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# On the $\sigma$-Length of Maximal Subgroups of Finite $\sigma$-Soluble Groups

**Abd El-Rahman Heliel** [1,2,*], **Mohammed Al-Shomrani** [1] and **Adolfo Ballester-Bolinches** [3]

1 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; malshamrani@kau.edu.sa
2 Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt
3 Departament de Matemàtiques, Universitat de València, Dr. Moliner 50, Burjassot, 46100 València, Spain; Adolfo.Ballester@uv.es
* Correspondence: aahsalem1@kau.edu.sa

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**Abstract:** Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all prime numbers and let $G$ be a finite group. We say that $G$ is $\sigma$-primary if all the prime factors of $|G|$ belong to the same member of $\sigma$. $G$ is said to be $\sigma$-*soluble* if every chief factor of $G$ is $\sigma$-primary, and $G$ is $\sigma$-nilpotent if it is a direct product of $\sigma$-primary groups. It is known that $G$ has a largest normal $\sigma$-nilpotent subgroup which is denoted by $F_\sigma(G)$. Let $n$ be a non-negative integer. The $n$-term of the $\sigma$-Fitting series of $G$ is defined inductively by $F_0(G) = 1$, and $F_{n+1}(G)/F_n(G) = F_\sigma(G/F_n(G))$. If $G$ is $\sigma$-soluble, there exists a smallest $n$ such that $F_n(G) = G$. This number $n$ is called the $\sigma$-*nilpotent length* of $G$ and it is denoted by $l_\sigma(G)$. If $\mathfrak{F}$ is a subgroup-closed saturated formation, we define the $\sigma$-$\mathfrak{F}$-*length* $n_\sigma(G, \mathfrak{F})$ of $G$ as the $\sigma$-nilpotent length of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of $G$. The main result of the paper shows that if $A$ is a maximal subgroup of $G$ and $G$ is a $\sigma$-soluble, then $n_\sigma(A, \mathfrak{F}) = n_\sigma(G, \mathfrak{F}) - i$ for some $i \in \{0, 1, 2\}$.

**Keywords:** finite group; $\sigma$-solubility; $\sigma$-nilpotency; $\sigma$-nilpotent length

## 1. Introduction

All groups considered in this paper are finite.

Skiba [1] (see also [2]) generalised the concepts of solubility and nilpotency by introducing $\sigma$-solubility and $\sigma$-nilpotency, in which $\sigma$ is a partition of $\mathbb{P}$, the set of all primes. Hence $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, with $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

In the sequel, $\sigma$ will be a partition of the set of all primes $\mathbb{P}$.

A group $G$ is called $\sigma$-*primary* if all the prime factors of $|G|$ belong to the same member of $\sigma$.

**Definition 1.** *A group $G$ is said to be* $\sigma$-soluble *if every chief factor of $G$ is $\sigma$-primary. $G$ is said to be* $\sigma$-nilpotent *if it is a direct product of $\sigma$-primary groups.*

We note in the special case that $\sigma$ is the partition of $\mathbb{P}$ containing exactly one prime each, the class of $\sigma$-soluble groups is just the class of all soluble groups and the class of $\sigma$-nilpotent groups is just the class of all nilpotent groups.

Many normal and arithmetical properties of soluble groups and nilpotent groups still hold for $\sigma$-soluble and $\sigma$-nilpotent groups (see [2]) and, in fact, the class $\mathcal{N}_\sigma$ of all $\sigma$-nilpotent groups behaves in $\sigma$-soluble groups as nilpotent groups in soluble groups. In addition, every $\sigma$-soluble group has a conjugacy class of Hall $\sigma_i$-subgroups and a conjugacy class of Hall $\sigma_i'$-subgroups, for every $\sigma_i \in \sigma$.

Recall that a class of groups $\mathfrak{F}$ is said to be a *formation* if $\mathfrak{F}$ is closed under taking epimorphic images and every group $G$ has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of $G$ and it is denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is called *subgroup-closed* if $X^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$ for all subgroups $X$ of every group $G$; $\mathfrak{F}$ is *saturated* if it is closed under taking Frattini extensions.

A class of groups $\mathfrak{F}$ is said to be a *Fitting class* if $\mathfrak{F}$ is closed under taking normal subgroups and every group $G$ has a largest normal subgroup in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*radical* of $G$.

The following theorem which was proved in [1] (Corollary 2.4 and Lemma 2.5) turns out to be crucial in our study.

**Theorem 1.** *$\mathcal{N}_\sigma$ is a subgroup-closed saturated Fitting formation.*

The $\mathcal{N}_\sigma$-radical of a group $G$ is called the *$\sigma$-Fitting subgroup* of $G$ and it is denoted by $F_\sigma(G)$. Clearly, $F_\sigma(G)$ is the product of all normal $\sigma$-nilpotent subgroups of $G$. If $\sigma$ is the partition of $\mathbb{P}$ containing exactly one prime each, then $F_\sigma(G)$ is just the Fitting subgroup of $G$.

If $G$ is $\sigma$-soluble, then every minimal normal subgroup $N$ of $G$ is $\sigma$-primary so that $N$ is $\sigma$-nilpotent and it is contained in $F_\sigma(G)$. In particular, $F_\sigma(G) \neq 1$ if $G \neq 1$.

Let $n$ be a non-negative integer. The $n$-term of the $\sigma$-Fitting series of $G$ is defined inductively by $F_0(G) = 1$, and $F_{n+1}(G)/F_n(G) = F_\sigma(G/F_n(G))$. If $G$ is $\sigma$-soluble, there exists a smallest $n$ such that $F_n(G) = G$. This number $n$ is called the *$\sigma$-nilpotent length* of $G$ and it is denoted by $l_\sigma(G)$ (see [3,4]). The nilpotent length $l(G)$ of a group $G$ is just the $\sigma$-nilpotent length of $G$ for $\sigma$ the partition of $\mathbb{P}$ containing exactly one prime each.

The $\sigma$-nilpotent length is quite useful in the structural study of $\sigma$-soluble groups (see [3,4]), and allows us to extend some known results.

The central concept of this paper is the following:

**Definition 2.** *Let $\mathfrak{F}$ be a saturated formation. The* $\sigma$-$\mathfrak{F}$-length $n_\sigma(G,\mathfrak{F})$ *of a group $G$ is defined as the $\sigma$-nilpotent length of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of $G$.*

Applying [5] (Chapter IV, Theorem (3.13) and Proposition (3.14)) (see also [3] (Lemma 4.1)), we have the following useful result.

**Proposition 1.** *The class of all $\sigma$-soluble groups of $\sigma$-length at most $l$ is a subgroup-closed saturated formation.*

It is clear that the $\mathfrak{F}$-length $n_{\mathfrak{F}}(G)$ of a group $G$ studied in [6] is just the $\sigma$-$\mathfrak{F}$-length of $G$ for $\sigma$ the partition of $\mathbb{P}$ containing exactly one prime each, and the $\sigma$-nilpotent length of $G$ is just the $\sigma$-$\mathfrak{F}$-length of $G$ for $\mathfrak{F} = \{1\}$.

Ballester-Bolinches and Pérez-Ramos [6] (Theorem 1), extending a result by Doerk [7] (Satz 1), proved the following theorem:

**Theorem 2.** *Let $\mathfrak{F}$ be a subgroup-closed saturated formation and $M$ be a maximal subgroup of a soluble group $G$. Then $n_{\mathfrak{F}}(M) = n_{\mathfrak{F}}(G) - i$ for some $i \in \{0,1,2\}$.*

Our main result shows that Ballester-Bolinches and Pérez-Ramos' theorem still holds for the $\sigma$-$\mathfrak{F}$-length of maximal subgroups of $\sigma$-soluble groups.

**Theorem A.** *Let $\mathfrak{F}$ be a saturated formation. If $A$ is a maximal subgroup of a $\sigma$-soluble group $G$, then $n_\sigma(A,\mathfrak{F}) = n_\sigma(G,\mathfrak{F}) - i$ for some $i \in \{0,1,2\}$.*

## 2. Proof of Theorem A

**Proof.** Suppose that the result is false. Let $G$ be a counterexample of the smallest possible order. Then $G$ has a maximal subgroup $A$ such that $n_\sigma(A,\mathfrak{F}) \neq n_\sigma(G,\mathfrak{F})) - i$ for every $i \in \{0,1,2\}$. Since $A^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$ because $\mathfrak{F}$ is subgroup-closed, we have that $G^{\mathfrak{F}} \neq 1$. Moreover,

$n_\sigma(A,\mathfrak{F}) \leq n_\sigma(G,\mathfrak{F}) = n$ and $n \geq 1$. We proceed in several steps, the first of which depends heavily on the fact that the $\mathfrak{F}$-residual is epimorphism-invariant.

**Step 1.** *If $N$ is a normal $\sigma$-nilpotent subgroup of $G$, then $N$ is contained in $A$, $n_\sigma(A,\mathfrak{F}) = n_\sigma(A/N,\mathfrak{F})$ and $n_\sigma(G/N,\mathfrak{F}) = n-1$.*

Let $N$ be a normal $\sigma$-nilpotent subgroup of $G$. Applying [7] (Chapter II, Lemma (2.4)), we have that $G^{\mathfrak{F}}N/N = (G/N)^{\mathfrak{F}}$. Consequently, either $n_\sigma(G/N,\mathfrak{F}) = n$ or $n_\sigma(G/N,\mathfrak{F}) = n-1$.

Assume that $N$ is not contained in $A$. Then $G = AN$ and so $G/N \cong A/A\cap N$. Observe that either $n_\sigma(A/A\cap N,\mathfrak{F}) = n_\sigma(G/N,\mathfrak{F}) = n$ or $n_\sigma(A/A\cap N,\mathfrak{F}) = n_\sigma(G/N,\mathfrak{F}) = n-1$. Therefore $n-1 \leq n_\sigma(A,\mathfrak{F}) \leq n$. Consequently, either $n_\sigma(A,\mathfrak{F}) = n$ or $n_\sigma(A,\mathfrak{F}) = n-1$, contrary to assumption.

Therefore, $N$ is contained in $A$. The minimal choice of $G$ implies that $n_\sigma(A/N,\mathfrak{F}) = n_\sigma(G/N,\mathfrak{F}) - i$ for some $i \in \{0,1,2\}$, and so either $n_\sigma(A/N,\mathfrak{F}) = n-i$ or $n_\sigma(A/N,\mathfrak{F}) = n-i-1$. Suppose that $n_\sigma(A,\mathfrak{F}) \neq n_\sigma(A/N,\mathfrak{F})$. Then $n_\sigma(A,\mathfrak{F}) = n_\sigma(A/N,\mathfrak{F}) + 1$. Hence either $n_\sigma(A,\mathfrak{F}) = n-i+1$ or $n_\sigma(A,\mathfrak{F}) = n-i$. In the first case, $i > 0$ because $n \geq n_\sigma(A,\mathfrak{F})$. Hence $n_\sigma(A,\mathfrak{F}) = n-j$ for some $j \in \{0,1,2\}$, which contradicts our supposition. Consequently, $n_\sigma(A,\mathfrak{F}) = n_\sigma(A/N,\mathfrak{F})$.

Suppose that $n_\sigma(G/N,\mathfrak{F}) = n$. The minimality of $G$ yields $n_\sigma(A/N,\mathfrak{F}) = n-i$ for some $i \in \{0,1,2\}$. Therefore $n_\sigma(A,\mathfrak{F}) = n_\sigma(G,\mathfrak{F}) - i$ for some $i \in \{0,1,2\}$. This is a contradiction since we are assuming that $G$ is a counterexample. Consequently, $n_\sigma(G/N,\mathfrak{F}) = n-1$.

**Step 2.** $\mathrm{Soc}(G)$ *is a minimal normal subgroup of $G$ which is not contained in $\Phi(G)$, the Frattini subgroup of $G$.*

Assume that $N$ and $L$ are two distinct minimal normal subgroups of $G$. Then, by Step 1, $n_\sigma(G/L,\mathfrak{F}) = n-1$. Since the class of all $\sigma$-soluble groups of $\sigma$-$\mathfrak{F}$-length at most $n-1$ is a saturated formation by Proposition 1 and $N \cap L = 1$, it follows that $n_\sigma(G,\mathfrak{F}) = n-1$. This contradiction proves that $N = \mathrm{Soc}(G)$ is the unique minimal normal subgroup of $G$.

Assume that $N$ is contained in $\Phi(G)$. Since $n_\sigma(G/N,\mathfrak{F}) = n-1$ and the class of all $\sigma$-soluble groups of $\sigma$-$\mathfrak{F}$-length at most $n-1$ is a saturated formation by Proposition 1, we have that $n_\sigma(G,\mathfrak{F}) = n-1$, a contradiction. Therefore $N$ is not contained in $\Phi(G)$ as desired.

According to Step 2, we have that $N = Soc(G)$ is a minimal normal subgroup of $G$ which is not contained in $\Phi(G)$. Hence $G$ has a core-free maximal subgroup, $M$ say. Then $G = NM$ and, by [5] (Chapter A, (15.2)), either $N$ is abelian and $\mathrm{C}_G(N) = N$ or $N$ is non-abelian and $\mathrm{C}_G(N) = 1$. Since $G$ is $\sigma$-soluble, it follows that $N$ is $\sigma$-primary. Thus, $N$ is a $\sigma_i$-group for some $\sigma_i \in \sigma$.

**Step 3.** *Let $H$ be a subgroup of $G$ such that $N \subseteq H$. Then $F_\sigma(H) = \mathrm{O}_{\sigma_i}(H)$.*

Since $N$ is contained in $F_\sigma(H)$, it follows that every Hall $\sigma_i'$-subgroup of $F_\sigma(H)$ centralises $N$. Since $\mathrm{C}_H(N) = N$ or $\mathrm{C}_H(N) = 1$, we conclude that $F_\sigma(H)$ is a $\sigma_i$-group, i.e., $F_\sigma(H) = \mathrm{O}_{\sigma_i}(H)$.

**Step 4.** *We have a contradiction.*

Let $X = F_\sigma(G)$, and $T/X = F_\sigma(G/X)$. Suppose that $T$ is not contained in $A$. Then $G = AT$, $G/T \cong A/A\cap T$, and $n_\sigma(G/T,\mathfrak{F}) = n_\sigma(A/A\cap T,\mathfrak{F})$. By Step 1, $n_\sigma(G/X,\mathfrak{F}) = n-1$. Hence $n_\sigma(G/T,\mathfrak{F}) \in \{n-2, n-1\}$. Now, $X \subseteq A$ and $n_\sigma(A,\mathfrak{F}) = n_\sigma(A/X,\mathfrak{F})$ by Step 1. Consequently, $n_\sigma(A/A\cap T,\mathfrak{F}) \in \{n_\sigma(A,\mathfrak{F}) - 1, n_\sigma(A,\mathfrak{F})\}$. This means that $n_\sigma(A,\mathfrak{F}) = n-j$ for some $j \in \{0,1,2\}$. This contradiction yields $T \subseteq A$.

By Step 3, we have that $X = \mathrm{O}_{\sigma_i}(G)$. Assume that $E/X$ and $F/X$ are the Hall $\sigma_i$-subgroup and the Hall $\sigma_i'$-subgroup of $T/X$ respectively. Then $T/X = E/X \times F/X$ and $E$ and $F$ are normal subgroups of $G$. Since $X$ and $E/X$ are $\sigma_i$-groups, it follows that $E$ is a $\sigma_i$-group and hence $E \subseteq X$. In particular, $T/X$ is a $\sigma_i'$-group.

On the other hand, $F_\sigma(A) = \mathrm{O}_{\sigma_i}(A)$ by Step 3. Consequently $F_\sigma(A)/X \subseteq \mathrm{C}_A(T/X)$. Applying [1] (Corollary 11), we conclude that $\mathrm{C}_A(T/X) \subseteq T/X$. Therefore $X = F_\sigma(A)$.

By Step 1, $n_\sigma(A,\mathfrak{F}) = n_\sigma(A/X,\mathfrak{F})$. Now $n_\sigma(A/X,\mathfrak{F}) = l_\sigma(A^{\mathfrak{F}}X/X)$. Since $A^{\mathfrak{F}}/A^{\mathfrak{F}} \cap X = A^{\mathfrak{F}}/F_\sigma(A^{\mathfrak{F}})$, it follows that $n_\sigma(A/X,\mathfrak{F}) = n_\sigma(A,\mathfrak{F}) - 1$ which yields the desired contradiction. $\square$

## 3. Applications

As it was said in the introduction, the $\mathfrak{F}$-length $n_{\mathfrak{F}}(G)$ of a group $G$ which is defined in [6] is just the $\sigma$-$\mathfrak{F}$-length of $G$ for $\sigma$ the partition of $\mathbb{P}$ containing exactly one prime each, and the $\sigma$-nilpotent length of $G$ is just the $\sigma$-$\mathfrak{F}$-length of $G$ for $\mathfrak{F} = \{1\}$.

Therefore the following results are direct consequences of our Theorem A.

**Corollary 1.** *If $A$ is a maximal subgroup of a $\sigma$-soluble group $G$, then $l_\sigma(A) = l_\sigma(G) - i$ for some $i \in \{0,1,2\}$.*

**Corollary 2** ([6] (Theorem 1))**.** *If $A$ is a maximal subgroup of a soluble group $G$ and $\mathfrak{F}$ is a saturated formation, then $n_{\mathfrak{F}}(A) = n_{\mathfrak{F}}(G) - i$ for some $i \in \{0,1,2\}$.*

**Corollary 3** ([7] (Satz 1))**.** *If $A$ is a maximal subgroup of a soluble group $G$, then $l(A) = l(G) - i$ for some $i \in \{0,1,2\}$.*

## 4. An Example

In [6], some examples showing that each case of Corollary 2 is possible for the partition $\sigma$ of $\mathbb{P}$ containing exactly one prime each. We give an example of slight different nature.

**Example 1.** *Assume that $\sigma = \{\{2,3,5,7\}, \{211\}, \{2,3,5,7,211\}'\}$. Let $X$ be a cyclic group of order 7 and let $Y$ be an irreducible and faithful $X$-module over the finite field of 211 elements. Applying [5] (Chapter B, Theorem (9.8)), $Y$ is a cyclic group of order 211. Let $L = [Y]X$ be the corresponding semidirect product. Consider now $G = A_5 \wr L$ the regular wreath product of $A_5$, the alternating group of degree 5, with $L$. Then $F_\sigma(G) = A_5^*$, the base group of $G$. Then $l_\sigma(G) = 3$. Let $A_1 = A_5^* X$. Then $A_1$ is a maximal subgroup of $G$ and $l_\sigma(A_1) = 1$. Let $A_2 = A_5^* Y$. Then $A_2$ is a maximal subgroup of $G$ and $l_\sigma(A_2) = 2$.*

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