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Exponential and Hypoexponential Distributions: Some Characterizations

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Abstract: The (general) hypoexponential distribution is the distribution of a sum of independent exponential random variables. We consider the particular case when the involved exponential variables have distinct rate parameters. We prove that the following converse result is true. If for some $n \geq 2$, $X_1, X_2, \ldots, X_n$ are independent copies of a random variable X with unknown distribution F and a specific linear combination of X_j's has hypoexponential distribution, then F is exponential. Thus, we obtain new characterizations of the exponential distribution. As corollaries of the main results, we extend some previous characterizations established recently by Arnold and Villaseñor (2013) for a particular convolution of two random variables.

Keywords: exponential distribution; hypoexponential distribution; characterizations

MSC: 62G30; 62E10

1. Introduction and Main Results

Sums of exponentially distributed random variables play a central role in many stochastic models of real-world phenomena. *Hypoexponential distribution* is the convolution of k exponential distributions each with their own rate λ_i, the rate of the i^{th} exponential distribution. As an example, consider the distribution of the time to absorption of a finite state Markov process. If we have a $k+1$ state process, where the first k states are transient and the state $k+1$ is an absorbing state, then the time from the start of the process until the absorbing state is reached is *phase-type distributed*. This becomes the hypoexponential if we start in state 1 and move skip-free from state i to $i+1$ with rate λ_i until state k transitions with rate λ_k to the absorbing state $k+1$.

We write $Z_i \sim \text{Exp}(\lambda_i)$ for $\lambda_i > 0$, if Z_i has density

$$f_i(z) = \lambda_i e^{-\lambda_i z}, \quad z \geq 0 \quad \textit{(exponential distribution)}.$$

The distribution of the sum $S_n := Z_1 + Z_2 + \ldots + Z_n$, where λ_i for $i = 1, \ldots, n$ are not all identical, is called (general) *hypoexponential distribution* (see [1,2]). It is absolutely continuous and we denote by g_n its density. It is called the *hypoexponetial* distribution as it has a coefficient of variation less than one, compared to the *hyper-exponential* distribution which has coefficient of variation greater than one and the *exponential* distribution which has coefficient of variation of one. In this paper, we deal with a particular case of the hypoexponential distribution when all λ_i are distinct, i.e., $\lambda_i \neq \lambda_j$ when $i \neq j$. In this case, it is known ([3], p. 311; [4], Chapter 1, Problem 12) that

$$S_n = Z_1 + Z_2 + \ldots + Z_n \quad \text{has density} \quad g_n(z) := \sum_{j=1}^{n} \ell_j f_j(z), \quad z \geq 0. \tag{1}$$

Here the weight ℓ_j is defined as

$$\ell_j = \prod_{i=1, i \neq j}^{n} \frac{\lambda_i}{\lambda_i - \lambda_j}.$$

Please note that $\ell_j := \ell_j(0)$, where $\ell_1(x), \ldots, \ell_n(x)$ are identified (see [5]) as the Lagrange basis polynomials associated with the points $\lambda_1, \ldots, \lambda_n$. The convolution density g_n in (1) is the weighted average of the values of the densities of $Z_1, Z_2, \ldots, Z_n$, where the weights ℓ_j sum to 1 (see [5]). Notice, however, since the weights can be both positive or negative, g_n is not a "usual" mixture of densities. If we place λ_j's in increasing or decreasing order, then the corresponding coefficients ℓ_j's alternate in sign.

Consider the Laplace transforms $\varphi_i(t) := \mathsf{E}[\mathrm{e}^{-tZ_i}]$, $t \geq 0$, $i = 1, 2, \ldots, n$. They are well-defined and will play a key role in the proofs of the main results.

To begin with, let us look at the case when all Z_i's are identically distributed, i.e., $\lambda_i = \lambda$ for $i = 1, 2, \ldots, n$, so we can use φ for the common Laplace transform. The sum $S_n = Z_1 + Z_2 + \ldots + Z_n$ has Erlang distribution whose Laplace transform $\tilde{\varphi}$, because of the independence, is expressed as follows:

$$\tilde{\varphi}(t) = \mathsf{E}\left[\mathrm{e}^{-tS_n}\right] = \varphi^n(t) = \left(\frac{\lambda}{\lambda + t}\right)^n.$$

If we go in the opposite direction, assuming that S_n has Erlang distribution with Laplace transform $\tilde{\varphi}$, then we conclude that $\varphi_i(t) = \lambda(\lambda + t)^{-1}$ for each $i = 1, 2, \ldots, n$, which in turn implies that $Z_i \sim \mathrm{Exp}(\lambda)$. By words, if Z_i are independent and identically distributed random variables and their sum has Erlang distribution, then the common distribution is exponential.

Does a similar characterization hold when the rate parameters λ_i are all different? The answer to this question is not obvious. It is our goal in this paper to show that the answer is positive.

Let $\mu_1, \mu_2, \ldots, \mu_n$ be positive real numbers, such that $\lambda_i = \lambda / \mu_i$. Without loss of generality suppose that $\mu_1 > \mu_2 > \ldots > \mu_n > 0$. Assume that $X_1, X_2, \ldots, X_n$, for fixed $n \geq 2$, are independent and identically distributed as a random variable X with density f, $f(x) = \lambda \mathrm{e}^{-\lambda x}$, $x > 0$. Then (1) is equivalent to the following:

$$S_n := \mu_1 X_1 + \mu_2 X_2 + \cdots + \mu_n X_n \quad \text{has density} \quad g_n(x) = \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} f\left(\frac{x}{\mu_j}\right), \quad x \geq 0. \tag{2}$$

Here the coefficients/weights are given as follows:

$$\ell_j = \prod_{i=1, i \neq j}^{n} \frac{\mu_i^{-1}}{\mu_i^{-1} - \mu_j^{-1}} = \prod_{i=1, i \neq j}^{n} \frac{\mu_j}{\mu_j - \mu_i}, \quad j = 1, 2, \ldots, n. \tag{3}$$

We use now the common Laplace transform $\varphi(t) := E[\mathrm{e}^{-tX_i}]$. Please note that since $\mu_i \neq \mu_j$ for $i \neq j$, relation (2) implies that

$$\begin{aligned} \varphi(\mu_1 t)\varphi(\mu_2 t) \cdots \varphi(\mu_n t) &= \int_0^\infty \mathrm{e}^{-tx} g_n(x)\, dx \\ &= \int_0^\infty \mathrm{e}^{-tx} \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} f\left(\frac{x}{\mu_j}\right) dx \\ &= \sum_{j=1}^{n} \ell_j \int_0^\infty \mathrm{e}^{-tx} \frac{1}{\mu_j} f\left(\frac{x}{\mu_j}\right) dx = \sum_{j=1}^{n} \ell_j \varphi(\mu_j t). \end{aligned} \tag{4}$$

The idea now is to start with an arbitrary non-negative random variable X with unknown density f and Laplace transform φ. If the Laplace transform of the linear combination $S_n = \sum_{i=1}^{n} \mu_i X_i$ satisfies (4), we will derive that $\varphi(t) = \lambda(\lambda + t)^{-1}$. Thus, the common distribution of X_j, $j = 1, 2, \ldots, n$ is exponential. More precisely, the following characterization result holds.

Theorem 1. *Suppose that $X_1, X_2, \ldots, X_n$, $n \geq 2$, are independent copies of a non-negative random variable X with density f. Assume further that X satisfies Cramér's condition: there is a number $t_0 > 0$ such that $\mathsf{E}[e^{-tX}] < \infty$ for all $t \in (-t_0, t_0)$. If relation (2) is satisfied for fixed $n \geq 2$ and fixed positive mutually different numbers $\mu_1, \mu_2, \ldots, \mu_n$, then $X \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$.*

The studies of characterization properties of exponential distributions are abundant. Comprehensive surveys can be found in [6–9]. More recently, Arnold and Villaseñor [10] obtained a series of exponential characterizations involving sums of two random variables and conjectured possible extensions for sums of more than two variables (see also [11]). Corollary 1 below extends the characterizations in [10,11] to sums of n variables, for any fixed $n \geq 2$.

Consider the special case of (2) when $\mu_j = 1/j$ for $j = 1, 2, \ldots, n$. Under this choice of μ_j's, the formula for the weight ℓ_j simplifies to (see [4], Chapter 1, Problem 13)

$$\ell_j = \prod_{i=1, i \neq j}^{n} \frac{i}{i-j} = \binom{n}{j} (-1)^{j-1}.$$

Therefore, Theorem 1 reduces to the following corollary.

Corollary 1. *Suppose that $X_1, X_2, \ldots, X_n$, $n \geq 2$, are independent copies of a non-negative random variable X with density f. Assume further that X satisfies Cramér's condition: there is a number $t_0 > 0$ such that $\mathsf{E}[e^{-tX}] < \infty$ for all $t \in (-t_0, t_0)$. If for fixed $n \geq 2$,*

$$X_1 + \frac{1}{2} X_2 + \ldots + \frac{1}{n} X_n \quad \textit{has density} \quad \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} j f(jx), \quad x \geq 0, \tag{5}$$

then $X \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$.

The exponential distribution has the striking property that if $\lambda = 1$ *(unit exponential)*, then the density f equals the survival function (the tail of the cumulative distribution function) $\overline{F} = 1 - F$. Therefore, in case of unit exponential distribution, (2) can be written as follows:

$$\tilde{S}_n := \mu_1 X_1 + \mu_2 X_2 + \cdots + \mu_n X_n \quad \text{has density} \quad \tilde{g}_n(x) := \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \overline{F} \left(\frac{x}{\mu_j} \right), \quad x \geq 0. \tag{6}$$

We will show that (6) is a sufficient condition for $X_1, X_2, \ldots, X_n$ to be *unit exponential*.

Theorem 2. *Suppose that $X_1, X_2, \ldots, X_n$, $n \geq 2$, are independent copies of a non-negative random variable X with distribution function F. Assume also that X satisfies Cramér's condition: there is a number $t_0 > 0$ such that $\mathsf{E}[e^{-tX}] < \infty$ for all $t \in (-t_0, t_0)$. If relation (6) is satisfied for fixed $n \geq 2$, then $X \sim \mathrm{Exp}(1)$.*

Setting $\mu_j = 1/j$ for $j = 1, 2, \ldots, n$, we obtain the following corollary of Theorem 2.

Corollary 2. *Suppose that $X_1, X_2, \ldots, X_n$, $n \geq 2$, are independent copies of a non-negative random variable X with distribution function F. Assume also that X satisfies Cramér's condition: there is a number $t_0 > 0$ such that $\mathsf{E}[e^{-tX}] < \infty$ for all $t \in (-t_0, t_0)$. If for fixed $n \geq 2$,*

$$X_1 + \frac{1}{2} X_2 + \ldots + \frac{1}{n} X_n \quad \textit{has density} \quad \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} j \overline{F}(jx) \qquad x > 0, \tag{7}$$

then $X \sim \mathrm{Exp}(1)$ for some $\lambda > 0$.

We organize the rest of the paper as follows. Section 2 contains preliminaries needed in the proofs of the theorems. The proofs themselves are given in Section 3. We discuss the findings in the concluding Section 4.

2. Auxiliaries

We will need the Leibniz rule for differentiating a product of functions. Denote by $v^{(k)}$ the kth derivative of $v(x)$ with $v^{(0)}(x) := v(x)$. Let us define a multi-index set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ as an n-tuple of non-negative integers, and denote $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Leibniz considered the problem of determining the kth derivative of the product of n smooth functions $v_1(t)v_2(t)\cdots v_n(t)$ and obtained the formula (e.g., [12])

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\prod_{i=1}^{n} v_i(t)\right) = \sum_{|\boldsymbol{\alpha}|=k}\left(\frac{k!}{\alpha_1!\alpha_2!\cdots\alpha_n!}\prod_{i=1}^{n} v_i^{(\alpha_i)}(t)\right). \tag{8}$$

Here the summation is taken over all multi-index sets $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = k$. Formula (8) can easily be proved by induction.

Lemma 1. *Assume that $v(t) = \sum_{i=0}^{\infty} a_i t^i$ is a functional series, such that for some $\tilde{t}_0 > 0$, the k^{th} order derivative $v^{(k)}(t)$ exists for all $t \in (-\tilde{t}_0, \tilde{t}_0)$. Then for arbitrary positive real constants $\mu_1, \mu_2, \ldots, \mu_n$, we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\prod_{i=1}^{n} v(\mu_i t)\right)\Big|_{t=0} = k!\sum_{|\boldsymbol{\alpha}|=k}\prod_{i=1}^{n}\mu_i^{\alpha_i} a_{\alpha_i}. \tag{9}$$

Proof. Formula (9) is proved by applying Leibniz rule (8) to $\prod_{i=1}^{n} v(\mu_i t)$. □

In addition to (9), we will need some properties of Lagrange basis polynomials ℓ_j collected below.

Lemma 2 (see [13])**.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers, such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Denote*

$$\ell_j = \prod_{i=1, i\neq j}^{n}\frac{\lambda_i}{\lambda_i - \lambda_j} \qquad j = 1, 2, \ldots, n.$$

Then, for $n \geq 2$, we have the following:

(i) $\displaystyle\sum_{j=1}^{n}\ell_j = 1.$

(ii) $\displaystyle\sum_{j=1}^{n}\ell_j\lambda_j^k = 0$ *for any k, $1 \leq k \leq n-1$.*

(iii) $\displaystyle\sum_{j=1}^{n}\frac{\ell_j}{\lambda_j^k} \geq \sum_{j=1}^{n}\frac{1}{\lambda_j^k}$ *for any k, $1 \leq k \leq n-1$, where the equality holds if and only if $k = 1$.*

Proof. Claim (i) follows by integrating (1) over $z > 0$. Claim (ii) is proved in Corollary 1 of [13]. To prove claim (iii) we involve $\boldsymbol{\alpha}$, the multi-index set as in (8). For $k \geq 1$, we have $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \cup \boldsymbol{\alpha}''$, where

$$\begin{aligned}
\boldsymbol{\alpha}' &= \{|\boldsymbol{\alpha}| = k : \text{ only one index in } \boldsymbol{\alpha} \text{ equals } k \text{ and all others are zeros}\} \\
\boldsymbol{\alpha}'' &= \{|\boldsymbol{\alpha}| = k : \text{ no single index in } \boldsymbol{\alpha} \text{ equals } k\}.
\end{aligned}$$

According to Proposition 5 in [13] we obtain, for $n \geq 2$ and $k \geq 1$, the following chain of relations:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\ell_j}{\lambda_j^k} &= \sum_{|\boldsymbol{\alpha}|=k} \frac{1}{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}} \\
&= \sum_{|\boldsymbol{\alpha}'|} \frac{1}{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}} + \sum_{|\boldsymbol{\alpha}''|} \frac{1}{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}} \\
&= \sum_{j=1}^{n} \frac{1}{\lambda_j^k} + \sum_{|\boldsymbol{\alpha}''|} \frac{1}{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}} \\
&\geq \sum_{j=1}^{n} \frac{1}{\lambda_j^k}.
\end{aligned} \tag{10}$$

Clearly, the equality in (10) holds if and only if $k = 1$. The proof is complete. □

The properties in Lemma 2 can be easily verified, as an illustration, for $n = 2$, $k = 1$, and $k = 2$. Indeed,

$$\begin{aligned}
\sum_{j=1}^{2} \ell_j &= \frac{\lambda_2}{\lambda_2-\lambda_1} + \frac{\lambda_1}{\lambda_1-\lambda_2} = 1, \qquad \sum_{j=1}^{2} \ell_j\lambda_j = \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} = 0, \\
\sum_{j=1}^{2} \frac{\ell_j}{\lambda_j} &= \frac{\lambda_2}{(\lambda_2-\lambda_1)\lambda_1} + \frac{\lambda_1}{(\lambda_1-\lambda_2)\lambda_2} = \frac{\lambda_2+\lambda_1}{\lambda_1\lambda_2} = \sum_{i=1}^{2} \frac{1}{\lambda_i}, \\
\sum_{j=1}^{2} \frac{\ell_j}{\lambda_j^2} &= \frac{\lambda_2}{(\lambda_2-\lambda_1)\lambda_1^2} + \frac{\lambda_1}{(\lambda_1-\lambda_2)\lambda_2^2} = \frac{\lambda_2^2+\lambda_2\lambda_1+\lambda_1^2}{\lambda_1^2\lambda_2^2} = \sum_{i=1}^{2} \frac{1}{\lambda_i^2} + \frac{1}{\lambda_1\lambda_2}.
\end{aligned}$$

3. Proofs of the Characterization Theorems

In the proofs of both theorems we follow the four-step scheme.

- Consider $X_1, X_2, \ldots, X_n$ for $n \geq 2$ to be independent copies of a non-negative random variable X with density f. Suppose $\mu_1 > \mu_2 > \ldots > \mu_n$ are positive real numbers.
- Assume the characterization property

$$S_n = \mu_1 X_1 + \mu_2 X_2 + \cdots + \mu_n X_n \quad \text{has density} \quad g_n(x) = \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} f\left(\frac{x}{\mu_j}\right), \quad x \geq 0,$$

  where ℓ_j is given in (3).
- For the Laplace transform $\varphi(t) = \mathsf{E}[\mathrm{e}^{-tX}]$, $t \geq 0$, obtain the equation

$$\varphi(\mu_1 t)\varphi(\mu_2 t)\cdots\varphi(\mu_n t) = \sum_{j=1}^{n} \ell_j \varphi(\mu_j t). \tag{11}$$

- Using Leibniz rule for differentiating product of functions and properties of Lagrange basis polynomials, show that (11) has a unique solution given by $\varphi(t) = (1+\lambda^{-1}t)^{-1}$ for some $\lambda > 0$ and conclude that

$$X_1, X_2, \ldots, X_n \quad \text{are} \quad \mathrm{Exp}(\lambda) \quad \text{random variables.}$$

Proof of Theorem 1. Recall that (see (4))

$$\varphi(\mu_1 t)\varphi(\mu_2 t)\cdots\varphi(\mu_n t) = \sum_{j=1}^{n} \ell_j \varphi(\mu_j t).$$

Dividing both sides of this equation by $\varphi(\mu_1 t)\varphi(\mu_2 t)\cdots\varphi(\mu_n t)$, we obtain

$$1 = \sum_{j=1}^{n}\left(\ell_j \prod_{i=1, i\neq j}^{n} \psi(\mu_i t)\right), \tag{12}$$

where $\psi := 1/\varphi$. Consider the series

$$\psi(t) = \sum_{k=0}^{\infty} a_k t^k, \tag{13}$$

which, as a consequence of Cramér's condition for φ, is convergent in a proper neighborhood of $t = 0$. To prove the theorem, it is sufficient to show that

$$\psi(t) = 1 + \lambda^{-1} t, \qquad \lambda > 0. \tag{14}$$

We will prove that (12) implies (14) by showing that the coefficients $\{a_k\}_{k=0}^{\infty}$ in (13) satisfy $a_0 = 1$, $a_1 = \lambda^{-1} > 0$, and $a_k = 0$ for $k \geq 2$. Notice first that

$$a_0 = \frac{1}{\varphi(0)} = 1. \tag{15}$$

Denote

$$\Psi_j(t) := \prod_{i=1, i\neq j}^{n} \psi(\mu_i t) \quad \text{and} \quad H(t) := \sum_{j=1}^{n} \ell_j \Psi_j(t) = \sum_{k=0}^{\infty} h_k t^k.$$

By (12) we have $H(t) \equiv 1$ and therefore $h_0 = 1$ and $h_k = 0$ for all $k \geq 1$. Equating h_k's to the corresponding coefficients of the series in the right-hand side of (12), we will obtain equations for $\{a_k\}_{k=0}^{\infty}$. As a first step, note that

$$h_k = \frac{1}{k!} H^{(k)}(t)|_{t=0} = \frac{1}{k!}\sum_{j=1}^{n} \ell_j \Psi_j^{(k)}(t)\big|_{t=0}, \qquad k \geq 1. \tag{16}$$

Next, we apply Leibniz rule for differentiation. To fix the notation, let us define a multi-index set $\boldsymbol{\alpha}_{-j} = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)$, $1 \leq j \leq n$ as a set of $(n-1)$-tuples of non-negative integer numbers, with $|\boldsymbol{\alpha}_{-j}| = \alpha_1 + \ldots + \alpha_{j-1} + \alpha_{j+1} + \ldots + \alpha_n$. Applying Lemma 1 for fixed $k \geq 1$ and fixed $1 \leq j \leq n$, we obtain

$$\Psi_j^{(k)}(t)\big|_{t=0} = k! \sum_{\{|\boldsymbol{\alpha}_{-j}|=k\}} \prod_{i=1, i\neq j}^{n} \mu_i^{\alpha_i} a_{\alpha_i}. \tag{17}$$

Introduce the set $\Lambda_{k,j} := \{\boldsymbol{\alpha}_{-j} : |\boldsymbol{\alpha}_{-j}| = k\}$ and partition it into three disjoint subsets as follows:

$$\Lambda_{k,j} = \Lambda'_{k,j} \cup \Lambda''_{k,j} \cup \Lambda'''_{k,j},$$

where for $k \geq 1$

$$\begin{aligned}
\Lambda'_{k,j} &= \{|\boldsymbol{\alpha}_{-j}| = k : \text{ only one index in } \boldsymbol{\alpha}_{-j} \text{ equals } k \text{, all others are zeros}\} \\
\Lambda''_{k,j} &= \{|\boldsymbol{\alpha}_{-j}| = k : \ k \geq 2 \text{ and exactly } k \text{ of the indices in } \boldsymbol{\alpha}_{-j} \text{ equal 1, all others are zeros}\} \\
\Lambda'''_{k,j} &= \{|\boldsymbol{\alpha}_{-j}| = k : \ k \geq 3 \ \text{ and there is an index } \alpha_i \text{ with } 2 \leq \alpha_i < k\}.
\end{aligned}$$

For example, if $n = 5$, $k = 3$, and $j = 5$, then $\Lambda'_{3,5} = \{(3,0,0,0), (0,3,0,0), (0,0,3,0), (0,0,0,3)\}$, $\Lambda''_{3,5} = \{(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)\}$, and $\Lambda'''_{3,5} = \{(1,2,0,0), (1,0,2,0), \ldots, (0,0,2,1)\}$. Referring to (16) and (17), we have for $k \geq 1$

$$\begin{aligned} h_k &= \sum_{j=1}^{n} \left(\ell_j \sum_{\Lambda_{k,j}} \prod_{i=1, i \neq j}^{n} \mu_i^{\alpha_i} a_{\alpha_i} \right) \\ &= \sum_{j=1}^{n} \ell_j \left(\sum_{\Lambda'_{k,j}} (\cdot) + \sum_{\Lambda''_{k,j}} (\cdot) + \sum_{\Lambda'''_{k,j}} (\cdot) \right) \\ &=: \sum_{j=1}^{n} \ell_j \left(S_{1,j} + S_{2,j} + S_{3,j} \right), \quad \text{say.} \end{aligned} \tag{18}$$

For the term $S_{2,j}$ in the middle, since $a_0 = 1$, we have $S_{2,j} = 0$ when $k = 1$ and for any $k \geq 2$

$$\begin{aligned} S_{2,j} &= \sum_{\Lambda''_{k,j}} \prod_{i=1, i \neq j}^{n} \mu_i^{\alpha_i} a_{\alpha_i} \\ &= a_0^{n-1-k} a_1^k \sum{}' (\mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}) \\ &= a_1^k \mu_j^{-1} \sum{}' (\mu_j \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}) \end{aligned}$$

where the summation in $\sum'$ is over all k-tuples (with i_jth component dropped) $i_1, \ldots, i_{j-1}, i_{j+1} \ldots, i_k$, such that $i_m \in \{1, 2, \ldots, n\}$ and $i_1 < i_2 < \ldots < i_k$. Using that $\sum_{j=1}^{n} \ell_j \mu_j^{-1} = 0$ by Lemma 2(ii) with $\lambda_i = \mu_i^{-1}$, we obtain for any $k \geq 2$

$$\sum_{j=1}^{n} \ell_j S_{2,j} = a_1^k \left(\sum_{j=1}^{n} \ell_j \mu_j^{-1} \right) \sum{}'' (\mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}) = 0. \tag{19}$$

Here the summation in $\sum''$ is over all k-tuples $i_1, i_2, \ldots, i_k$, such that $i_m \in \{1, 2, \ldots, n\}$ and $i_1 < i_2 < \ldots < i_k$. For the first term $S_{1,j}$ in the last expression of (18), we have for any $k \geq 1$

$$\begin{aligned} S_{1,j} &= \sum_{\Lambda'_{k,j}} \prod_{i=1, i \neq j}^{n} \mu_i^{\alpha_i} a_{\alpha_i} = a_0^{n-2} a_k \sum_{i=1, i \neq j}^{n} \mu_i^k \\ &= a_k \left(\sum_{i=1}^{n} \mu_i^k - \mu_j^k \right). \end{aligned}$$

Furthermore, since $\sum_{j=1}^{n} \ell_j = 1$ by Lemma 2(i) with $\lambda_i = \mu_i^{-1}$, we have for any $k \geq 1$

$$\begin{aligned} \sum_{j=1}^{n} \ell_j S_{1,j} &= a_k \sum_{j=1}^{n} \ell_j \left(\sum_{i=1}^{n} \mu_i^k - \mu_j^k \right) \\ &= a_k \sum_{i=1}^{n} \mu_i^k \sum_{j=1}^{n} \ell_j - a_k \sum_{j=1}^{n} \ell_j \mu_j^k \\ &= a_k \left(\sum_{i=1}^{n} \mu_i^k - \sum_{j=1}^{n} \ell_j \mu_j^k \right) \\ &=: a_k c_k. \end{aligned} \tag{20}$$

Lemma 2(iii) with $\lambda_i = \mu_i^{-1}$ implies that $c_1 = 0$ and $c_k < 0$ for any $k \geq 2$. It follows from (18)–(20) that

$$h_k = c_k a_k + \sum_{j=1}^{n} \ell_j S_{3,j}, \tag{21}$$

where $c_1 = 0$ and $c_k < 0$ for $k \geq 2$.

Let $k = 1$. Since $h_1 = 0$ and the sets Λ_1'' and Λ_2''' are empty, we obtain $c_1 a_1 = 0$, where $c_1 = 0$. Hence, there are no restrictions on the coefficient a_1, other than $a_1 > 0$, since X has positive mean. Therefore, there is a number $\lambda^{-1} > 0$ such that

$$a_1 = \lambda^{-1} > 0. \tag{22}$$

Let $k = 2$. Since the set Λ_2''' is empty, Equation (21) yields $h_2 = c_2 a_2 = 0$, where recall that $c_2 < 0$. Thus, $a_2 = 0$. Next, applying (21) and taking into account that $h_k = 0$ for $k \geq 2$, we will show by induction that $a_k = 0$ for any $k \geq 2$. Assuming $a_k = 0$ for $k = 2, 3, \ldots, r$, we will show that $a_{r+1} = 0$. Indeed, by (21) we have

$$h_{r+1} = c_{r+1} a_{r+1} + \sum_{j=1}^{n} \left(\ell_j \sum_{\Lambda_{r+1,j}'''} \prod_{i=1, i \neq j}^{n} \mu_i^{\alpha_i} a_{\alpha_i} \right) = c_{r+1} a_{r+1},$$

because at least one index α_i, satisfies $2 \leq \alpha_i \leq r$ and hence $a_{\alpha_i} = 0$, by assumption. Therefore, $h_{r+1} = c_{r+1} a_{r+1} = 0$ and, since $c_{r+1} < 0$, we have $a_{r+1} = 0$, which completes the induction. Hence,

$$a_k = 0 \quad \text{for any} \quad k \geq 2. \tag{23}$$

The Equations (15) and (22)–(23) imply (14), which completes the proof of the theorem. □

Proof of Theorem 2. Taking into account (6), similarly to (4) and using integration-by-parts, we obtain

$$\begin{aligned} \varphi(\mu_1 t)\varphi(\mu_2 t) \cdots \varphi(\mu_n t) &= \int_0^\infty \mathrm{e}^{-tx} g_n(x)\, dx = \int_0^\infty \mathrm{e}^{-tx} \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \overline{F}\left(\frac{x}{\mu_j}\right) dx \\ &= \sum_{j=1}^{n} \ell_j \int_0^\infty \mathrm{e}^{-tx} \frac{1}{\mu_j} \overline{F}\left(\frac{x}{\mu_j}\right) dx \\ &= \frac{1}{t} \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \left(1 - \varphi(\mu_j t)\right). \end{aligned}$$

Using the fact that $\sum_{j=1}^{n} \ell_j / \mu_j = 0$ (see Lemma 2(ii)), this simplifies to

$$\varphi(\mu_1 t)\varphi(\mu_2 t) \cdots \varphi(\mu_n t) = -\frac{1}{t} \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \varphi(\mu_j t). \tag{24}$$

Dividing both sides of (24) by $-\varphi(\mu_1 t)\varphi(\mu_2 t) \cdots \varphi(\mu_n t)/t$, for $t > 0$, we obtain

$$-t = \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \prod_{i=1, i \neq j}^{n} \psi(\mu_i t), \tag{25}$$

where, as before, $\psi = 1/\varphi$. Consider the series $\psi(t) = \sum_{k=0}^{\infty} a_k t^k$, which is convergent by assumption. To prove the theorem, it is sufficient to show that $\psi(t) = 1 + t$, $t \geq 0$, or, equivalently, that the coefficients $\{a_k\}_{k=0}^{\infty}$ of the above series satisfy $a_0 = 1$, $a_1 = 1$, and $a_k = 0$ for $k \geq 2$. Clearly, $a_0 = 1/\varphi(0) = 1$. Recall that

$$\Psi_j(t) := \prod_{i=1, i \neq j}^{n} \psi(\mu_i t) \quad \text{and denote} \quad -Q(t) := \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \Psi_j(t) = -\sum_{k=0}^{\infty} q_k t^k.$$

By (25) we have $Q(t) \equiv t$ and therefore $q_1 = 1$ and $q_k = 0$ for all $k \neq 1$. We will express q_k in terms of a_j's. Proceeding as in the proof of Theorem 1, applying Leibniz rule for differentiating a product of functions, and using the same notation, we obtain for $k \geq 1$ that

$$-\,q_k = \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \left(S_{1,j} + S_{2,j} + S_{3,j}\right).$$

As with (19), applying Lemma 2(ii), we obtain

$$\sum_{j=1}^{n} \frac{\ell_j}{\mu_j} S_{2,j} = a_1^k \left(\sum_{j=1}^{n} \frac{\ell_j}{\mu_j^2}\right) \sum{}'' (\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}) = 0, \tag{26}$$

where the summation in $\sum''$ is over all k-tuples $i_1, \ldots, i_k$, such that $i_m \in \{1, \ldots, n\}$ and $i_1 < \ldots < i_k$. Furthermore, since $\sum_{j=1}^{n} \ell_j = 1$ and $\sum_{j=1}^{n} \ell_j/\mu_j^2 = 0$ by Lemma 2, we have for any $k \geq 1$

$$\begin{aligned} \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} S_{1,j} &= a_k \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} \left(\sum_{i=1}^{n} \mu_i^k - \mu_j^k\right) \\ &= a_k \sum_{i=1}^{n} \mu_i^k \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} - a_k \sum_{j=1}^{n} \ell_j \mu_j^{k-1} \\ &= -a_k \sum_{j=1}^{n} \ell_j \mu_j^{k-1} \\ &=: -a_k d_k. \end{aligned} \tag{27}$$

It follows from (26) and (27) that for $k \geq 1$,

$$-\,q_k = -a_k d_k + \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} S_{3,j}. \tag{28}$$

Let $k = 1$. Since $q_1 = 1$ and the set Λ_1''' is empty, we obtain $a_1 d_1 = 1$, where $d_1 = 1$ by Lemma 2(iii). Therefore, $a_1 = 1$. Let $k = 2$. Since Λ_2''' is empty, Equation (28) yields $q_2 = d_2 a_2 = 0$, where $d_2 > 0$ by Lemma 2(iii). Thus, $a_2 = 0$. Assuming $a_k = 0$ for $2 \leq k \leq r$, we will show that $a_{r+1} = 0$. Indeed,

$$q_{r+1} = d_{r+1} a_{r+1} + \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} S_{3,j} = d_{r+1} a_{r+1},$$

because at least one index α_i, satisfies $2 \leq \alpha_i \leq r$, in which case $a_{\alpha_i} = 0$, by assumption. Therefore, $q_{r+1} = d_{r+1} a_{r+1} = 0$ and, since $d_{r+1} < 0$, we have $a_{r+1} = 0$, which completes the induction proof. Hence, $a_k = 0$ for any $k \geq 2$. Since $a_0 = a_1 = 1$ and $a_k = 0$ for $k \geq 2$, we obtain $\psi(t) = 1 + t$, which clearly completes the proof of the theorem. □

4. Concluding Remarks

Arnold and Villaseñor [10] proved that if X_1 and X_2 are two independent and non-negative random variables with common density f and $\mathsf{E}[X_1] < \infty$, then

$$X_1 + \frac{1}{2} X_2 \qquad \text{has density} \qquad 2f(x) - 2f(2x), \quad x > 0,$$

if and only if $X_1 \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. Motivated by this result, we extended it in two directions considering: (i) arbitrary number $n \geq 2$ of independent identically distributed non-negative random variables and (ii) linear combination of independent variables with arbitrary positive and distinct coefficients $\mu_1, \mu_2, \ldots, \mu_n$. Namely, our main result is that

$$S_n = \mu_1 X_1 + \mu_2 X_2 + \ldots + \mu_n X_n \quad \text{has density} \quad g_n(x) = \sum_{j=1}^{n} \frac{\ell_j}{\mu_j} f\left(\frac{x}{\mu_j}\right) \quad x \geq 0,$$

where $\ell_j = \prod_{i=1, i\neq j}^{n} \mu_j(\mu_j - \mu_i)^{-1}$, if and only if $X_i \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

In this paper, we dealt with the situation where the rate parameters λ_i are all distinct from each other. The other extreme case of equal λ_i's is trivial. The obtained characterization seems of interest on its own, but it can also serve as a basis for further investigations of intermediate cases of mixed type with some ties and at least two distinct parameters (see [2]). Of certain interest is also the case where not all weights μ_i's are positive (see [1]).

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References

1. Li, K.-H.; Li, C.T. Linear combination of independent exponential random variables. *Methodol. Comput. Appl. Probab.* **2019**, *21*, 253–277. [CrossRef]
2. Smaili, K.H.; Kadri, T.H.; Kadry, S. Finding the PDF of the hypoexponential random variable using the Kad matrix similar to the general Vandermonde matrix. *Commun. Statist. Theory Methods* **2016**, *45*, 1542–1549. [CrossRef]
3. Ross, S.M. *Introduction to Probability Models*, 12th ed.; Academic Press: New York, NY, USA, 2019.
4. Feller, W. *An Introduction to Probability Theory and Its Applications*, 2nd ed.; Wiley: New York, NY, USA, 1971; Volume II.
5. Sen, A.; Balakrishnan, N. Convolution of geometrics and a reliability problem. *Stat. Probab. Lett.* **1999**, *43*, 421–426. [CrossRef]
6. Ahsanullah, M. *Characterizations of Univariate Continuous Distributions*; Atlantic Press: Amsterdam, The Netherlands, 2017.
7. Arnold, B.C.; Huang, J.S. Characterizations. In *The Exponential Distribution: Theory, Methods and Applications*; Balakrishnan, N., Basu, A.P., Eds.; Gordon and Breach: Amsterdam, The Netherlands, 1995; pp. 79–95.
8. Azlarov, T.; Volodin, N.A. *Characterization Problems Associated with the Exponential Distribution*; Springer: Berlin, Germany, 1986.
9. Nagaraja, H.N. Characterizations of probability distributions. In *Springer Handbook of Engineering Statistics*; Pham, H., Ed.; Springer: Berlin, Germany, 2006; pp. 395–402.
10. Arnold, B.C.; Villaseñor, J.A. Exponential characterizations motivated by the structure of order statistics in sample of size two. *Stat. Probab. Lett.* **2013**, *83*, 596–601. [CrossRef]
11. Yanev, G.P. On Arnold-Villaseñor conjectures for characterizing exponential distribution based on sample of size three. *REVSTAT* **2020**, *18*, 177–188.
12. Thaheem, A.B.; Laradji, A. Classroom note: A generalization of Leibniz rule for higher derivatives. *Intern. J. Math. Educ. Sci. Technol.* **2003**, *34*, 739–742. [CrossRef]
13. Smaili, K.H.; Kadri, T.H.; Kadry, S. Hypoexpponential distribution with different parameters. *Appl. Math.* **2013**, *4*, 624–631. [CrossRef]