergence (30) still holds true for $f \in \mathcal{C}(\overline{Q}; B^{p',\infty}_{\mathcal{A}}\left(\mathbb{R}^d\right))$, where $B^{p',\infty}_{\mathcal{A}}\left(\mathbb{R}^d\right) = B^{p'}_{\mathcal{A}}\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$, $\frac{1}{p}+\frac{1}{p'}=1$.*

The following results are the main properties of sigma-convergence and they can be found in [6,14,17]. Before we can state them, we need to define what we call a fundamental sequence. By a fundamental sequence, we term any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ (denoted here, below, by E) of real numbers satisfying $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$.

$(SC)_1$ For $1 < p < \infty$, any sequence that is bounded in $L^p(Q)$ possesses a weakly Σ-convergent subsequence.

$(SC)_2$ Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(0,T;W^{1,p}_0(\Omega))$ $(1 < p < \infty)$ be a bounded sequence in $L^p(0,T;W^{1,p}_0(\Omega))$. Afterwards, up to a subsequence E' from E, there exists a couple (u_0,u_1) with $u_0 \in L^p(0,T;W^{1,p}_0(\Omega))$ and $u_1 \in L^p(Q;B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d))$, such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p(Q) \text{-weak } \Sigma$$

and

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^p(Q)\text{-weak } \Sigma,\ 1 \leq i \leq d. \tag{32}$$

$(SC)_3$ Let $1 < p,q < \infty$ and $r \geq 1$ be such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q} \leq 1$. Assume that $(u_\varepsilon)_{\varepsilon>0} \subset L^q(Q)$ is weakly Σ-convergent in $L^q(Q)$ to some $u_0 \in L^q(Q;\mathcal{B}^q_{\mathcal{A}}(\mathbb{R}^d))$ and $(v_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ is strongly Σ-convergent in $L^p(Q)$ to some $v_0 \in L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$. Subsequently, the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon>0}$ is weakly Σ-convergent in $L^r(Q)$ to $u_0 v_0$.

4. Homogenization Result

4.1. Passage to the Limit

First, we set

$$\mathbb{V}=\left\{\mathbf{u}\in L^2(0,T;H_0^1(\Omega)^2):\mathbf{u}'=\frac{\partial\mathbf{u}}{\partial t}\in L^2(0,T;H^{-1}(\Omega)^2)\right\};$$
$$\mathbb{H}=H^1(0,T;L^2(\Omega)).$$

The spaces $\mathbb{V}$ and $\mathbb{H}$ are Hilbert spaces with obvious norms. Moreover, the imbedding $\mathbb{V}\hookrightarrow L^2(0,T;L^2(\Omega)^2)$ is compact.

Now, in view of a priori estimates in Lemma 2, the sequences $(\mathbf{u}_\varepsilon)_\varepsilon$ and $(z_\varepsilon)_\varepsilon$ are bounded in $\mathbb{V}$ and in $\mathbb{H}$, respectively. Thus, given a fundamental sequence E, there exist a subsequence E' of E and a couple $(\mathbf{u}_0,z_0)\in\mathbb{V}\times\mathbb{H}$, such that, as $E'\ni\varepsilon\to 0$,

$$\mathbf{u}_\varepsilon\to\mathbf{u}_0 \text{ in } \mathbb{V}\text{-weak}; \tag{33}$$
$$\mathbf{u}_\varepsilon\to\mathbf{u}_0 \text{ in } L^2(0,T;L^2(\Omega)^2)\text{-strong}; \tag{34}$$
$$z_\varepsilon\to z_0 \text{ in } \mathbb{H}\text{-weak}; \tag{35}$$

Using the estimates (13)–(17), it follows that there exist a subsequence of E' (not relabeled) and a function $\mathbf{u}_1\in L^2(Q;B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$, such that, as $E'\ni\varepsilon\to 0$,

$$\frac{\partial\mathbf{u}_\varepsilon}{\partial x_i}\to\frac{\partial\mathbf{u}_0}{\partial x_i}+\frac{\partial\mathbf{u}_1}{\partial y_i} \text{ in } L^2(Q)^2\text{-weak } \Sigma,\ i=1,2. \tag{36}$$

It follows that $(\mathbf{u}_0,\mathbf{u}_1)\in\mathbb{F}_0^1=\mathbb{V}\times L^2(Q;B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$.

Now, for an element $\mathbf{v}=(\mathbf{v}_0,\mathbf{v}_1)\in\mathbb{F}_0^1$, we set

$$\mathbb{D}\mathbf{v}=\nabla\mathbf{v}_0+\nabla_y\mathbf{v}_1=(\mathbb{D}_i\mathbf{v})_{1\le i\le 2} \text{ where } \mathbb{D}_i\mathbf{v}=\frac{\partial\mathbf{v}_0}{\partial x_i}+\frac{\partial\mathbf{v}_1}{\partial y_i},\ i=1,2$$

with $\frac{\partial\mathbf{v}_0}{\partial x_i}+\frac{\partial\mathbf{v}_1}{\partial y_i}=\left(\frac{\partial v_0^j}{\partial x_i}+\frac{\partial v_1^j}{\partial y_i}\right)_{1\le j\le 2}$. The smooth counterpart of $\mathbb{F}_0^1$ is defined by $\mathcal{F}_0^\infty=\mathcal{C}_0^\infty(Q)^2\otimes\mathcal{C}_0^\infty(Q;(\mathcal{A}^\infty/\mathbb{R})^2)$.

Proposition 1. *Let* $\mathbf{u}=(\mathbf{u}_0,\mathbf{u}_1)\in\mathbb{F}_0^1$ *and* $z_0\in\mathbb{H}$. *Afterwards,* $\mathbf{u}$ *and* z_0 *solve the following variational problem:*

$$\begin{aligned}&-\int_Q\mathbf{u}_0\frac{\partial\varphi_0}{\partial t}dxdt+\int_Q M\left(A_0\mathbb{D}\mathbf{u}\cdot\mathbb{D}\varphi\right)dxdt+\int_Q\mathbf{B}\left(\mathbf{u}_0\right)\varphi_0dxdt+\int_Q g\nabla z_0\varphi_0dxdt\\&=\int_0^T\left(\mathbf{f}(t),\varphi_0(t)\right)dt\end{aligned} \tag{37}$$

$$-\int_Q z_0\frac{\partial\psi_0}{\partial t}dxdt-\int_Q h\mathbf{u}_0\cdot\nabla\psi_0dxdt=0 \tag{38}$$

for all $\varphi=(\varphi_0,\varphi_1)\in\mathcal{F}_0^\infty$ *and* $\psi_0\in\mathcal{C}_0^\infty(Q)$.

Proof. Let $\varphi=(\varphi_0,\varphi_1)$ and ψ_0 be, as above, and define

$$\varphi_\varepsilon(x,t)=\varphi_0(x,t)+\varphi_1\left(x,t,\frac{x}{\varepsilon}\right) \text{ for } (x,t)\in Q.$$

Taking $(\varphi_\varepsilon, \psi_0)$ as a test function in the variational form of (2)–(5), we obtain

$$-\int_Q \mathbf{u}_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} dxdt + \int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi_\varepsilon dxdt + \int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi_\varepsilon dxdt + \int_Q g\nabla z_\varepsilon \varphi_\varepsilon dxdt$$
$$= \int_0^T (\mathbf{f}(t), \varphi_\varepsilon(t))\, dt \tag{39}$$

and

$$-\int_Q z_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt - \int_Q h\mathbf{u}_\varepsilon \cdot \nabla \psi_0 dxdt = 0. \tag{40}$$

While using the identities

$$\frac{\partial \varphi_\varepsilon}{\partial t} = \frac{\partial \varphi_0}{\partial t} + \varepsilon \left(\frac{\partial \varphi_1}{\partial t} \right)^\varepsilon \text{ and } \nabla \varphi_\varepsilon = \nabla \varphi_0 + (\nabla_y \varphi_1)^\varepsilon + \varepsilon (\nabla \varphi_1)^\varepsilon,$$

we infer that, as $\varepsilon \to 0$,

$$\frac{\partial \varphi_\varepsilon}{\partial t} \to \frac{\partial \varphi_0}{\partial t} \text{ in } L^2\left(0, T; H^{-1}(\Omega)^2\right) \text{-weak} \tag{41}$$
$$\nabla \varphi_\varepsilon \to \nabla \varphi_0 + \nabla_y \varphi_1 \text{ in } L^2(Q)^{2\times 2}\text{-strong } \Sigma \tag{42}$$
$$\varphi_\varepsilon \to \varphi_0 \text{ in } L^2(Q)^2\text{-strong.} \tag{43}$$

Let us consider each of the Equations (39) and (40) separately. We first consider (39) and, using the convergence results (34) and (41), we obtain

$$\int_Q \mathbf{u}_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} dxdt \to \int_Q \mathbf{u}_0 \frac{\partial \varphi_0}{\partial t} dxdt. \tag{44}$$

Because $A_0 \in \mathcal{C}(\overline{Q}; B_{\mathcal{A}}^{2,\infty}(\mathbb{R}^2)^{2\times 2})$, we use it as test function together with property (SC)$_3$ (recall that we have (36) and (42)) to obtain

$$\int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi_\varepsilon dxdt \to \int_Q M\left(A_0 \mathbb{D}\mathbf{u} \cdot \mathbb{D}\varphi\right) dxdt. \tag{45}$$

Let us show that

$$\int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi_\varepsilon dxdt \to \int_Q \mathbf{B}(\mathbf{u}_0)\varphi_0 dxdt. \tag{46}$$

First, we have, from (34), that, up to a subsequence of E' not relabeled, $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ a.e. in Q. Hence, from the continuity of $\mathbf{B}$, we entail

$$\mathbf{B}(\mathbf{u}_\varepsilon) \to \mathbf{B}(\mathbf{u}_0) \text{ a.e. in } Q.$$

we infer, from the boundedness of the sequence $(\mathbf{B}(\mathbf{u}_\varepsilon))_{\varepsilon>0}$, that $\mathbf{B}(\mathbf{u}_\varepsilon) \to \mathbf{B}(\mathbf{u}_0)$ in $L^2(Q)^2$-weak. Putting this together with (43), we obtain (46). We also easily obtain

$$\int_0^T (\mathbf{f}(t), \varphi_\varepsilon(t))\, dt \to \int_0^T (\mathbf{f}(t), \varphi_0(t)) dt. \tag{47}$$

Next, the convergence results (35) and (43) yield

$$\int_Q g\nabla z_\varepsilon \varphi_\varepsilon dxdt \to \int_Q g\nabla z_0 \varphi_0 dxdt. \tag{48}$$

As for Equation (40), we use the weak convergence (35) that is associated to (43) to obtain

$$\int_Q z_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt \to \int_Q z_0 \frac{\partial \psi_0}{\partial t} dxdt.$$

Concerning the second term in (40), we infer, from (34), that

$$\int_Q h\mathbf{u}_\varepsilon \cdot \nabla\psi_0 dxdt \to \int_Q h\mathbf{u}_0 \cdot \nabla\psi_0 dxdt,$$

thereby completing the proof of the proposition. □

4.2. Homogenized Problem

Here, we intend to derive the problem whose the couple $(\mathbf{u}_0, z_0)$ is solution. In order to achieve this, we first uncouple Equation (37), which is equivalent to the system consisting of (49) and (50), below:

$$\begin{aligned} &-\int_Q \mathbf{u}_0 \frac{\partial\varphi_0}{\partial t} dxdt + \int_Q M\left(A_0\mathbb{D}\mathbf{u}\cdot\nabla\varphi_0\right) dxdt + \int_Q \mathbf{B}\left(\mathbf{u}_0\right)\varphi_0 dxdt + \int_Q g\nabla z_0 \cdot \varphi_0 dxdt \\ &= \int_0^T \left(\mathbf{f}(t), \varphi_0(t)\right) dt; \end{aligned} \tag{49}$$

$$\int_Q M\left(A_0\mathbb{D}\mathbf{u}\cdot\nabla_y\varphi_1\right) dxdt = 0. \tag{50}$$

Choosing in (50)

$$'_1(x,t,y) = \theta(x,t)\mathbf{v}(y) \text{ where } \theta \in \mathcal{C}_0^\infty(Q), \mathbf{v} \in (\mathcal{A}^\infty)^2, \tag{51}$$

we obtain

$$M\left(A_0\mathbb{D}\mathbf{u}\cdot\nabla\mathbf{v}\right) = 0 \text{ for all } \mathbf{v} \in (\mathcal{A}^\infty)^2. \tag{52}$$

Let us deal with (52). To this end, fix $\xi \in \mathbb{R}^{2\times 2}$ and consider the corrector problem:

$$\begin{cases} \text{Find } \text{ß}(\xi) \in \mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2) \text{ such that}: \\ -\operatorname{div}_y[A_0(x,\cdot)(\xi + \nabla_y \text{ß}(\xi))] = 0 \text{ in } \mathbb{R}^2. \end{cases} \tag{53}$$

Subsequently, in view of the properties of the matrix $A_0(x,\cdot)$, we infer, from [19,20], that (53) possesses a unique solution in $\mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$. Coming back to (53) and taking there $\xi = \nabla\mathbf{u}_0(x,t)$, testing the resulting equation with φ_1 as in (51), we get, by the uniqueness of the solution of (53), that $\mathbf{u}_1(x,t,y) = \text{ß}(\nabla\mathbf{u}_0(x,t)(x,y)$. This shows that $\text{ß}(\nabla\mathbf{u}_0)$ belongs to $L^2(0,T;\mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2))$. Clearly, if χ_j^ℓ is the solution of (53) corresponding to $\xi = \xi_j^\ell = (\delta_{ij}\delta_{k\ell})_{1\le i,k\le 2}$ (that is all of the entries of ξ are zero, except the entry occupying the jth row and the ℓth column, which is equal to 1), then

$$\mathbf{u}_1 = \sum_{j,\ell=1}^{2} \frac{\partial u_0^\ell}{\partial x_j}\chi_j^\ell \text{ where } \mathbf{u}_0 = (u_0^\ell)_{1\le\ell\le 2}. \tag{54}$$

We recall again that χ_j^ℓ depends on x, as it is the case for A_0. In the variational form of (49), we insert the value of $\mathbf{u}_1$ that was obtained in (54) to obtain the equation

$$\frac{\partial\mathbf{u}_0}{\partial t} - \operatorname{div}(\widehat{A}_0(x)\nabla\mathbf{u}_0) + \mathbf{B}\left(\mathbf{u}_0\right) + g\nabla z_0 = \mathbf{f} \text{ in } Q. \tag{55}$$

where $\widehat{A}_0(x) = (\widehat{a}_{ij}^{k\ell}(x))_{1\le i,j,k,\ell\le 2}$, $\widehat{a}_{ij}^{k\ell}(x) = a_{\text{hom}}(\chi_j^\ell + P_j^\ell, \chi_i^k + P_i^k)$ with $P_j^\ell = y_j e^\ell$ (e^ℓ the ℓth vector of the canonical basis of $\mathbb{R}^2$) and

$$a_{\text{hom}}(\boldsymbol{u},\boldsymbol{v}) = \sum_{i,j,k=1}^{2} M\left(a_{ij}\frac{\partial u^k}{\partial y_j}\frac{\partial v^k}{\partial y_i}\right) \text{ where } A_0 = (a_{ij})_{1\le i,j\le 2}.$$

Additionally, Equation (38) is equivalent to

$$\frac{\partial z_0}{\partial t} + \operatorname{div}(h\mathbf{u}_0) = 0 \text{ in } Q. \tag{56}$$

Finally, putting together the Equations (55) and (56) associated to the boundary and initial conditions, we are led to the homogenized problem, viz.

$$\begin{cases} \dfrac{\partial \mathbf{u}_0}{\partial t} - \operatorname{div}(\widehat{A}_0(x)\nabla \mathbf{u}_0) + \mathbf{B}(\mathbf{u}_0) + g\nabla z_0 = \mathbf{f} \text{ in } Q \\ \dfrac{\partial z_0}{\partial t} + \operatorname{div}(h\mathbf{u}_0) = 0 \text{ in } Q \\ \mathbf{u}_0 = 0 \text{ on } \partial\Omega \times (0,T) \\ \mathbf{u}_0(x,0) = \mathbf{u}^0(x), z_0(x,0) = z^0(x) \text{ in } \Omega. \end{cases} \tag{57}$$

It can be easily shown that the matrix $\widehat{A}_0$ of homogenized coefficients has entries in $\mathcal{C}(\overline{\Omega})$, and it is uniformly elliptic, so that, under the conditions **(A1)–(A2)**, the problem (57) possesses a unique solution $(\mathbf{u}_0, z_0)$ with $\mathbf{u}_0 \in L^2\left(0,T;H_0^1(\Omega)^2\right)$ and $z_0 \in L^2\left(0,T;L^2(\Omega)\right)$. Because the solution of (57) is unique, we infer that the whole sequence $(\mathbf{u}_\varepsilon, z_\varepsilon)$ converges in a suitable space towards $(\mathbf{u}_0, z_0)$, as stated in the following result, which is the main result of the work.

Theorem 2. *Assume that* (**A1**) *to* (**A3**) *hold. For any* $\varepsilon > 0$, *let* $(\mathbf{u}_\varepsilon, z_\varepsilon)$ *be the unique solution of problem* (2) *to* (5). *Subsequently,* $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ *in* $L^2(Q)^2$*-strong and* $z_\varepsilon \to z_0$ *in* $L^2(Q)$*-weak, where* $(\mathbf{u}_0, z_0)$ *is the unique solution of problem* (57).

Proof. The proof is a consequence of the previous steps. □

5. Some Concrete Applications of Theorem 2

The homogenization of problem has been made possible under the fundamental assumption (**A3**). Some of the physical situations that lead to (**A3**) are listed below.

Problem 1 (**Periodic Homogenization**). *The homogenization of (2)–(5) holds under the periodicity assumption that the matrix-function* $A_0(x,\cdot)$ *is periodic with period* 1 *in each coordinate, for any* $x \in \overline{\Omega}$. *In that case, we have* $\mathcal{A} = \mathcal{C}_{per}(Y)$, *where* $Y = (0,1)^2$ *and* $\mathcal{C}_{per}(Y)$ *is the algebra of continuous* Y*-periodic functions defined in* $\mathbb{R}^2$. *It is easy to see that* $B^2_{\mathcal{A}}(\mathbb{R}^2) = L^2_{per}(Y) \equiv \{u \in L^2_{loc}(\mathbb{R}^2) : u \text{ is } Y\text{-periodic}\}$, *and the mean value expresses as* $M(u) = \int_Y u(y)dy$. *Hence, the homogenized matrix is defined by* $\widehat{A}_0(x) = (\widehat{a}_{ij}^{k\ell}(x))_{1\le i,j,k,\ell\le 2}$, $\widehat{a}_{ij}^{k\ell}(x) = a_{\text{hom}}(\chi_j^\ell + P_j^\ell, \chi_i^k + P_i^k)$ *with* $P_j^\ell = y_j e^\ell$ *(*e^ℓ *the* ℓ*th vector of the canonical basis of* $\mathbb{R}^2$*) and*

$$a_{\text{hom}}(\boldsymbol{u},\boldsymbol{v}) = \sum_{i,j,k=1}^{2} \int_Y a_{ij} \frac{\partial u^k}{\partial y_j}\frac{\partial v^k}{\partial y_i} dy \text{ where } A_0 = (a_{ij})_{1\le i,j\le 2}.$$

where, here, χ_j^ℓ *is the solution of the cell problem*

$$\chi_j^\ell(x,\cdot) \in H^1_\#(Y)^2 : -\operatorname{div}_y(A_0(x,\cdot)(\xi_j^\ell + \nabla_y \chi_j^\ell(x,\cdot))) = 0 \text{ in } Y$$

with $H^1_\#(Y) = \{v \in H^1_{per}(Y) : \int_Y v dy = 0\}$, $H^1_{per}(Y) = \{v \in L^2_{per}(Y) : \nabla_y v \in L^2_{per}(Y)^2\}$ *and* $\xi_j^\ell = (\delta_{ij}\delta_{k\ell})_{1\le i,k\le 2}$.

Problem 2 (**Almost periodic Homogenization**). *We may consider the homogenization problem for (2)–(5) under the assumption that the coefficients of the matrix* $A_0(x,\cdot)$ *are Besicovitch almost periodic functions [21]. In that case, hypothesis (**A3**) holds true, with* $\mathcal{A} = \mathrm{AP}(\mathbb{R}^2)$, *where* $\mathrm{AP}(\mathbb{R}^2)$ *is the algebra of Bohr almost* *periodic functions on $\mathbb{R}^2$ [22]. The mean value of a function $u \in \mathrm{AP}\left(\mathbb{R}^2\right)$ is the unique constant that belongs to the close convex hull of the family of the translates $(u(\cdot + a))_{a\in\mathbb{R}^2}$.*

Problem 3 (Weakly almost periodic Homogenization). *We may solve the homogenization problem for (2)–(5) under the assumption: the function $A_0(x,\cdot)$ is weakly almost periodic, which is, the matrix $A_0(x,\cdot)$ has its entries in the algebra with mean value $\mathcal{A} = \mathrm{WAP}\left(\mathbb{R}^2\right)$ (where $\mathrm{WAP}\left(\mathbb{R}^2\right)$ is the algebra of continuous weakly almost periodic functions on $\mathbb{R}^2$; see, e.g., [23]).*

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