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The Residual Lifetime of Surviving Components of Coherent System under Periodical Inspections

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Abstract: In this manuscript, we gain a mixture representation for reliability function of the residual lifetime of unfailed components in a coherent system under periodical inspections, given that the number of failed components before time t_1 is $r (\geq 0)$, but the system is still operating at time t_1 , and the system eventually failed at time $t_2 (> t_1)$. Some aging properties and stochastic orders of the residual lifetime on survival components are also established. Finally, some numerical examples and graphs are given in order to confirm the theoretical results.

Keywords: residual lifetime; coherent system; periodical inspection; signature vector; stochastic orders

MSC: 90B25; 60E15; 60K10

1. Introduction

Reliability is one of the most crucial properties of coherent systems; it is widely used in many engineering fields and is significant to scholars and engineers. A system that consists of $n (n \geq 1)$ components is called a coherent system, such that the structure function of the system is monotone at each component and there are no irrelevant components in the system (i.e., irrelevant components mean that their performance will not affect the performance of the system), see [1] to know more details regarding the coherent systems.

Suppose that a coherent system has n components with independent and identically distributed (i.i.d.) lifetimes $X_1, X_2, \dots, X_n$, where X_i has an absolutely continuous distribution function (CDF) F . Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the ordered statistics of components lifetimes and τ is structure function of coherent system, then $T = \tau(X_1, X_2, \dots, X_n)$ can express the lifetime of coherent system. For example, a series system has lifetime $\tau(X_1, X_2, \dots, X_n) = \min(X_1, X_2, \dots, X_n)$ and a parallel system has lifetime $\tau(X_1, X_2, \dots, X_n) = \max(X_1, X_2, \dots, X_n)$. If $\bar{F}_T$ denotes the reliability function of coherent systems lifetime T , then $\bar{F}_T$ is denoted by

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{F}_{i:n}(t), \quad (1)$$

where $s_i = P(T = X_{i:n})$ is the element of probability vector $\mathbf{s}$, $\bar{F}_{i:n}(t)$ is survival function of $X_{i:n}, i = 1, 2, \dots, n$. We call the probability vector $\mathbf{s} = (s_1, s_2, \dots, s_n)$ as the signature vector of the coherent system, and it is a vital tool for studying the lifetime of coherent systems, see [2,3] for more details regarding the signature vector of coherent system. Ref. [4] demonstrated that the representation (3) is still valid when the components of coherent system are exchangeable. Ref. [5] developed the concept of signature of coherent systems with dependent components. Equation (1) is

a great tool for scholars and engineers to research the reliability properties of coherent systems. The residual lifetime and past lifetime of coherent system are some significantly concepts in reliability theory and survival analysis. In the past decades, a lot of scholars have shown extreme interest in investigating the lifetimes of a coherent system or the components of a coherent system. There have been many articles regarding the residual lifetime and past lifetime of coherent systems in the literature; one can refer to [6–15]. However, there are few results in the literature about the residual lifetime and past lifetime of components in a coherent system. To the best of our knowledge, only few results in the literature, for example, ref. [16–20].

Consider a coherent system with n i.i.d. components, and suppose that the signature vector of the system, as following:

$$\mathbf{s} = (s_1, s_2, \dots, s_i, 0, \dots, 0), i = 1, \dots, n-1. \quad (2)$$

It is obvious that a coherent system has a signature vector of form (2), which has the property when the system failed, the components with lifetime $X_{k:n}$ ($k = i+1, \dots, n$) remains alive in a coherent system; meanwhile, these unfailed components can be removed from the original system and used for other points, such as placing on another system (cf. [18,19]). Therefore, the investigation of the stochastic and aging properties of these unfailed components in system may be of interest for engineers.

Under a coherent system with signature vector of form (2), ref. [18] considered the reliability of the residual lifetime of unfailed components under the assumption that the system has failed before time t , but the component with lifetime $X_{k:n}$, $k = i+1, \dots, n$ is still alive at time t ; they researched the stochastic orders and aging properties of the residual lifetime of unfailed components by random variables, as follows:

$$(X_{k:n} - t | T < t < X_{k:n}), k = i+1, \dots, n. \quad (3)$$

Ref. [19] investigated the reliability properties of the residual lifetimes on unfailed components in a coherent system on the conditional that the failure time of the system is time t . They explored the following conditional random variables in order to consider the residual lifetime of unfailed components in coherent system:

$$(X_{k:n} - t | T = t), k = i+1, \dots, n. \quad (4)$$

In real-life case, a system is planned to be monitored at different times. Under these periodical monitors, engineers would obtain some information regarding the system. It is impossible or expensive in practical life because of continuous inspection; thus, a coherent system is planned to be monitored twice at different times. In more recent years, some scholars have considered the residual lifetime of coherent systems under double monitoring, one can see [21–27].

Our main work extends two conditional random variables of (3) and (4) to the case of double monitoring (5), and considers the statistical properties of the residual lifetime on unfailed components in a coherent system, such as the stochastic orders and aging properties. As shown in Figure 1, assume that the system is monitored once at time t_1 , and it is observed that there are r components ($0 \leq r < i$) have failed and the system is still functioning. Subsequently, the system failed at time t_2 ($> t_1$). This manuscript mainly considers a coherent system with n i.i.d. components where the signature vector of the system has the form (2), and we are interested in investigating the residual lifetime of unfailed components in coherent system

$$RX_{k:n}(t_1, t_2) = (X_{k:n} - t_2 | X_{r:n} < t_1 < X_{r+1:n}, T = t_2), \text{ for } 0 \leq r < k \leq n. \quad (5)$$

In fact, $RX_{k:n}(t_1, t_2)$ is the residual lifetime of unfailed components $X_{k:n}$ ($k = i+1, \dots, n$) in the coherent system at time t_2 ($> t_1$).

Figure 1. The number of failed comments is exactly $r (< i < n)$ at time $t_1 (> 0)$, and the system failed at time $t_2 (> t_1)$.

The remainder of this manuscript follows, as follows. In Section 2, we provide some definitions of the stochastic orders and useful lemmas. In Section 3, we will derive a mixture representation for the survival function of the residual lifetime on unfailed components, obtain several stochastic comparisons between two coherent systems, and some aging properties that are based on the proposed conditional random variable. Finally, some conclusions and remarks are made in Section 4.

2. Preliminaries

The term increasing and decreasing are used instead of monotone non-decreasing and monotone non-increasing, respectively. For the sake of discussion, we denote $\mathbb{R}^+ = (0, +\infty)$. Before introducing the results of this manuscript, we will recall some basic definitions on the stochastic orders, and it will be discussed in our subsequent discussion.

Definition 1. Suppose that X and Y are two random variables with distribution functions $F(x)$ and $G(x)$, survival functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, and density functions $f(x)$ and $g(x)$, respectively. If X is said to be smaller than Y in the stochastic orders, then:

- (i) the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}^+$;
- (ii) the hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}^+$;
- (iii) the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}^+$;
- (iv) the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}^+$.

Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be two discrete probability distributions, if $\mathbf{p}$ is said to be smaller than $\mathbf{q}$ in the stochastic orders, then

- (i) $\mathbf{p} \leq_{st} \mathbf{q}$, if $\sum_{i=j}^n p_i \leq \sum_{i=j}^n q_i, j = 1, 2, \dots, n$;
- (ii) $\mathbf{p} \leq_{lr} \mathbf{q}$, if q_j/p_j is increasing in j .

As is well known, the likelihood ratio order means that the (reversed) hazard rate order and the usual stochastic order; the (reversed) hazard rate order also implies the usual stochastic order, but the reversed statement is not true in general. One can refer to [28,29] in order to know more properties and details about the various stochastic orders.

The following lemma plays a crucial role in founding the main results.

Lemma 1 ([28]). (i) Let $X_1, X_2, \dots, X_n$ be n independent random variables, all with absolutely continuous distribution functions, all having the same support which is an interval of the real line, and all having differentiable densities. If $X_1 \leq_{lr} X_2 \leq_{lr} \dots \leq_{lr} X_n$, then $X_{k-1:n} \leq_{lr} X_{k:n}, 2 \leq k \leq n$, and $X_{k-1:n-1} \leq_{lr} X_{k:n}, 2 \leq k \leq n$.

(ii) If $X \leq_{st} Y$ and g is any increasing [decreasing] function, then $g(X) \leq_{st} [\geq_{st}] g(Y)$.

(iii) Let α and β be the real valued function such that β is nonnegative, and α/β and β are decreasing. If $X_1 \leq_{rh} X_2$, then

$$E[\alpha(X_1)]E[\beta(X_2)] \geq E[\alpha(X_2)]E[\beta(X_1)].$$

(iii) Let α and β be the real valued function, such that β is nonnegative, and α/β and β are increasing. If $X_1 \leq_{hr} X_2$, then

$$E[\alpha(X_1)]E[\beta(X_2)] \leq E[\alpha(X_2)]E[\beta(X_1)].$$

3. Main Result

Let T be the lifetime of a coherent system that has n i.i.d. components with lifetimes $X_1, X_2, \dots, X_n$, and X_i has distribution function F . When considering the coherent system with the signature vector

of form (2) under the assumption that the number of failed components is r ($0 \leq r < i$) at time t_1 , the system is still functioning at time t_1 , and the system failed at time t_2 ($> t_1$).

The following theorem provides the representation of the reliability function for the residual lifetimes $RX_{k:n}(t_1, t_2)$ of unfailed components with lifetime $X_{k:n}, k = i + 1, \dots, n$ in a coherent system. Following Equation (6), the reliability function of $RX_{k:n}(t_1, t_2)$ can be expressed as a mixture representation of the residual lifetime on the unfailed components with lifetime $X_{k:n}, k = i + 1, \dots, n$ in a $(n - m + 1)$ -out-of- n system on the condition that exactly r ($r < i$) components have failed before time t_1 ($t_1 \geq 0$), and the system has failed at time t_2 (cf. [18,19,30]). Hence, by this mixture representation of the reliability function on the residual lifetime $RX_{k:n}(t_1, t_2)$, we can further investigate the statistical properties of the residual lifetime on unfailed components in a coherent system under the scenario of double monitoring, such as the stochastic orders and aging properties.

Theorem 1. Assume that $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ are the order statistics of the random variables $X_1, X_2, \dots, X_n$. Subsequently, for any $0 \leq r < k, i = 2, \dots, n$ and $0 < t_1 < t_2$, the reliability function of $RX_{k:n}(t_1, t_2)$ can be represented as

$$\begin{aligned} \mathbb{P}(RX_{k:n}(t_1, t_2) > x) &= \mathbb{P}(X_{k:n} - t_2 > x | X_{r:n} < t_1 < X_{r+1:n}, T = t_2) \\ &= \sum_{m=r+1}^i q_m(r, t_1, t_2) P(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2), \end{aligned} \quad (6)$$

where $Ar(t_1)$ denotes the event $\{X_{r:n} < t_1 < X_{r+1:n}\}$, and

$$q_m(r, t_1, t_2) = \mathbb{P}(T = X_{m:n} | Ar(t_1), T = t_2).$$

Proof. For any $x \in \mathbb{R}^+$ and $0 < t_1 < t_2$, the reliability function of $RX_{k:n}(t_1, t_2)$ can be represented as

$$\begin{aligned} &\mathbb{P}(RX_{k:n}(t_1, t_2) > x) \\ &= P(X_{k:n} - t_2 > x | X_{r:n} < t_1 < X_{r+1:n}, T = t_2) \\ &= \sum_{m=1}^i P(X_{k:n} - t_2 > x, T = X_{m:n} | Ar(t_1), T = t_2) \\ &= \sum_{m=r+1}^i P(X_{k:n} - t_2 > x | Ar(t_1), T = t_2, T = X_{m:n}) \mathbb{P}(T = X_{m:n} | Ar(t_1), T = t_2) \\ &= \sum_{m=r+1}^i q_m(r, t_1, t_2) P(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2). \end{aligned}$$

Let $h_i > 0$ ($i = 1, 2$), we have

$$\begin{aligned} q_m(r, t_1, t_2) &= \mathbb{P}(T = X_{m:n} | Ar(t_1), T = t_2) \\ &= \frac{\mathbb{P}(T = X_{m:n}, Ar(t_1), T = t_2)}{\mathbb{P}(Ar(t_1), T = t_2)} \\ &= \frac{\mathbb{P}(Ar(t_1), T = t_2 | T = X_{m:n}) \mathbb{P}(T = X_{m:n})}{\sum_{j=r+1}^i \mathbb{P}(Ar(t_1), T = t_2, T = X_{j:n})} \\ &= \frac{\mathbb{P}(Ar(t_1), T = t_2 | T = X_{m:n}) \mathbb{P}(T = X_{m:n})}{\sum_{j=r+1}^i \mathbb{P}(Ar(t_1), T = t_2 | T = X_{j:n}) \mathbb{P}(T = X_{j:n})} \\ &= \frac{s_m \lim_{h_1 \rightarrow 0} \mathbb{P}(Ar(t_1), t_2 \leq X_{m:n} \leq t_2 + h_1)}{\sum_{j=r+1}^i s_j \lim_{h_2 \rightarrow 0} \mathbb{P}(Ar(t_1), t_2 \leq X_{j:n} \leq t_2 + h_2)} \\ &= \frac{s_m \frac{n!}{r!(m-r-1)!(n-m)!} F^r(t_1) (F(t_2) - F(t_1))^{m-r-1} \bar{F}^{n-m}(t_2)}{\sum_{j=r+1}^i s_j \frac{n!}{r!(j-r-1)!(n-j)!} F^r(t_1) (F(t_2) - F(t_1))^{j-r-1} \bar{F}^{n-j}(t_2)} \end{aligned}$$

$$\begin{aligned}
&= \frac{s_m C_{r,m}^n (F(t_2) - F(t_1))^{m-r-1} \bar{F}^{n-m}(t_2)}{\sum_{j=r+1}^i s_j C_{r,j}^n (F(t_2) - F(t_1))^{j-r-1} \bar{F}^{n-j}(t_2)} \\
&= \frac{s_m C_{r,m}^n \left(\frac{\bar{F}(t_1) - \bar{F}(t_2)}{\bar{F}(t_2)} \right)^m}{\sum_{j=r+1}^i s_j C_{r,j}^n \left(\frac{\bar{F}(t_1) - \bar{F}(t_2)}{\bar{F}(t_2)} \right)^j} \\
&= \frac{s_m C_{r,m}^n (\phi(t_1, t_2))^m}{\sum_{j=r+1}^i s_j C_{r,j}^n (\phi(t_1, t_2))^j},
\end{aligned}$$

where $\phi(t_1, t_2) = \frac{\bar{F}(t_1) - \bar{F}(t_2)}{\bar{F}(t_2)}$, $C_{r,m}^n = \frac{n!}{r!(m-r-1)!(n-m)!}$. $\square$

Remark 1. It is noticed that the probability vector

$$\mathbf{q}(t_1, t_2) = (0, \dots, 0, p_{r+1}(r, t_1, t_2), \dots, p_i(r, t_1, t_2), 0, \dots, 0)$$

can be considered to be the conditional signature vector of the coherent system, and it only depends on the distribution function F . It follows from Equation (6) that the conditional residual lifetime $RX_{k:n}(t_1, t_2)$ of unfailed components under double monitoring can be expressed as the mixture of the residual lifetime $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$ of the unfailed components with $X_{k:n}(k > i)$ in a $(n - m + 1)$ -out-of- n system, under the assumption that the number of failed components is r ($r < i$) at time t_1 ($t_1 \geq 0$), and the system failed at time t_2 . Theorem 1 extends the residual lifetime of the unfailed components in [19] to the scenario of double monitoring.

Next, consider the reliability function of $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$. It is noted that $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$ and $Z_{k-m:n-m}^{t_2}$ have the same distribution function. We have the following lemma.

Lemma 2. For $m = r + 1, \dots, i$, and $0 \leq r < k$, we have

$$(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2) \stackrel{d}{=} Z_{k-m:n-m}^{t_2}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $Z_{k-m:n-m}^{t_2}$ is the $(k - m)$ th order statistics in $(n - m)$ i.i.d. random variables with survival function

$$\bar{G}_{t_2}(x) = \frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)}, \quad x \in \mathbb{R}^+.$$

Proof. Let h be a small positive real number, the reliability function of $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$ is given as

$$\begin{aligned}
&\mathbb{P}(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2) \\
&= \lim_{h \rightarrow 0} \frac{\mathbb{P}(X_{k:n} - t_2 > x, Ar(t_1), t_2 \leq X_{m:n} \leq t_2 + h)}{\mathbb{P}(Ar(t_1), t_2 \leq X_{m:n} \leq t_2 + h)} \\
&= \sum_{j=n-k+1}^{n-m} \binom{n-m}{j} \frac{(F(t_2 + x) - F(t_2))^{n-m-j} \bar{F}^j(t_2 + x)}{\bar{F}^{n-m}(t_2)} \\
&= \sum_{j=n-k+1}^{n-m} \binom{n-m}{j} (\bar{G}_{t_2}(x))^j (1 - \bar{G}_{t_2}(x))^{n-m-j},
\end{aligned} \tag{7}$$

where $\bar{G}_{t_2}(x) = \frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)}$. Thus, from (7), we obtain the equality in distribution

$$(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2) \stackrel{d}{=} Z_{k-m:n-m}^{t_2}.$$

The proof of the result is completed. $\square$

Example 1. Assume that X_1, X_2, X_3 , and X_4 are i.i.d. random variables. Let $T = \min(X_1, X_2, X_3)$ be the lifetime of a coherent system, where the signature vector of the system is $\mathbf{s} = (\frac{3}{4}, \frac{1}{4}, 0, 0)$. Suppose that there is one component failed before time t_1 , but the system failed at time $t_2 (> t_1)$. The residual lifetime of unfailed components $X_{k:n}, k = i + 1, \dots, n$ is given by

$$RX_{k:n}(t_1, t_2) = (X_{k:n} - t_2 | Ar(t_1), T = t_2).$$

Now, according to Theorem 1, the survival function of $RX_{k:n}(t_1, t_2)$ can be written as

$$\begin{aligned} \mathbb{P}(RX_{k:n}(t_1, t_2) > x) &= \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T = t_2) \\ &= \sum_{m=r+1}^i q_m(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2). \end{aligned}$$

For $r = 1, k = 3$, we have

$$\mathbb{P}(X_{3:4} - t_2 > x | Ar(t_1), T = t_2) = q_2(1, t_1, t_2) \mathbb{P}(X_{3:4} - t_2 > x | Ar(t_1), X_{2:4} = t_2),$$

where

$$q_2(1, t_1, t_2) = 1, \quad \mathbb{P}(X_{3:4} - t_2 > x | Ar(t_1), X_{2:4} = t_2) = \left(\frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2, \quad x \in \mathbb{R}^+.$$

Thus,

$$\mathbb{P}(RX_{3:4}(t_1, t_2) > x) = \mathbb{P}(X_{3:4} - t_2 > x | Ar(t_1), T = t_2) = \left(\frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2, \quad x \in \mathbb{R}^+.$$

For $r = 1, k = 4$, we have

$$\mathbb{P}(X_{4:4} - t_2 > x | Ar(t_1), T = t_2) = q_2(1, t_1, t_2) \mathbb{P}(X_{4:4} - t_2 > x | Ar(t_1), X_{2:4} = t_2),$$

where

$$q_2(1, t_1, t_2) = 1,$$

and

$$\mathbb{P}(X_{4:4} - t_2 > x | Ar(t_1), X_{2:4} = t_2) = 1 - \left(\frac{\bar{F}(t_2) - \bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2, \quad x \in \mathbb{R}^+.$$

Thus,

$$\mathbb{P}(RX_{4:4}(t_1, t_2) > x) = \mathbb{P}(X_{4:4} - t_2 > x | Ar(t_1), T = t_2) = 1 - \left(\frac{\bar{F}(t_2) - \bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2, \quad x \in \mathbb{R}^+.$$

Suppose that X_i follows the Weibull distribution (cf. [31]), i.e., $\bar{F}(x) = e^{-\lambda^\alpha x^\alpha}, x \in \mathbb{R}^+$. Subsequently, we have

$$\mathbb{P}(RX_{3:4}(t_1, t_2) > x) = \left(\frac{\bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2 = e^{2\lambda^\alpha(t_2^\alpha - (t_2 + x)^\alpha)}, \quad x \in \mathbb{R}^+,$$

$$\mathbb{P}(RX_{4:4}(t_1, t_2) > x) = 1 - \left(\frac{\bar{F}(t_2) - \bar{F}(t_2 + x)}{\bar{F}(t_2)} \right)^2 = 2e^{\lambda^\alpha(t_2^\alpha - (t_2 + x)^\alpha)} - e^{2\lambda^\alpha(t_2^\alpha - (t_2 + x)^\alpha)}, \quad x \in \mathbb{R}^+.$$

Figures 2 and 3 show the graphs of the reliability function of $RX_{3:4}(t_1, t_2)$ and $RX_{4:4}(t_1, t_2)$ in Example 1 when $\alpha = 0.6, \lambda = 1.1$, and $\alpha = 1.3, \lambda = 0.8$.

Figure 2. The graphs of the reliability function of $RX_{3:4}(t_1, t_2)$, for the Weibull ($\alpha = 0.6, \lambda = 1.1$) (left) and ($\alpha = 1.3, \lambda = 0.8$) (right).

Figure 3. The graphs of the reliability function of $RX_{4:4}(t_1, t_2)$, for the Weibull ($\alpha = 0.6, \lambda = 1.1$) (left) and ($\alpha = 1.3, \lambda = 0.8$) (right).

The following Theorem 2 compares the conditional residual lifetimes $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$ of the unfailed components in two $(n - m + 1)$ -out-of- n systems by the likelihood ratio order.

Theorem 2. For $l < m$, we have

$$(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2) \leq_{lr} (X_{k:n} - t_2 | Ar(t_1), X_{l:n} = t_2).$$

Proof. Combining Lemma 1(i) with Lemma 2, it holds that

$$Z_{k-m:n-m}^{t_2} \leq_{lr} Z_{k-l:n-l}^{t_2}.$$

□

Remark 2. Theorem 2 shows a $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$ decrease in m from the view of the likelihood ratio.

The following theorem provides a sufficient condition of the usual stochastic order for conditional signature vector $\mathbf{q}(t_1, t_2)$.

Theorem 3. Suppose that $\mathbf{q}(t_1, t_2)$ is the vector of conditional signature $q_m(r, t_1, t_2)$. If $t_1 \leq t_2 \leq t_3$, then $\mathbf{q}(t_1, t_2) \leq_{st} \mathbf{q}(t_1, t_3)$.

Proof. For obtaining the required conclusion, we need to prove that for all $m \geq r + 1$,

$$\frac{\sum_{j=m}^n s_j C_{r,j}^n(\phi(t_1, t_2))^j}{\sum_{l=r+1}^n s_l C_{r,l}^n(\phi(t_1, t_2))^l} \leq \frac{\sum_{j=m}^n s_j C_{r,j}^n(\phi(t_1, t_3))^j}{\sum_{l=r+1}^n s_l C_{r,l}^n(\phi(t_1, t_3))^l},$$

which is equivalent to

$$\begin{aligned}
 & \sum_{j=m}^n \sum_{l=r+1}^n s_j s_l C_{r,j}^n C_{r,l}^n [(\phi(t_1, t_3))^j ((\phi(t_1, t_2))^l - (\phi(t_1, t_2))^j (\phi(t_1, t_3))^l)] \\
 = & \sum_{j=m}^n \sum_{l=r+1}^{m-1} s_j s_l C_{r,j}^n C_{r,l}^n (\phi(t_1, t_3))^l (\phi(t_1, t_2))^l [(\phi(t_1, t_3))^{j-l} - (\phi(t_1, t_2))^{j-l}] \\
 & + \sum_{j=m}^n \sum_{l=m}^n s_j s_l C_{r,j}^n C_{r,l}^n [(\phi(t_1, t_3))^j ((\phi(t_1, t_2))^l - (\phi(t_1, t_2))^j (\phi(t_1, t_3))^l)] \geq 0.
 \end{aligned} \quad (8)$$

It is easily seen that the second term of (8) equals zero, and for $t_1 < t_2 < t_3$ and $l < j$, due to $\phi(t_1, t_2)$ increasing in t_2 , thus (8) is nonnegativity. The proof of the result is completed. $\square$

Example 2. Assume that X_1, X_2, X_3 , and X_4 are i.i.d. random variables. Consider a coherent system with lifetime $T = \min(X_1, \max(X_2, X_3, X_4))$, the signature vector of the system is $\mathbf{s} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$. Suppose that there is one component failed at time t_1 , and the system failed at time $t_2 (> t_1)$. It is easy to compute the conditional signature vector of the system that can be written as

$$\mathbf{q}(t_1, t_2) = \left(0, \frac{\bar{F}(t_2)}{4\bar{F}(t_1) - 3\bar{F}(t_2)}, \frac{4\bar{F}(t_1) - 4\bar{F}(t_2)}{4\bar{F}(t_1) - 3\bar{F}(t_2)}, 0 \right).$$

Suppose that X has a Weibull distribution, i.e., $\bar{F}(\lambda, \alpha; x) = e^{-\lambda^\alpha x^\alpha}$, let $\lambda = 1.7$, $\alpha = 0.6$ and $m = 3$. It is follows from Figure 4 that verifies the nonnegativeness of (8); hence, the validity of Theorem 3 is confirmed.

Figure 4. The graphs of the function of $H(t_1, t_2, t_3)$, for the $t_1 = 1, t_2 = 3$ (left) and $t_1 = 1, t_2 = 4$ (right).

Some sufficient conditions for the stochastic orders of two coherent systems with i.i.d. components but having different structures are given by the following Theorem 4.

Theorem 4. Let $T_j = \tau_j(X_1, X_2, \dots, X_n)$ be the lifetime of two coherent systems, where $X_1, X_2, \dots, X_n$ are i.i.d random variables with common distribution function F . Suppose that coherent system T_j has signature vector $\mathbf{s}^{(j)} = (s_1^{(j)}, \dots, s_i^{(j)}, 0, \dots, 0)$ and conditional signature vector $\mathbf{q}^{(j)}(t_1, t_2) = (0, \dots, 0, q_{r+1}^{(j)}(r, t_1, t_2), \dots, q_i^{(j)}(r, t_1, t_2), 0, \dots, 0), j = 1, 2$. Subsequently, for any $0 < t_1 < t_2$, we have

(i) If $\mathbf{q}^{(1)}(t_1, t_2) \leq_{st} \mathbf{q}^{(2)}(t_1, t_2)$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_1 = t_2) \geq_{st} (X_{k:n} - t_2 | Ar(t_1), T_2 = t_2);$$

(ii) If $\mathbf{q}^{(1)}(t_1, t_2) \leq_{rh} \mathbf{q}^{(2)}(t_1, t_2)$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_1 = t_2) \geq_{hr} (X_{k:n} - t_2 | Ar(t_1), T_2 = t_2);$$

(iii) If $\mathbf{q}^{(1)}(t_1, t_2) \leq_{hr} \mathbf{q}^{(2)}(t_1, t_2)$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_1 = t_2) \geq_{rh} (X_{k:n} - t_2 | Ar(t_1), T_2 = t_2);$$

(iv) If $\mathbf{q}^{(1)}(t_1, t_2) \leq_{lr} \mathbf{q}^{(2)}(t_1, t_2)$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_1 = t_2) \geq_{lr} (X_{k:n} - t_2 | Ar(t_1), T_2 = t_2).$$

Proof. (i) According to Theorem 1 and Lemma 2, the reliability function of $(X_{k:n} - t_2 | Ar(t_1), T_j = t_2)$, $j = 1, 2$, can be represented as

$$\mathbb{P}(X_{k:n} - t_2 | Ar(t_1), T_j = t_2) = \sum_{m=r+1}^i q_m^{(j)}(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2). \quad (9)$$

Because $\mathbf{q}^{(1)}(t_1, t_2) \leq_{st} \mathbf{q}^{(2)}(t_1, t_2)$, thus, from Theorem 2 and Lemma 1(ii), it implies that

$$\sum_{m=r+1}^i q_m^{(1)}(r, t_1, t_2) \mathbb{P}(Z_{k-m:n-m}^{t_2} > x) \geq \sum_{m=r+1}^i q_m^{(2)}(r, t_1, t_2) \mathbb{P}(Z_{k-m:n-m}^{t_2} > x). \quad (10)$$

Hence, from (9) and (10), we have

$$\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T_1 = t_2) \geq \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T_2 = t_2).$$

The proof of part(i) is completed.

(ii) In order to obtain the required result, it suffices to show that

$$\frac{\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T_1 = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T_2 = t_2)} \quad (11)$$

is increasing in $x \in \mathbb{R}^+$, that is, for any $0 < x_1 < x_2$,

$$\frac{\mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), T_1 = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), T_2 = t_2)} \geq \frac{\mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), T_1 = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), T_2 = t_2)},$$

which is equivalent to

$$\begin{aligned} & \frac{\sum_{m=r+1}^i q_m^{(1)}(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), X_{m:n} = t_2)}{\sum_{m=r+1}^i q_m^{(1)}(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), X_{m:n} = t_2)} \\ & \geq \frac{\sum_{m=r+1}^i q_m^{(2)}(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), X_{m:n} = t_2)}{\sum_{m=r+1}^i q_m^{(2)}(r, t_1, t_2) \mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), X_{m:n} = t_2)}. \end{aligned}$$

Define, for $0 < x_1 < x_2, m = r + 1, \dots, i$,

$$\alpha(m) = \mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), X_{m:n} = t_2), \quad \beta(m) = \mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), X_{m:n} = t_2).$$

According to Theorem 2,

$$\frac{\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m+1:n} = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2)}$$

is decreasing in $x \in \mathbb{R}^+$. That is, for any $x_1 \leq x_2$, we have

$$\frac{\mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), X_{m+1:n} = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), X_{m+1:n} = t_2)} \leq \frac{\mathbb{P}(X_{k:n} - t_2 > x_2 | Ar(t_1), X_{m:n} = t_2)}{\mathbb{P}(X_{k:n} - t_2 > x_1 | Ar(t_1), X_{m:n} = t_2)},$$

thus, $\alpha(m)/\beta(m)$ is decreasing in m , note that $\beta(m)$ is also decreasing in m . Because $\mathbf{q}^{(1)}(t_1, t_2) \leq_{rh} \mathbf{q}^{(2)}(t_1, t_2)$, hence, from Lemma 1(iii), (11) is increasing in $x \in \mathbb{R}^+$. This completes the proof of part(ii).

(iii) By Lemma 1 (iv), the proof of part (iii) is similar to that of part (ii), and is, thus, is omitted here.

(iv) For any $0 \leq t_1 < t_2$, let $f_{T_j}(t_1, t_2, x)$ denote the density function of $(X_{k:n} - t_2 | Ar(t_1), T_j = t_2)$, $j = 1, 2$. In order to obtain the required conclusion, we need to prove that

$$\frac{f_{T_1}(t_1, t_2, x)}{f_{T_2}(t_1, t_2, x)}$$

is increasing in $x \in \mathbb{R}^+$,; hence, for any $x_1 \leq x_2$,

$$\frac{\sum_{m=r+1}^i q_m^{(1)}(r, t_1, t_2) h_m(t_1, t_2, x_1)}{\sum_{m=r+1}^i q_m^{(2)}(r, t_1, t_2) h_m(t_1, t_2, x_1)} \leq \frac{\sum_{m=r+1}^i q_m^{(1)}(r, t_1, t_2) h_m(t_1, t_2, x_2)}{\sum_{m=r+1}^n q_m^{(2)}(r, t_1, t_2) h_m(t_1, t_2, x_2)},$$

which is equivalent to

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{l=r+1}^i q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)] \\ &= \sum_{m=r+1}^i \sum_{l=r+1}^m q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)] \\ &+ \sum_{m=r+1}^i \sum_{l=m}^i q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)] \end{aligned} \quad (12)$$

is nonnegative, where $h_m(t_1, t_2, x)$ is the density function of $(X_{k:n} - t_2 | Ar(t_1), X_{m:n} = t_2)$. Note that the second term in the right side of (12) can be represented as

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{l=m}^i q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)] \\ &= \sum_{l=r+1}^i \sum_{m=r+1}^l q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)] \\ &= \sum_{m=r+1}^i \sum_{l=r+1}^m q_l^{(1)}(r, t_1, t_2) q_m^{(2)}(r, t_1, t_2) [h_l(t_1, t_2, x_2) h_m(t_1, t_2, x_1) - h_l(t_1, t_2, x_1) h_m(t_1, t_2, x_2)]. \end{aligned}$$

Thus, the right side of (12) can be rewritten by

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{l=r+1}^m [q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) - q_m^{(2)}(r, t_1, t_2) q_l^{(1)}(r, t_1, t_2)] \\ & \quad \times [h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2)]. \end{aligned} \quad (13)$$

Due to $\mathbf{q}^{(1)}(t_1, t_2) \leq_{lr} \mathbf{q}^{(2)}(t_1, t_2)$ implying that $(q_m^{(2)}(r, t_1, t_2)) / (q_m^{(1)}(r, t_1, t_2))$ is increasing in $m(m = r + 1, \dots, i)$, which is,

$$q_m^{(1)}(r, t_1, t_2) q_l^{(2)}(r, t_1, t_2) - q_m^{(2)}(r, t_1, t_2) q_l^{(1)}(r, t_1, t_2) \leq 0, \text{ for } l < m. \quad (14)$$

Additionally, according to Theorem 2, for $l < m$, we have

$$h_m(t_1, t_2, x_2) h_l(t_1, t_2, x_1) - h_m(t_1, t_2, x_1) h_l(t_1, t_2, x_2) \leq 0, \text{ for } x_1 \leq x_2. \quad (15)$$

Hence, it is follows from (14) and (15) that (13) is nonnegative. The proof is completed. $\square$

Remark 3. It is noted that Theorem 4 extends Theorem 2.4 of [19] to the stochastic orders of residual lifetimes of the unfailed components under double monitoring.

The following example illustrates the results of Theorem 4.

Example 3. Assume that X_1, X_2, X_3 , and X_4 are i.i.d. random variables. Consider two coherent systems with lifetime $T_1 = \max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$ and $T_2 = \min(X_1, \max(X_2, X_3, X_4))$, which have the signature vectors $\mathbf{s}^{(1)} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $\mathbf{s}^{(2)} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, respectively. Suppose that there is one component failed at time t_1 , and the system failed at time $t_2 (> t_1)$. Set X has a Weibull distribution, i.e., $\bar{F}(\lambda, \alpha; x) = e^{-\lambda x^\alpha}$, $\lambda = 0.4$, $\alpha = 1.2$ and $r = 1, k = 4$.

- (i) Figure 5 plots the reliability functions of $RX_{4:4}^{T_1}$ and $RX_{4:4}^{T_2}$, it is easily see that $\bar{F}(RX_{4:4}^{T_1}; x) \geq \bar{F}(RX_{4:4}^{T_2}; x)$;
 - (ii) As in Figure 6, it is clearly seen that $(\bar{F}(RX_{4:4}^{T_1}; x)) / (\bar{F}(RX_{4:4}^{T_2}; x))$ is increasing in $x \in \mathbb{R}^+$;
 - (iii) Figure 7 clearly shows that $(F(RX_{4:4}^{T_1}; x)) / (F(RX_{4:4}^{T_2}; x))$ is increasing in $x \in \mathbb{R}^+$;
 - (iv) From Figure 8, we clearly see that $(f(RX_{4:4}^{T_1}; x)) / (f(RX_{4:4}^{T_2}; x))$ is increasing in $x \in \mathbb{R}^+$.
- Hence, it follows from (i), (ii), (iii), and (iv) that confirms the validity of Theorem 4.

Figure 5. The graphs of the reliability function of $RX_{4:4}^{T_1}(t_1, t_2)$ and $RX_{4:4}^{T_2}(t_1, t_2)$, for the $t_1 = 1$ (left) and $t_2 = 8$ (right).

Figure 6. The graphs of the $(\bar{F}(RX_{4:4}^{T_1}; x)) / (\bar{F}(RX_{4:4}^{T_2}; x))$, for the $t_1 = 1$ (left) and $t_2 = 8$ (right).

Figure 7. The graphs of the $(F(RX_{4:4}^{T_1}; x))/(F(RX_{4:4}^{T_2}; x))$, for the $t_1 = 1$ (left) and $t_2 = 8$ (right).

Figure 8. The graphs of the $(f(RX_{4:4}^{T_1}; x))/(f(RX_{4:4}^{T_2}; x))$, for the $t_1 = 1$ (left) and $t_2 = 8$ (right).

In the next, we will consider aging property of the residual lifetimes on survival components in a coherent system. Before giving the theorem, we first introduce a lemma.

Lemma 3. Assume that $X_1, X_2, \dots, X_n$ are i.i.d. random variables that have common distribution function F . If X_i is IFR (increasing failure rate), then for $0 \leq r < k$, $\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2)$ is decreasing in $t_2 (> t_1)$.

Proof. For any $x \in \mathbb{R}^+$, and $0 < t_1 < t_2$, note that

$$\begin{aligned} & \mathbb{P}(t_2 - X_{k:n} > x | X_{r:n} = t_1, X_{m:n} = t_2) \\ &= \sum_{j=n-k+1}^{n-m} \binom{n-m}{j} (\bar{G}_{t_2}(x))^j (1 - \bar{G}_{t_2}(x))^{n-m-j} \\ &= \int_{1-\bar{G}_{t_2}(x)}^1 (k-m) \binom{n-m}{k-m} u^{k-m-1} (1-u)^{n-k+m} du, \end{aligned} \quad (16)$$

where $\bar{G}_{t_2}(x) = \frac{F(t_2+x)}{F(t_2)}$. Note that X_i is IFR, which is equivalent to $\bar{G}_{t_2}(x) = \frac{F(t_2+x)}{F(t_2)}$ is decreasing in t_2 . Thus, it follows from (16) that $P(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2)$ is decreasing in t_2 . $\square$

Theorem 5. Consider that a coherent system is composed of n i.i.d. components with lifetimes $X_1, X_2, \dots, X_n$. Suppose that the system has signature vector $\mathbf{s} = (s_1, s_2, \dots, s_i, 0, \dots, 0)$. If X_i is IFR, then $\mathbb{P}(RX_{k:n}(t_1, t_2) > x)$ is decreasing in $t_2 (> t_1)$.

Proof. First, taking the partial derivative of reliability function of $RX_{k:n}(t_1, t_2)$ with respect to t_2 gives rise to

$$\begin{aligned} & \frac{\partial \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T = t_2)}{\partial t_2} \\ = & \frac{\partial \sum_{m=r+1}^i q_m(r, t_1, t_2) p(X_{k:n} - t_2 > x | Ar(t_1), T = t_2)}{\partial t_2} \\ = & \sum_{m=r+1}^i \frac{\partial q_m(r, t_1, t_2)}{\partial t_2} \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T = t_2) \\ & + \sum_{m=r+1}^i q_m(r, t_1, t_2) \frac{\partial \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2)}{\partial t_2}. \end{aligned} \quad (17)$$

Because F is IFR, it follows from Lemma 3 that the second term in (17) is nonpositive. Note that

$$\begin{aligned} \frac{\partial q_m(r, t_1, t_2)}{\partial t_2} &= \frac{\partial \left(\frac{s_m C_{r,m}^n (\phi(t_1, t_2))^m}{\sum_{j=r+1}^i s_j C_{r,j}^n (\phi(t_1, t_2))^j} \right)}{\partial t_2} \\ &= \frac{\frac{\partial \phi(t_1, t_2)}{\partial t_2} \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j)}{\left(\sum_{j=r+1}^i s_j C_{r,j}^n (\phi(t_1, t_2))^j \right)^2}. \end{aligned}$$

Let $g(m) = \mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), X_{m:n} = t_2)$, then

$$\begin{aligned} & \sum_{m=r+1}^i \frac{\partial q_m(r, t_1, t_2)}{\partial t_2} g(m) \\ = & \sum_{m=r+1}^i \frac{\frac{\partial \phi(t_1, t_2)}{\partial t_2} \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j)}{\left(\sum_{j=r+1}^i s_j C_{r,j}^n (\phi(t_1, t_2))^j \right)^2} g(m) \\ \stackrel{sgn}{=} & \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ = & \sum_{m=r+1}^i \sum_{j=r+1}^m s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ & + \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ = & \sum_{m=r+1}^i \sum_{j=r+1}^m s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ & + \sum_{j=r+1}^i \sum_{m=r+1}^j s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ = & \sum_{m=r+1}^i \sum_{j=r+1}^m s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(m) \\ & - \sum_{m=r+1}^i \sum_{j=r+1}^m s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} (m-j) g(j) \\ = & \sum_{m=r+1}^i \sum_{j=r+1}^m s_m s_j C_{r,m}^n C_{r,j}^n (\phi(t_1, t_2))^{m+j-1} [(m-j)(g(m) - g(j))]. \end{aligned} \quad (18)$$

Hence, according to Theorem 2, $g(m)$ is decreasing in m , which implies $(m-j)(g(m) - g(j)) \leq 0$, that is, (18) is nonpositive. Thus, (17) is nonpositive. The proof is completed. $\square$

Remark 4. It should be noted that Theorem 5 generalizes Theorem 2.6 in [19] to the case of coherent system under double monitoring.

Notice that random variable X is IFR in Example 1 when $\alpha > 1$. From the right of Figures 2 and 3, we can clearly see that $\mathbb{P}(RX_{k:n}(t_1, t_2) > x)$ is decreasing in $t_2 (> t_1)$, which confirms the validity of Theorem 5.

Theorem 6. Suppose that two coherent systems are composed of n i.i.d. components with lifetimes $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$, which have distribution functions F and G , respectively. Let $T_1 = \tau(X_1, X_2, \dots, X_n)$ and $T_2 = \tau(Y_1, Y_2, \dots, Y_n)$ be the lifetime of two coherent systems that have common signature vector $\mathbf{s} = (s_1, \dots, s_i, 0, \dots, 0)$. If $X_1 \leq_{hr} Y_1$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_X = t_2) \leq_{st} (Y_{k:n} - t_2 | Y_{r:n} = t_1, T_Y = t_2).$$

Proof. For $m = r+1, \dots, i$, and $0 < t_1 < t_2$, let

$$q_m^X(r, t_1, t_2) = \frac{s_m C_{r,m}^n (\phi^X(t_1, t_2))^m}{\sum_{j=r+1}^i s_j C_{r,j}^n (\phi^X(t_1, t_2))^j}, \quad q_m^Y(r, t_1, t_2) = \frac{s_m C_{r,m}^n (\phi^Y(t_1, t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi^Y(t_1, t_2))^j},$$

where $\phi^X(t_1, t_2) = \frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1$ and $\phi^Y(t_1, t_2) = \frac{\bar{G}(t_1)}{\bar{G}(t_2)} - 1$. According to Theorem 1 and Lemma 2, the reliability function of inactivity times $RX_{k:n}(t_1, t_2)$ and $RY_{k:n}(t_1, t_2)$ are

$$\mathbb{P}(X_{k:n} - t_2 > x | Ar(t_1), T_X = t_2) = \sum_{m=r+1}^i \frac{s_m C_{r,m}^n (\phi^X(t_1, t_2))^m}{\sum_{j=r+1}^i s_j C_{r,j}^n (\phi^X(t_1, t_2))^j} \mathbb{P}(Z_{k-m:n-m}^{t_2, X} > x),$$

and

$$\mathbb{P}(Y_{k:n} - t_2 > x | Ar(t_1), T_Y = t_2) = \sum_{m=r+1}^i \frac{s_m C_{r,m}^n (\phi^Y(t_1, t_2))^m}{\sum_{j=r+1}^i s_j C_{r,j}^n (\phi^Y(t_1, t_2))^j} \mathbb{P}(Z_{k-m:n-m}^{t_2, Y} > x),$$

respectively, where $Z_{k-m:n-m}^{t_2, X}$ is the $(k-m)$ th order statistics in $(n-m)$ i.i.d. random variables with distributed distribution $(F(t_2 + x) - F(t_2)) / (1 - F(t_2))$, and $Z_{k-m:n-m}^{t_2, Y}$ is the $(k-m)$ th order statistics in $(n-m)$ i.i.d. random variables with distributed distribution $(G(t_2 + x) - G(t_2)) / (1 - G(t_2))$.

Thus, in order to show the desired conclusion, we only need to prove that

$$\sum_{m=r+1}^i q_m^Y(r, t_1, t_2) \mathbb{P}(Z_{k-m:n-m}^{t_2, Y} > x) \geq \sum_{m=r+1}^i q_m^X(r, t_1, t_2) \mathbb{P}(Z_{k-m:n-m}^{t_2, X} > x),$$

it is equivalent to

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j \mathbb{P}(Z_{k-m:n-m}^{t_2, Y} > x) \\ & - \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j \mathbb{P}(Z_{k-m:n-m}^{t_2, X} > x) \geq 0, \end{aligned}$$

that is,

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n (\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j [\mathbb{P}(Z_{k-m:n-m}^{t_2,Y} > x) - \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x)] \\ & + \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j \\ & - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \geq 0. \end{aligned} \quad (19)$$

Owing to $X_1 \leq_{hr} Y_1$ meaning that $\bar{F}(t_2 + x)/\bar{F}(t_2) \leq \bar{G}(t_2 + x)/\bar{G}(t_2)$, it follows from (16) that

$$\mathbb{P}(Z_{k-m:n-m}^{t_2,Y} > x) - \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) \geq 0,$$

which implies that the first term in the left side of (19) is positive. It is easily observed that the second term in the left side of (19) can be written as

$$\begin{aligned} & \sum_{m=r+1}^i \sum_{j=r+1}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ = & \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ & + \sum_{m=r+1}^i \sum_{j=r+1}^{m-1} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ = & \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ & + \sum_{j=r+1}^i \sum_{m=j}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ = & \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ & - \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{k-j:n-j}^{t_2,X} > x) [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ = & \sum_{m=r+1}^i \sum_{j=m}^i s_m s_j C_{r,m}^n C_{r,j}^n [(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j] \\ & \times [\mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) - \mathbb{P}(Z_{k-j:n-j}^{t_2,X} > x)]. \end{aligned} \quad (20)$$

From Theorem 2, for any $m < j$, it holds that

$$\mathbb{P}(Z_{k-m:n-m}^{t_2,X} > x) - \mathbb{P}(Z_{k-j:n-j}^{t_2,X} > x) \geq 0. \quad (21)$$

Because of $X_1 \leq_{hr} Y_1$, it is easy to see that

$$(\phi^Y(t_1, t_2))^m (\phi^X(t_1, t_2))^j - (\phi^X(t_1, t_2))^m (\phi^Y(t_1, t_2))^j \geq 0. \quad (22)$$

According to (21) and (22), the nonnegativity of (20) is verified, which yields the result. $\square$

Remark 5. Theorem 6 provides a sufficient condition for the usual stochastic order holds between two coherent systems having the same structure, but with two heterogeneous sets of i.i.d components. This extends Theorem 2.8 of [19] to the scenario of double monitoring.

The next example confirms the result of Theorem 6.

Example 4. Based on the Example 1, suppose that X_1 and Y_1 have a Weibull distribution, i.e., $\bar{F}(\lambda_1, \alpha_1; x) = e^{-\lambda_1^{\alpha_1} x^{\alpha_1}}$, $\bar{G}(\lambda_2, \alpha_2; x) = e^{-\lambda_2^{\alpha_2} x^{\alpha_2}}$. Let $\lambda_1 = 1.5$, $\lambda_2 = 0.4$, $\alpha_1 = \alpha_2 = 0.6$. Owing to $\bar{G}(\lambda_2, \alpha_2; x) / \bar{F}(\lambda_1, \alpha_1; x) = e^{0.698345x^{0.6}}$ is increasing in $x \in \mathbb{R}^+$, obviously, $X_1 \leq_{hr} Y_1$.

(i) For $r = 1, k = 3$, Figure 9 plots the reliability functions of $RX_{3:4}$ and $RY_{3:4}$, it is easily seen that $\bar{F}(RX_{3:4}; x) \leq \bar{G}(RY_{3:4}; x)$;

(ii) For $r = 1, k = 4$, from which it is clear that the survival functions $\bar{F}(RX_{4:4}; x) \leq \bar{G}(RY_{4:4}; x)$ in Figure 9.

Therefore, it follows from (i) and (ii), which confirms the validity of Theorem 6.

Figure 9. The reliability functions of $RX_{3:4}$ and $RY_{3:4}$.

In combination with Theorem 4(i) and Theorem 6, it is easy to gain the next corollary.

Corollary 1. Suppose that $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$ are i.i.d. random variables with common distribution function F and G , respectively. Let $T_1 = \tau_1(X_1, X_2, \dots, X_n)$, $T_2 = \tau_2(X_1, X_2, \dots, X_n)$, and $T_3 = \tau_2(Y_1, Y_2, \dots, Y_n)$ be the lifetimes of three coherent systems, where T_1 has the signature vector $\mathbf{s}^{(1)} = (s_1^{(1)}, \dots, s_i^{(1)}, 0, \dots, 0)$, and T_2 and T_3 have a common signature vector $\mathbf{s}^{(2,3)} = (s_1^{(2,3)}, \dots, s_i^{(2,3)}, 0, \dots, 0)$. If $\mathbf{p}^{(1)} \leq_{st} \mathbf{p}^{(2)}$ and $X_1 \geq_{hr} Y_1$, for any $t_1 < t_2$, then

$$(X_{k:n} - t_2 | Ar(t_1), T_1 = t_2) \geq_{st} (X_{k:n} - t_2 | Ar(t_1), T_2 = t_2) \geq_{st} (Y_{k:n} - t_2 | Ar(t_1), T_3 = t_2).$$

4. Conclusions

In this manuscript, we consider the residual lifetime on unfailed components in a coherent system, which has n i.i.d. components, under the assumption that there are r ($r < i$) failed components at time t_1 , but the system did not fail at time t_1 , and the failure time of the system is time t_2 ($> t_1$). The survival function of the residual lifetimes on unfailed components in a coherent system can be expressed as a mixture representation of the survival function for the residual lifetime on the unfailed components with lifetime $X_{k:n}$, $k = i + 1, \dots, n$ in a $(n - m + 1)$ -out-of- n system on the conditional that the failure time of the system is time t_2 ($> t_1$). With the help of the reliability function, some stochastic comparisons of condition residual lifetime of unfailed components between two coherent systems are obtained. We also provide the aging properties of the residual lifetime of unfailed components. In fact, we provide conditional random variable $(X_{k:n} - t_2 | Ar(t_1), T = t_2)$ that is a generalization of conditional random variable $(X_{k:n} - t | T = t)$, which is proposed in ([18,19]).

In the practical case, the components of the system may be statistically dependent. In fact, it will be interesting to consider the residual lifetimes of unfailed components in a coherent system with dependent components. However, these problems remain open and merit further discussion, owing to the complexity of modeling for the system with dependent components.

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