mathematics

MDPI

Article

Solution of Euler's Differential Equation in Terms of Distribution Theory and Fractional Calculus

Tohru Morita [1,*] **and Ken-ichi Sato** [2]

[1] Graduate School of Information Sciences, Tohoku University, Sendai 980-8577, Japan
[2] Kurume Library on Mathematics, Koriyama 963-8846, Japan; kensatokurume@ybb.ne.jp
* Correspondence: senmm@jcom.home.ne.jp; Tel.: +81-22-278-6186

Received: 3 November 2020; Accepted: 24 November 2020; Published: 26 November 2020

check for updates

Abstract: For Euler's differential equation of order n, a theorem is presented to give n solutions, by modifying a theorem given in a recent paper of the present authors in J. Adv. Math. Comput. Sci. 2018; 28(3): 1–15, and then the corresponding theorem in distribution theory is given. The latter theorem is compared with recent studies on Euler's differential equation in distribution theory. A supplementary argument is provided on the solutions expressed by nonregular distributions, on the basis of nonstandard analysis and Laplace transform.

Keywords: linear differential equations with polynomial coefficients; Euler's differential equation; distribution theory; fractional calculus; nonstandard analysis; Laplace transform

1. Introduction

In the preceding papers [1,2], linear differential equations of order $n \in \mathbb{Z}_{>0}$, with polynomial coefficients, are studied. They take the form:

$$\sum_{k=0}^{n}\sum_{m=0}^{\infty} a_{k,m}t^m \frac{d^k}{dt^k}u(t) = \sum_{k=0}^{n}(a_{k,0} + a_{k,1}\cdot t + a_{k,2}\cdot t^2 + a_{k,3}\cdot t^3 + \cdots)\cdot \frac{d^k}{dt^k}u(t) = 0, \quad t > 0, \tag{1}$$

where $t \in \mathbb{R}$, and $a_{k,m} \in \mathbb{C}$ for $k \in \mathbb{Z}_{[0,n]}$ and $m \in \mathbb{Z}_{>-1}$ are constants. We assume that a finite number of the constants are nonzero.

Here, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all integers, all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{m \in \mathbb{Z} | m > a\}$ for $a \in \mathbb{Z}$, and $\mathbb{Z}_{[a,b]} = \{m \in \mathbb{Z} | a \le m \le b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{Z}_{<b} = \{m \in \mathbb{Z} | m < b\}$ for $b \in \mathbb{Z}$, $\mathbb{R}_{>r} = \{x \in \mathbb{R} | x > r\}$ for $r \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$.

We use $(z)_k$ and $(z)^-_k$ for $z \in \mathbb{C}$, $k \in \mathbb{Z}_{>-1}$, which denote $(z)_k = \prod_{m=0}^{k-1}(z+m)$ if $k \in \mathbb{Z}_{>0}$, and $(z)_0 = 1$, as usual, and

$$(z)^-_k = \prod_{m=0}^{k-1}(z-m) = (-1)^k(-z)_k, \ \ k \in \mathbb{Z}_{>0}, \tag{2}$$

and $(z)^-_0 = 1$.

We reassemble the terms of Equation (1) as

$$\sum_{l=-\infty}^{n} D^l_t u(t) = 0, \quad t > 0, \tag{3}$$

where

$$D_t^l u(t) = \sum_{k=\max\{0,l\}}^{n} a_{k,k-l} \cdot t^{k-l} \frac{d^k}{dt^k} u(t). \tag{4}$$

We call $D_t^l u(t)$ a block of classified terms. When $n = 2$, Equation (3) is expressed as

$$D_t^2 u(t) + D_t^1 u(t) + D_t^0 u(t) + D_t^{-1} u(t) + D_t^{-2} u(t) + \cdots = 0, \quad t > 0, \tag{5}$$

where

$$\begin{aligned} &D_t^2 = a_{2,0} \frac{d^2}{dt^2}, \quad D_t^1 = a_{2,1} t \cdot \frac{d^2}{dt^2} + a_{1,0} \frac{d}{dt}, \\ &D_t^0 = a_{2,2} t^2 \cdot \frac{d^2}{dt^2} + a_{1,1} t \cdot \frac{d}{dt} + a_{0,0}, \quad D_t^{-1} = a_{1,2} t^2 \cdot \frac{d}{dt} + a_{0,1} t, \\ &D_t^{-2} = a_{2,4} t^4 \cdot \frac{d^2}{dt^2} + a_{1,3} t^3 \cdot \frac{d}{dt} + a_{0,2} t^2, \quad \cdots . \end{aligned} \tag{6}$$

In Reference [1,2], discussions are focused on the solution of Equation (3) with two blocks of classified terms. Kummer's and the hypergeometric differential equations are special examples of them.

Equation (3), which consists of only one block of classified terms for $l = 0$, is expressed as

$$D_t^0 u(t) := \sum_{k=0}^{n} a_k \cdot t^k \frac{d^k}{dt^k} u(t) = 0, \quad t > 0, \tag{7}$$

where $n \in \mathbb{Z}_{>0}$ and a_k, which represent $a_{k,k}$ in Equation (6), are constants, among which $a_n \neq 0$. This equation is called Euler's differential equation (Section 6.3, Reference [3]), (Chapter II, Section 7, Reference [4]). In recent papers [5–7], the solution of the equation in distribution theory, which corresponds to Equation (7), is discussed. In Reference [5], special attention is focused to the cases where the coefficients satisfy $a_k = 1$ for $k \in \mathbb{Z}_{[1,n]}$, and $a_0 \in \mathbb{Z}$. In Reference [1], a theorem is given on the solution of Equation (7). It is the purpose of the present paper to present a theorem which provides n solutions of Equation (7), by modifying Theorem 1.1 given in Reference [1], and then the corresponding theorem in distribution theory. It is shown that the results in Reference [5–7] are obtained as special results of that theorem given in Section 4.

In the Appendix A, a theorem is presented to show that there exist n and only n complementary solutions of a linear differential equation of order n, with constant coefficients, in terms of distribution theory. It guarantees the corresponding theorem on (7).

In Section 2, we present three theorems on the solution of Equation (7), two of which are related with the theorems given in Reference [5–7]. In Section 4, we give the corresponding theorems on the solution of the corresponding differential equation in distribution theory. In Section 3, formulas in distribution theory, are presented, which are used in Section 4. In Section 5, an argument is given to show a relation of the solutions of Equation (7), and the solutions expressed by nonregular distributions, of the corresponding equation in distribution theory, on the basis of nonstandard analysis. In Section 5.1, a brief discussion is given on the Laplace transform of Euler's differential equation.

2. Theorems on the Solution of Euler's Differential Equation

When D_t^0 given in (7) is operated to t^α for $\alpha \in \mathbb{C}$, we have

$$D_t^0 t^\alpha = A_0(\alpha) t^\alpha, \tag{8}$$

where

$$A_0(\alpha) := \sum_{k=0}^{n} a_k \cdot (\alpha)_k^- . \tag{9}$$

By modifying Theorem 1.1 given in Reference [1], we obtain

Theorem 1. *Let $A_0(\alpha)$ for D_t^0 in Equation (7) be given by (9). Then, $A_0(\alpha)$ is a polynomial of degree n. Let $k_x \in \mathbb{Z}_{>0}$ be the total number of distinct roots of $A_0(\alpha) = 0$, which are α_k for $k \in \mathbb{Z}_{[0,k_x]}$. Then, $A_0(\alpha)$ is expressed as*

$$A_0(\alpha) = a_n \prod_{k=1}^{k_x} (\alpha - \alpha_k)^{m_k}, \tag{10}$$

where $m_k \in \mathbb{Z}_{>0}$ for $k \in \mathbb{Z}_{[1,k_x]}$ satisfy $\sum_{k=1}^{k_x} m_k = n$, and we have k_x series of solutions of (7). In the kth series, if $m_k = 1$, we have one solution given by t^{α_k}, and if $m_k \geq 2$, we have m_k solutions given by

$$t^{\alpha_k},\ t^{\alpha_k} \log_e t,\ \cdots,\ t^{\alpha_k} (\log_e t)^{m_k - 1}. \tag{11}$$

Remark 1. *In Reference [1], Section 6.3 in Reference [3], and Chapter II, Section 7, in Reference [4], Equation (7) is reduced to a linear differential equation with constant coefficients, that is*

$$\prod_{k=1}^{k_x} (\frac{d}{dx} - \alpha_k)^{m_k} y(x) = 0, \tag{12}$$

by the change of variable from t to $x = \log_e t$. In the Appendix A, an argument is given to show that there exist n complementary solutions of a linear differential equation of order n, with constant coefficients, in terms of distribution theory.

Remark 2. *By the change of variable from back x to $t = e^x$ in (12), we obtain*

$$\prod_{k=1}^{k_x} (t\frac{d}{dt} - \alpha_k)^{m_k} u(t) = D_t^0 u(t) = 0, \tag{13}$$

which has the n solutions given in Theorem 1.

Lemma 1. *Let $p \in \mathbb{C}$. Then, $u(t) = t^p$ is a solution of the following equation:*

$$(t\frac{d}{dt} - p)u(t) = 0, \quad t > 0. \tag{14}$$

Lemma 2. *Let $p \in \mathbb{C}$, $m \in \mathbb{Z}_{>1}$, and*

$$(t\frac{d}{dt} - p)^m u(t) = 0, \quad t > 0. \tag{15}$$

Then, we have m solutions of (15), which are given by

$$t^p, t^p \log_e t,\ t^p (\log_e t)^2, \cdots, t^p (\log_e t)^{m-1}. \tag{16}$$

Proof. Lemma 1 shows that t^p is a solution of (15). We note that, if $l \in \mathbb{Z}_{>0}$,

$$(t\frac{d}{dt} - p)[t^p (\log_e t)^l] = l \cdot t^p (\log_e t)^{l-1}. \tag{17}$$

By applying this formula l times and then using Lemma 1, for $l \in \mathbb{Z}_{[1,m-1]}$, we obtain

$$(t\frac{d}{dt}-p)^m[t^p(\log_e t)^l] = l(t\frac{d}{dt}-p)^{m-1}[t^p(\log_e t)^{l-1}] = \cdots = l!(t\frac{d}{dt}-p)^{m-l}t^p = 0, \tag{18}$$

which shows that we have $m-1$ solutions of (15), which involve $\log_e t$ and are given in (16). $\square$

Poof of Theorem 1. Noting that any pair of k_x operators $(t\frac{d}{dt}-\alpha_k)^{m_k}$ can be exchanged with each other, with the aid of Lemmas 1 and 2, we confirm that the n solutions given in Theorem 1 are the solutions of Equation (13). $\square$

Example 1. *Let $p \in \mathbb{C}$ and $q \in \mathbb{C}$. We then consider*

$$\begin{aligned} D_t^0 u(t) &:= (t^2 \cdot \frac{d^2}{dt^2} - (p+q-1)t \cdot \frac{d}{dt} + pq)u(t) \\ &= (t\frac{d}{dt}-p)(t\frac{d}{dt}-q)u(t) = 0, \quad t>0. \end{aligned} \tag{19}$$

If $u(t) = t^\alpha$, $D_t^0 u(t) = A_0(\alpha)u(t) = (\alpha(\alpha-1) - (p+q-1)\alpha + pq)u(t) = (\alpha-p)(\alpha-q)u(t)$; hence, we have two solutions of (19), given by $u(t) = t^p$ and $u(t) = t^q$ if $p \neq q$. If $p = q$, Lemma 2 shows that the solutions are $u(t) = t^p$ and $u(t) = t^p \log_e t$. If $p = 1$ and $q = 0$, the solutions of $t^2\frac{d^2}{dt^2}u(t) = 0$ are $u(t) = t$ and $u(t) = 1$.

Remark 3. *In Reference [8,9], Nishimoto developed a method of solving Kummer's and the hypergeometric differential equation, with the aid of the Liouville fractional derivative. In the method, the problem of solving a differential equation of the second order is reduced the one of solving an equation of the first order. Two examples of using the method with the aid of Riemann-Liouville fractional derivative are given in Reference [10,11]. That method is called the Euler method in Reference [12]. In Reference [13], Nishimoto applied his method to the solution of Euler's Equation (19) with p and q replaced by $-\nu$ and $1+a-\nu$. He obtained one of the solutions, which is $t^{1+a-\nu}$ multiplied by a constant. He also gave a solution of the inhomogeneous equation:*

$$(t^2 \cdot \frac{d^2}{dt^2} - (p+q-1)t \cdot \frac{d}{dt} + pq)u(t) = f(t), \quad t>0. \tag{20}$$

We notethat, if $\alpha \neq p$, $\alpha \neq q$ and $f(t) = t^\alpha$, a particular solution of (20) is $u(t) = \frac{1}{(\alpha-p)(\alpha-q)}t^\alpha$.

Here, we present, theorems which correspond to Theorem 3.1 given in Reference [5] and Theorems 1 and 2 given in [7] in distribution theory.

Theorem 2. *Let $p \in \mathbb{C}$ and $a_k \in \mathbb{C}$ for $k \in \mathbb{Z}_{[1,n]}$ in Equation (7) be given, and then a_0 in Equation (7) be chosen to be $a_0 = -A_1(p)$, where*

$$A_1(p) := \sum_{k=1}^{n} a_k \cdot (p)_k^-. \tag{21}$$

Then, t^p is a solution of (7).

Proof. In this case, $A_0(p) = A_1(p) + a_0 = 0$ by (9); hence, Theorem 1 guarantees that t^p is a solution of (7). $\square$

Example 2. *Let $n = 2$, $a_2 = a_1 = 1$, for which $A_1(\alpha) = \alpha(\alpha-1) + \alpha = \alpha^2$, and $p \in \mathbb{C}$. We choose $a_0 = -A_1(p) = -p^2$. Then, Equation (7) becomes*

$$D_t^0 u(t) := (t^2 \cdot \frac{d^2}{dt^2} + t \cdot \frac{d}{dt} - p^2)u(t) = (t\frac{d}{dt}-p)(t\frac{d}{dt}+p)u(t) = 0, \quad t>0, \tag{22}$$

and Theorem 1 gives two solutions t^p and t^{-p} of Equation (22) if $p \neq 0$, and $u(t) = 1$ and $\log_e t$ if $p = 0$.

Theorem 3. *Let the condition of Theorem 1 be satisfied. Then, we have k_x solutions expressed by t^{α_k}.*

In Section 4, the theorems in distribution theory, which correspond to Theorems 1–3, are given. Among these theorems, those corresponding to Theorems 2 and 3, are due to Reference [5,7].

3. Preliminaries on Distribution Theory

Distributions in the space $\mathcal{D}'$ are first introduced in Reference [14–17]. The distributions are either regular ones or their derivatives. A regular distribution in $\mathcal{D}'$ corresponds to a function $f(t)$ which is locally integrable on $\mathbb{R}$. We denote the distribution by $\tilde{f}(t)$.

A distribution $\tilde{u} \in \mathcal{D}'$ is a functional, to which $\langle \tilde{u}, \phi \rangle \in \mathbb{C}$ is associated with every $\phi \in \mathcal{D}$, where $\mathcal{D}$, that is dual to $\mathcal{D}'$, is the space of testing functions, which are infinitely differentiable and have a compact support on $\mathbb{R}$.

If $\tilde{f} \in \mathcal{D}'$ is a regular distribution, we have

$$\langle \tilde{f}, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt. \tag{23}$$

Operator$D$ is so defined that $\langle D\tilde{u}, \phi \rangle = \langle \tilde{u}, D_W \phi \rangle$ for $\tilde{u} \in \mathcal{D}'$, where $D_W = -\frac{d}{dt}$. Because of this definition of D, we can confirm the following lemma.

Lemma 3. *Let $\tilde{f}$ and $\tilde{g}$ be regular distributions in $\mathcal{D}'$, which correspond to $f(t)$ and $g(t) = \frac{d}{dt} f(t)$, respectively. Then, $\tilde{g} = D\tilde{f}$.*

Proof. In this condition, we have

$$\langle D\tilde{f}, \phi \rangle = \langle \tilde{f}, D_W \phi \rangle = -\int_{-\infty}^{\infty} f(t)[\frac{d}{dt}\phi(t)]dt = \int_{-\infty}^{\infty} [\frac{d}{dt} f(t)]\phi(t)dt = \langle \tilde{g}, \phi \rangle. \tag{24}$$

□

If $\tilde{u}_k \in \mathcal{D}'$ is not a regular one, it is expressed as $\tilde{u}_k(t) = D^k \tilde{f}(t)$, by $k \in \mathbb{Z}_{>0}$ and a regular distribution $\tilde{f} \in \mathcal{D}'$, and then we have

$$\langle \tilde{u}_k, \phi \rangle = \langle D^k \tilde{f}, \phi \rangle = \langle \tilde{f}, D_W^k \phi \rangle = \int_{-\infty}^{\infty} f(t)[D_W^k \phi(t)]dt. \tag{25}$$

If $\tilde{g} = D\tilde{f}$, operator D^{-1} is so defined that $\tilde{f} = D^{-1}\tilde{g}$; hence, if $D\tilde{f}$ exists, $\tilde{f} = D^{-1}D\tilde{f}$.

Lemma 4. *Let the condition in Lemma 3 be satisfied. Then, $\tilde{f} = D^{-1}\tilde{g}$, which corresponds to $f(t) = \int_{-\infty}^{t} g(x)dx = \int_{-\infty}^{t} \frac{df(x)}{dx} dx$.*

Lemma 5. *Let $\tilde{u} \in \mathcal{D}'$ and $m \in \mathbb{Z}$. Then,*

$$tD^m \tilde{u}(t) = D^m[t\tilde{u}(t)] - mD^{m-1}\tilde{u}(t). \tag{26}$$

In particular, when $m = 1$,

$$tD\tilde{u}(t) = D[t\tilde{u}(t)] - \tilde{u}(t). \tag{27}$$

Proof. (i) We first give a proof for the case of $m \in \mathbb{Z}_{>0}$, in terms of mathematical induction. When $m = 1$, we have

$$\langle tD\tilde{u}(t), \phi(t)\rangle = \langle \tilde{u}(t), D_W[t\phi(t)]\rangle = \langle \tilde{u}(t), -\phi(t) + tD_W\phi(t)\rangle = \langle -\tilde{u}(t) + D[t\tilde{u}(t)], \phi(t)\rangle, \quad (28)$$

which gives (27). If (26) holds for a value of $m \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} tD^{m+1}\tilde{u}(t) &= tD^m[D\tilde{u}(t)] = D^m[tD\tilde{u}(t)] - mD^{m-1}[D\tilde{u}(t)] \\ &= D^m[D[t\tilde{u}(t)] - \tilde{u}(t)] - mD^m\tilde{u}(t) = D^{m+1}[t\tilde{u}(t)] - (m+1)D^m\tilde{u}(t), \end{aligned}$$

which shows that (26) holds also for $m + 1$. (ii) We now assume $m \in \mathbb{Z}_{>0}$. We apply D^{-m} to Equation (26), and put $\tilde{v}(t) = D^m\tilde{u}(t)$ or $\tilde{u}(t) = D^{-m}\tilde{v}(t)$ in (26). We then obtain

$$D^{-m}[t\tilde{v}(t)] = tD^{-m}\tilde{v}(t) - mD^{-m-1}\tilde{v}(t). \quad (29)$$

This shows that (26) holds even when "$m > 0$" and $\tilde{u}(t)$ are replaced by "$-m < 0$" and $\tilde{v}(t) \in \mathcal{D}'$, respectively. □

Lemma 6. *Let $\tilde{u} \in \mathcal{D}'$ and $k \in \mathbb{Z}_{>1}$. Then,*

$$t^kD^k\tilde{u}(t) = (tD)(tD-1)\cdots(tD-(k-1))\tilde{u}(t) = (tD)_k^-\tilde{u}(t), \quad (30)$$

where in the last member, the notation defined by (2) is used for operator $z = tD$.

Proof. A proof by mathematical induction is given. When $k = 2$, by using Equation (27) in Lemma 5, we have

$$t^2D^2\tilde{u}(t) = t[tD[D\tilde{u}(t)]] = t[Dt[D\tilde{u}(t)] - D\tilde{u}(t)] = tD(tD-1)\tilde{u}(t), \quad (31)$$

which shows that (30) holds for $k = 2$. If (30) holds for a value of $k \in \mathbb{Z}_{>1}$ and for $k = 2$, by using Lemma 5, we have

$$\begin{aligned} t^{k+1}D^{k+1}\tilde{u}(t) &= t^k[tD^k[D\tilde{u}(t)]] = t^k[D^kt[D\tilde{u}(t)] - kD^{k-1}[D\tilde{u}(t)]] \\ &= t^kD^k[(tD-k)\tilde{u}(t)] = (tD)(tD-1)\cdots(tD-(k-1))(tD-k)\tilde{u}(t), \end{aligned}$$

which shows that (30) holds also for $k + 1$. □

Lemma 7. *Let $m \in \mathbb{Z}$ and $\tilde{v}(t) \in \mathcal{D}'_R$. Then,*

$$D^{-m}(tD)D^m\tilde{v}(t) = (tD-m)\tilde{v}(t). \quad (32)$$

Proof. By applying D^{-m} to Equation (26) and then replacing $\tilde{u}$ by $D\tilde{v}$, we obtain (32). □

3.1. Distributions in the Space $\mathcal{D}'_R$

We now consider the space of distributions $\mathcal{D}'_R$, which is a subspace of $\mathcal{D}'$. A regular distribution in $\mathcal{D}'_R$ is such a distribution that it corresponds to a function which is locally integrable on $\mathbb{R}$ and has a support bounded on the left. The space $\mathcal{D}_R$, that is dual to $\mathcal{D}'_R$, is the space of testing functions, which are infinitely differentiable on $\mathbb{R}$ and have a support bounded on the right.

The Heaviside step function $H(t)$ is such that $H(t) = 0$ for $t \leq 0$, and $H(t) = 1$ for $t > 0$. The corresponding distribution $\tilde{H}(t)$ is a regular distribution in the space $\mathcal{D}'$, as well as in $\mathcal{D}'_R$. Dirac's function $\delta(t)$ is the distribution, which is defined by $\delta(t) = D\tilde{H}(t)$.

In Reference [18,19], the solutions of special cases of Equation (3) or (1) were studied with the aid of Riemann-Liouville fractional integral and derivative, distribution theory and the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation. In the study, $g_\nu(t)$ for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ is defined by

$$g_\nu(t) := \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{33}$$

where $\Gamma(\nu)$ is the gamma function, and the following condition was adopted.

Condition 1. *$u(t)$ and $f(t)$ in (3) are expressed as a linear combination of $g_\nu(t)$ for $t > 0$ and $\nu \in S$, where S is a set of $\nu \in \mathbb{R}_{>-M}\backslash\mathbb{Z}_{<1}$ for some $M \in \mathbb{Z}_{>-1}$.*

As a consequence, $u(t)$ is expressed as follows:

$$u(t) = \sum_{\nu \in S} u_{\nu-1} \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{34}$$

where $u_{\nu-1} \in \mathbb{C}$ are constants. Because of this condition, obtained solutions are expressed by a power series of t multiplied by a power t^α:

$$u(t) = t^\alpha \sum_{k=0}^{\infty} p_k t^k, \tag{35}$$

where $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{<0}$, $p_k \in \mathbb{C}$ and $p_0 \neq 0$.

A basic method of solving Equation (1) is to assume the solution in the form (35) with $\alpha \notin \mathbb{Z}_{<0}$. The solution is obtained by determining the coefficients p_k recursively; see e.g., Section 10.3 in Reference [20].

In the space $\mathcal{D}'_R$, we define regular distribution $\tilde{g}_\nu(t) = g_\nu(t)\tilde{H}(t)$ for $\nu \in \mathbb{C}_+$, which corresponds to function $g_\nu(t)H(t)$, and then define operator of fractional integral and derivative D^β and distribution $\tilde{g}_{\nu-\beta}(t)$ for $\beta \in \mathbb{C}$ such that

$$D^\beta \tilde{g}_\nu(t) = D^{\beta-\nu}\delta(t) = \tilde{g}_{\nu-\beta}(t) = \begin{cases} \tilde{g}_{\nu-\beta}(t), & \nu - \beta \in \mathbb{C}, \\ D^k\delta(t), & k = \beta - \nu \in \mathbb{Z}_{>-1}. \end{cases} \tag{36}$$

Lemma 8. *$D^{-\nu-1}\delta(t)$ for $\nu \in \mathbb{C}$ represents*

$$D^{-\nu}\tilde{H}(t) = D^{-\nu-1}\delta(t) = \tilde{g}_{\nu+1}(t) = \begin{cases} \tilde{g}_{\nu+1}(t), & \nu \in \mathbb{C}, \\ \frac{1}{\Gamma(\nu+1)} t^\nu \tilde{H}(t), & \nu + 1 \in \mathbb{C}_+, \\ \delta^{(\mu-1)}(t) = D^\mu \tilde{H}(t), & \mu = -\nu \in \mathbb{Z}_{>0}. \end{cases} \tag{37}$$

Proof. The equation for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ is due to (36) for $\beta = -1$. □

3.2. Distributions in the Space $\mathcal{D}'_R$, as well as in the Space $\mathcal{D}'$

Remark 4. *When $\nu + 1 \in \mathbb{C}_+$, $D^{-\nu-1}\delta(t)$ is a regular distribution, and when $\nu + 1 \in \mathbb{C}\backslash\mathbb{C}_+$, it is a non-regular distribution, which are given by*

$$D^{-\nu-1}\delta(t) = \tilde{g}_{\nu+1}(t) = \begin{cases} g_{\nu+1}(t)\tilde{H}(t) = \frac{1}{\Gamma(\nu+1)} t^\nu \tilde{H}(t), & \nu + 1 \in \mathbb{C}_+, \\ D^\mu[D^{-\lambda-1}\delta(t)] = D^\mu[g_{\lambda+1}(t)\tilde{H}(t)], & \nu + 1 \in \mathbb{C}\backslash\mathbb{C}_+, \\ \delta^{(\mu-1)}(t) = D^\mu \tilde{H}(t), & \mu = -\nu \in \mathbb{Z}_{>0}, \end{cases} \tag{38}$$

where $\mu \in \mathbb{Z}_{>0}$ and $\lambda = \nu + \mu$ for which $\lambda + 1 \in \mathbb{C}_+$.

Remark 5. *In (38), $D^{-\nu-1}\delta(t)$ for $\nu+1 \in \mathbb{C}\backslash\mathbb{C}_+$ satisfying $\nu \notin \mathbb{Z}$, which is not a regular distribution, is represented by $D^{\mu}[g_{\lambda+1}(t)\tilde{H}(t)]$, in accordance with the definition.*

Lemma 9. *Let $\nu \in \mathbb{C}$. Then,*

$$tD^{-\nu-1}\delta(t) = (\nu+1)D^{-\nu-2}\delta(t). \tag{39}$$

Proof. If $\nu+1 \in \mathbb{C}_+$, by using (37) or (38), we have

$$tD^{-\nu-1}\delta(t) = tg_{\nu+1}(t)\tilde{H}(t) = \frac{1}{\Gamma(\nu+1)}t^{\nu+1}\tilde{H}(t) = (\nu+1)D^{-\nu-2}\delta(t). \tag{40}$$

If $\nu+1 \in \mathbb{C}\backslash\mathbb{C}_+$, we adopt (38), and then by using Lemma 5 and also Equation (40), we have

$$\begin{aligned} tD^{-\nu-1}\delta(t) &= tD^{\mu}[D^{-\lambda-1}\delta(t)] = D^{\mu}[tD^{-\lambda-1}\delta(t)] - \mu D^{\mu-1}[D^{-\lambda-1}\delta(t)] \\ &= (\lambda+1-\mu)D^{-\nu-2}\delta(t) = (\nu+1)D^{-\nu-2}\delta(t), \end{aligned} \tag{41}$$

where $\mu \in \mathbb{Z}_{>0}$ and $\lambda = \nu+\mu$ for which $\lambda+1 \in \mathbb{C}_+$. This equation and Equation (40) show that (39) holds for all $\nu = \lambda-\mu \in \mathbb{C}$. □

Lemma 10. *Let $k \in \mathbb{Z}_{>0}$ and $\nu \in \mathbb{C}$. Then,*

$$t^kD^k[D^{-\nu-1}\delta(t)] = (\nu)_k^{-}D^{-\nu-1}\delta(t) = (-1)^k(-\nu)_kD^{-\nu-1}\delta(t). \tag{42}$$

In particular, when $k=1$, we have

$$tD[D^{-\nu-1}\delta(t)] = \nu D^{-\nu-1}\delta(t). \tag{43}$$

Proof. Lemma 9 shows that, if $\tilde{u}(t) = D^{-\nu-1}\delta(t)$, $tD\tilde{u}(t) = tD^{-\nu}\delta(t) = \nu D^{-\nu-1}\delta(t)$, which gives (43). By using (43) in (30), we obtain (42). □

3.3. Regular Distributions in the Space $\mathcal{D}'_R$

In the present paper, we study the solution of the equation which corresponds to Equation (7), in distribution theory. When function $u(t)H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$, we introduce $\tilde{u}(t) = u(t)\tilde{H}(t) \in \mathcal{D}'_R$.

Lemma 11. *Let $l \in \mathbb{Z}_{>0}$, and function $u(t)$ be such that $\frac{d^k}{dt^k}u(t) \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$ for $k \in \mathbb{Z}_{[0,l]}$. Then,*

$$\frac{d^l}{dt^l}u(t) \cdot \tilde{H}(t) = D^l\tilde{u}(t) - u^{(l-1)}(0)\delta(t) - u^{(l-2)}(0)\delta'(t) - \cdots - u(0)\delta^{(l-1)}(t). \tag{44}$$

Proof. When $l=1$, we have

$$\begin{aligned} &\langle \frac{d}{dt}u(t) \cdot \tilde{H}(t), \phi(t)\rangle = \int_{-\infty}^{\infty} \frac{d}{dt}u(t) \cdot H(t)\phi(t)dt = \int_0^{\infty} \frac{d}{dt}u(t) \cdot \phi(t)dt \\ &= \int_0^{\infty} \frac{d}{dt}[u(t)\phi(t)]dt - \int_0^{\infty} u(t) \cdot \frac{d}{dt}\phi(t)dt = -u(0)\phi(0) - \int_{-\infty}^{\infty} u(t)H(t) \cdot \frac{d}{dt}\phi(t)dt \\ &= -\langle u(0)\delta(t), \phi(t)\rangle + \langle D[u(t)\tilde{H}(t)], \phi(t)\rangle, \end{aligned} \tag{45}$$

which gives (44) for $l=1$. When $l=2$, by using it, we obtain

$$\begin{aligned} \frac{d}{dt}u'(t) \cdot \tilde{H}(t) &= D[u'(t)\tilde{H}(t)] - u'(0)\delta(t) = D[D[u(t)\tilde{H}(t)] - u(0)\delta(t)] - u'(0)\delta(t) \\ &= D^2[u(t)\tilde{H}(t)] - u(0)\delta'(t) - u'(0)\delta(t), \end{aligned} \tag{46}$$

which gives (44) for $l = 2$. Equation (44) can be proved by mathematical induction. □

Lemma 12. *Let $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$ satisfy $0 \le k < l$. Then,*

$$t^l \delta^{(k)}(t) = 0. \tag{47}$$

Proof. In this case, we have

$$\langle t^l \delta^{(k)}(t), \phi(t)\rangle = (-1)^k \langle \delta(t), \frac{d^k}{dt^k}[t^l \phi(t)]\rangle = 0. \tag{48}$$

□

Lemma 13. *Let the condition of Lemma 11 be satisfied. Then,*

$$t^l \frac{d^l}{dt^l} u(t) \cdot \tilde{H}(t) = t^l D^l \tilde{u}(t). \tag{49}$$

Proof. By multiplying t^l to Equation (44) and then using (47), we obtain (49). □

Lemma 14. *Let $n \in \mathbb{Z}_{>0}$, function $u(t)$ be such that $\frac{d^k}{dt^k} u(t) \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$ for $k \in \mathbb{Z}_{[0,n]}$, and $\tilde{u}(t) = u(t)\tilde{H}(t)$. Then,*

$$\sum_{k=0}^{n} a_k \cdot t^k \frac{d^k}{dt^k} u(t) \cdot \tilde{H}(t) = \sum_{k=0}^{n} a_k \cdot t^k D^k \tilde{u}(t), \tag{50}$$

where a_k are constants.

Lemma 15. *Let $p \in \mathbb{C}$, function $u(t)$ be such that $u(t) \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$, $\tilde{u}(t) = u(t)\tilde{H}(t)$, and $t\frac{d}{dt}u(t) \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$. Then,*

$$(t\frac{d}{dt} - p)u(t) \cdot \tilde{H}(t) = (tD - p)\tilde{u}(t), \tag{51}$$

where the condition $t\frac{d}{dt}u(t) \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$ may be replaced by $\frac{d}{dt}[tu(t)] \cdot H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$, since

$$t\frac{d}{dt}u(t) = \frac{d}{dt}[tu(t)] - u(t). \tag{52}$$

4. Euler's Equation in the Space of Distributions $\mathcal{D}'_R$

Lemma 14 shows that the equation which corresponds to (7), in distribution theory, is

$$\tilde{D}^0_t \tilde{u}(t) := \sum_{k=0}^{n} a_k \cdot t^k D^k \tilde{u}(t) = 0. \tag{53}$$

Remark 6. *Let the l-th solution of (7) given in Theorem 1 be expressed by $u_l(t) := t^{\alpha_k}(\log_e t)^m$ for $m \in \mathbb{Z}_{>-1}$. Then, Lemma 14 shows that if $\mathrm{Re}\,\alpha_k > n-1$, $u_l(t)\tilde{H}(t)$ is a solution of (53).*

By using (30), (9), and then (10) with $a_n = 1$ in (53), we obtain

$$\tilde{D}^0_t \tilde{u}(t) = \sum_{k=0}^{n} a_k (tD)^-_k \tilde{u}(t) = A_0(tD)\tilde{u}(t) = \prod_{k=1}^{k_x} (tD - \alpha_k)^{m_k} \tilde{u}(t). \tag{54}$$

As a consequence, Equation (53) is expressed by

$$\tilde{D}_t^0 \tilde{u}(t) = \prod_{k=1}^{k_x} (tD - \alpha_k)^{m_k} \tilde{u}(t) = 0. \tag{55}$$

Lemma 16. *Let $\mu \in \mathbb{Z}$ and $\tilde{u}(t)$ be a solution of (55). Then, $\tilde{v}(t) = D^{-\mu}\tilde{u}(t)$ is a solution of*

$$\prod_{k=1}^{k_x} (tD - \alpha_k - \mu)^{m_k} \tilde{v}(t) = 0. \tag{56}$$

This shows that, if we choose μ such that $\mathrm{Re}\ \alpha_k + \mu > -1$ for all $k \in \mathbb{Z}_{[1,k_x]}$, $\tilde{u}(t) = D^{\mu}\tilde{v}(t)$ is expressed by the regular distribution $\tilde{v}(t)$.

Proof. This is confirmed with the aid of Lemma 7. □

Lemma 17. *Let $p \in \mathbb{C}$, $\tilde{u}(t)$ be the solution of*

$$(tD - p)\tilde{u}(t) = 0. \tag{57}$$

Equation (43) shows that $\tilde{u}(t) = D^{-p-1}\delta(t)$ is a solution of (57). Lemma 8 or Remark 4 shows that if $\mathrm{Re}\ p > -1$, the solution is given by $\tilde{u}(t) = t^p \tilde{H}(t)$. If $\mathrm{Re}\ p \le -1$, Lemma 16 shows that, if we choose μ such that $\mathrm{Re}\ p + \mu > -1$, the solution is given by $\tilde{u}(t) = D^{\mu}[t^{p+\mu}\tilde{H}(t)]$.

Remark 7. *An alternative proof of Lemma 17 is given for the case of $\mathrm{Re}\ p > -1$. Then, we put $u(t) = t^p$ and $\tilde{u}(t) = u(t)\tilde{H}(t)$, and, we confirm*

$$(tD - p)\tilde{u}(t) = ((t\frac{d}{dt} - p)u(t)) \cdot \tilde{H}(t) = 0. \tag{58}$$

The first and the second equalities are due to Lemmas 3 and 1, respectively.

Lemma 18. *Let $m \in \mathbb{Z}_{>1}$, $p \in \mathbb{C}$ and*

$$\tilde{D}_t^0 \tilde{u}(t) := (t \cdot D - p)^m \tilde{u}(t) = 0. \tag{59}$$

If $\mathrm{Re}\ p > -1$, we choose $\mu = 0$ or $\mu \in \mathbb{Z}_{>0}$, and, if $\mathrm{Re}\ p \le -1$, we choose $\mu \in \mathbb{Z}_{>0}$ which satisfies $\mu \ge -\lceil p \rceil$, where $\lceil p \rceil$ is the least integer which is not less than p. We then put $\lambda = p + \mu$, and obtain m solutions of (59) given by

$$D^{\mu}[D^{-\lambda-1}\delta(t)], D^{\mu}[(\log_e t)D^{-\lambda-1}\delta(t)], \cdots, D^{\mu}[(\log_e t)^{m-1}D^{-\lambda-1}\delta(t)]. \tag{60}$$

Proof. Lemma 17 shows that $\tilde{u}(t) = D^{-p-1}\delta(t)$ is a solution of (59). If $\mathrm{Re}\ p > -1$, $\mu = 0$ and $\lambda = p$, we note that, if $l > 0$, $D^{-p-1}\delta(t) = \frac{t^p}{\Gamma(p+1)}\tilde{H}(t)$, and

$$(tD - p)[t^p(\log_e t)^l \tilde{H}(t)] = (t\frac{d}{dt} - p)[t^p(\log_e t)^l] \cdot \tilde{H}(t) = l \cdot t^p(\log_e t)^{l-1}\tilde{H}(t). \tag{61}$$

By using this formula l times and Lemma 17 for $l \in \mathbb{Z}_{[1,m-1]}$, we obtain

$$(tD - p)^m[t^p(\log_e t)^l \tilde{H}(t)] = l!(tD - p)^{m-l}[t^p \tilde{H}(t)] = 0, \tag{62}$$

which shows that we have $m-1$ solutions of (59), which involve $\log_e t$ and are given in (60) for $\mu = 0$ and $\lambda = p$. If $\mu \in \mathbb{Z}_{>0}$, $\lambda = p + \mu$ satisfies $\mathrm{Re}\,\lambda > -1$, and then Lemma 16 shows that, if $\tilde{v}(t)$ is a solution of the following equation:

$$(t \cdot D - p - \mu)^m \tilde{v}(t) = 0, \tag{63}$$

$\tilde{u}(t) = D^{\mu}\tilde{v}(t)$ is a solution of (59). As a consequence of this fact, we obtain m solutions of (59) given by (60). □

The theorem which corresponds to Theorem 1 is as follows.

Theorem 4. *Let the condition of Theorem 1 be satisfied. Then, Equation (53) is expressed by (55), and we have k_x series of solutions of Equation (53). In the kth series, if $m_k = 1$, we have one solution given by $D^{-\alpha_k-1}\delta(t)$, and, if $m_k \geq 2$, we have m_k solutions given by*

$$D^{-\alpha_k-1}\delta(t),\ D^{\mu_k}[(\log_e t)D^{-\lambda_k-1}\delta(t)],\ \cdots, D^{\mu_k}[(\log_e t)^{m_k-1}D^{-\lambda_k-1}\delta(t)], \tag{64}$$

where $\mu_k \in \mathbb{Z}_{>-1}$ and $\lambda_k = \alpha_k + \mu_k$ satisfies $\mathrm{Re}\,\lambda_k > -1$, so that $(\log_e t)^l D^{-\lambda_k-1}\delta(t)$ for $l \in \mathbb{Z}_{1,m_k-1}$ are regular distributions. See Lemma 8 and Remark 4 for the expression $D^{-\alpha_k-1}\delta(t)$.

Remark 8. *If $\mathrm{Re}\,\alpha_k > -1$, we may choose $\mu_k = 0$ and $\lambda_k = \alpha_k$ in (64).*

Remark 9. *In (64), we may choose $\mu_k = \mu$ such that $\lambda_k = \alpha_k + \mu$ satisfies $\mathrm{Re}\,\lambda_k > -1$ for all $k \in \mathbb{Z}_{[1,k_x]}$.*

Poof of Theorem 4. (i) With the aid of Lemmas 17 and 18, we confirm that the n solutions given in Theorem 4 are the solutions of Equation (53) or (55). (ii) If $\mathrm{Re}\,\alpha_k > -1$ for all $k \in \mathbb{Z}_{[1,k_x]}$, we may choose $\mu_k = 0$ and $\lambda_k = \alpha_k$ for all k, and then this theorem is proved with the aid of Lemma 15 and Theorem 1. (iii) When we choose μ_k, as in Remark 9, this theorem is proved with the aid of Lemma 16 and Theorem 1. □

Example 3. *Let $n = 2$, $p \in \mathbb{C}$, $q \in \mathbb{C}$, and*

$$\begin{aligned}\tilde{D}_t^0\tilde{u}(t) &:= (t^2 \cdot D^2 - (p+q-1)t \cdot D + pq)\tilde{u}(t) \\ &= (tD - p)(tD - q)\tilde{u}(t) = 0,\end{aligned} \tag{65}$$

where the second equality is justified by Lemma 6. Then, if $\tilde{u}(t) = D^{-\alpha-1}\delta(t)$, $\tilde{D}_t^0\tilde{u}(t) = A_0(\alpha)\tilde{u}(t) = (\alpha(\alpha-1) - (p+q-1)\alpha + pq)\tilde{u}(t) = (\alpha - p)(\alpha - q)\tilde{u}(t)$; hence, we have two solutions of (65) given by $\tilde{u}(t) = D^{-p-1}\delta(t)$ and $\tilde{u}(t) = D^{-q-1}\delta(t)$ if $p \neq q$; see Lemma 8 or Remark 4 for their expressions. If $p = q$, we have two solutions $\tilde{u}(t) = D^{-p-1}\delta(t)$ and $\tilde{u}_1(t) := D^{\mu}[(\log_e t)D^{-\lambda-1}\delta(t)]$, as shown in Lemma 18.

Remark 10. *We now consider the inhomogeneous differential equation which corresponds to (65):*

$$(t^2 \cdot D^2 - (p+q-1)t \cdot D + pq)\tilde{u}(t) = \tilde{f}(t). \tag{66}$$

If $\alpha \neq p$, $\alpha \neq q$ and $\tilde{f}(t) = D^{-\alpha-1}\delta(t)$, a particular solution of this equation is $\frac{1}{(\alpha-p)(\alpha-q)}D^{-\alpha-1}\delta(t)$.

Example 4. *Let $n = 2$, $m \in \mathbb{Z}_{>-1}$ and*

$$\tilde{D}_t^0\tilde{u}(t) := (t^2 \cdot D^2 + t \cdot D - m^2)\tilde{u}(t) = (tD - m)(tD + m)\tilde{u}(t) = 0, \tag{67}$$

where the second equality is justified by Lemma 6. Equation (42) shows that, if $\tilde{u}(t) = D^{-\alpha-1}\delta(t)$, $D_t\tilde{u}(t) = A_0(\alpha)\tilde{u}(t) = (\alpha(\alpha-1) + \alpha - m^2)\tilde{u}(t) = (\alpha - m)(\alpha + m)\tilde{u}(t)$; hence, if $m > 0$, we have two solutions *of* (67)*, given by* $\tilde{u}(t) = D^{-m-1}\delta(t) = \tilde{g}_{m+1}(t) = \frac{1}{m!}t^m \cdot \tilde{H}(t)$ *and* $\tilde{u}(t) = D^{m-1}\delta(t) = \delta^{(m-1)}(t)$*. The case of* $m = 0$ *is Example* 3 *for* $p = q = 0$*, so that we have two solutions* $\tilde{H}(t)$ *and* $(\log_e t)\tilde{H}(t)$*.*

Here, we present a theorem which corresponds to Theorem 2, Theorem 3.1 given in Reference [5], and Theorems 1 and 2 given in Reference [7].

Theorem 5. *Let* $p \in \mathbb{C}$ *and* a_k *for* $k \in \mathbb{Z}_{[1,n]}$ *in Equation* (53) *be given, and then* a_0 *in Equation* (53) *be chosen to be* $a_0 = -A_1(p)$*, where* $A_1(p)$ *is given by* (21)*. Then,* $D^{-p-1}\delta(t)$ *is a solution of* (53)*. See Lemma* 8 *and Remark* 4 *for the expression* $D^{-p-1}\delta(t)$*.*

Proof. This theorem is proved with the aid of Theorem 4. □

Remark 11. *Equation* (67)*, for* $m = 1$ *and* $m = 3$*, is taken up in Reference [5]. When* $m = 1$*, we have two solutions* $t\tilde{H}(t)$ *and* $\delta(t)$*, but* $t\tilde{H}(t)$ *is not mentioned there, and when* $m = 3$*, we have two solutions* $\frac{1}{3!}t^3\tilde{H}(t)$ *and* $\delta''(t)$*, but* $\delta''(t)$ *is not mentioned there.*

Example 5. *Let* $n = 2$*,* $a_2 = a_1 = 1$*, for which* $A_1(\alpha) = \alpha(\alpha-1) + \alpha = \alpha^2$*, and* $m \in \mathbb{Z}_{>-1}$*. We choose* $a_0 = -A_1(m) = -m^2$*. Then, Equation* (53) *becomes* (67)*, and Theorem* 5 *gives two solutions given in Example* 4*.*

Here, we present a theorem which corresponds to Theorem 3 and Theorems 1 and 2 given in Reference [6].

Theorem 6. *Let the condition of Theorem* 1 *be satisfied. Then, we have* k_x *solutions of Equation* (53)*, which are expressed by* $D^{-\alpha_k-1}\delta(t)$*. See Lemma* 8 *and Remark* 4 *for the expression* $D^{-\alpha_k-1}\delta(t)$*.*

Remark 12. *In this theorem, when* $k_x \neq n$*, the solutions involving* $\log_e t$ *are not mentioned, among the* n *solutions given by* (64) *in Theorem* 4*. When* $k_x = n$*, we have all the solutions by this theorem. Such is the case for Theorem 2 given in Reference [6].*

We recall two examples from Reference [6].

Example 6. *We put* $n = 3$*,* $k_x = 1$*,* $m_1 = 3$ *and* $\alpha_1 = -2$ *in Theorem* 6*. Then, Equation* (53) *becomes*

$$(t^3D^3 + 9t^2D^2 + 19tD + 8)\tilde{u}(t) = (tD + 2)^3\tilde{u}(t) = 0. \tag{68}$$

We have three solutions $D^{2-1}\delta(t) = \delta'(t) = D^2\tilde{H}(t)$*,* $D^2[(\log_e t)\tilde{H}(t)]$ *and* $D^2[(\log_e t)^2\tilde{H}(t)]$*. In Reference [6], the last two are not mentioned.*

Example 7. *We put* $n = 3$*,* $k_x = 2$*,* $m_1 = 1$*,* $m_2 = 2$*,* $\alpha_1 = -4$ *and* $\alpha_2 = 2$ *in Theorem* 6*. Then, Equation* (53) *becomes*

$$(t^3D^3 + 3t^2D^2 - 11tD + 16)\tilde{u}(t) = (tD + 4)(tD - 2)^2\tilde{u}(t) = 0. \tag{69}$$

We have three solutions $D^{4-1}\delta(t) = \delta^{(3)}(t)$*,* $D^{-2-1}\delta(t) = \frac{t^2}{2!}\tilde{H}(t)$ *and* $(\log_e t)\frac{t^2}{2!}\tilde{H}(t)$*. In Reference [6], the last one is not mentioned.*

5. Euler's Equation Studied in Nonstandard Analysis

We first consider the solution of simple Euler's equation:

$$(t\frac{d}{dt} - \mu)u(t) = 0, \quad t > 0, \ \ \mu \in \mathbb{C}\backslash\mathbb{Z}_{<0}. \tag{70}$$

We denote the solution of (70) by $u_\mu(t)$. For $\mu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$, we adopt the solution:

$$u_\mu(t) = \frac{t^\mu}{\Gamma(\mu+1)}. \tag{71}$$

Then, we note that they are related by $u_{\mu-1}(t) = \frac{d}{dt}u_\mu(t)$ for $\mu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $\frac{d}{dt}u_0(t) = 0$.

In nonstandard analysis [21], we consider

$$(t\frac{d}{dt} - \mu - \epsilon)u(t) = 0, \quad t > 0, \ \mu \in \mathbb{Z}, \tag{72}$$

in place of (70), where ϵ is an infinitesimal number. When $\mu = 0$, the solution of (72) is

$$u_\epsilon(t) = \frac{t^\epsilon}{\Gamma(\epsilon+1)} = \frac{1 + \epsilon \log_e t + O(\epsilon^2)}{1 - \gamma\epsilon + O(\epsilon^2)}, \tag{73}$$

where $\gamma = -\Gamma'(1) = 0.5772\cdots$ is Euler's constant. From this, we have

$$u_{\epsilon-1}(t) = \frac{d}{dt}u_\epsilon(t) = \epsilon\frac{1}{t} + O(\epsilon^2), \tag{74}$$

$$u_{\epsilon-n}(t) = \frac{d^n}{dt^n}u_\epsilon(t) = \epsilon(-1)^{n-1}(n-1)!\frac{1}{t^n} + O(\epsilon^2), \quad n \in \mathbb{Z}_{>0}. \tag{75}$$

When $\mu = \epsilon - n$, (71) is expressed by

$$u_{\epsilon-n}(t) = \frac{t^{\epsilon-n}}{\Gamma(\epsilon-n+1)} = \frac{1}{\pi}(-1)^{n-1}\sin(\epsilon\pi)\cdot\Gamma(n-\epsilon)t^{-n+\epsilon}. \tag{76}$$

The leading term of $O(\epsilon)$ of this expression gives (75), since $\sin(\epsilon\pi) = \epsilon\pi + O(\epsilon^2)$.

We next consider

$$(t\frac{d}{dt} - \mu)^m u(t) = 0, \quad t > 0, \ m \in \mathbb{Z}_{>1}, \ \mu \in \mathbb{C}\backslash\mathbb{Z}_{<0}. \tag{77}$$

Lemma 2 shows that when the solution of Equation (70) is $u_\mu(t)$, we have m solutions of (77), which are given by

$$u_\mu(t), u_\mu(t)\log_e t, \ u_\mu(t)(\log_e t)^2, \cdots, u_\mu(t)(\log_e t)^{m-1}. \tag{78}$$

When $\mu = -n \in \mathbb{Z}_{<0}$, in place of (77), we consider

$$(t\frac{d}{dt} - \mu - \epsilon)^m u(t) = 0, \quad t > 0, \ m \in \mathbb{Z}_{>0}. \tag{79}$$

We now obtain the solutions:

$$u_{\epsilon-n}(t), u_{\epsilon-n}(t)\log_e t, \ u_{\epsilon-n}(t)(\log_e t)^2, \cdots, u_{\epsilon-n}(t)(\log_e t)^{m-1}, \tag{80}$$

where $u_{\epsilon-n}(t)$ is given by (75).

Theorem 7. *Let the condition in Theorem 1 be satisfied. Then, we have n solutions of Equation (7), which are classified into k_x series.* (i) *If $\alpha_k \notin \mathbb{Z}_{<0}$,*

$$u_{\alpha_k}(t) = \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \tag{81}$$

is a solution in the kth series, and if $m_k > 1$,

$$u_{\alpha_k}(t)\log_e t,\ u_{\alpha_k}(t)(\log_e t)^2, \cdots, u_{\alpha_k}(t)(\log_e t)^{m_k-1}, \tag{82}$$

are also solutions in the kth series. (ii) *If* $\alpha_k \in \mathbb{Z}_{<0}$,

$$u_{\alpha_k+\epsilon}(t) = \epsilon(-1)^{-\alpha_k}(-\alpha_k-1)!t^{\alpha_k} \tag{83}$$

is a solution in the kth series, and if $m_k > 1$,

$$u_{\alpha_k+\epsilon}(t)\log_e t,\ u_{\alpha_k+\epsilon}(t)(\log_e t)^2, \cdots, u_{\alpha_k+\epsilon}(t)(\log_e t)^{m_k-1}, \tag{84}$$

are also solutions in the kth series.

Proof. Every solution given in the theorem is confirmed to be a solution of Equation (13), which represents Equation (7). □

5.1. AC-Laplace Transform of Euler's Equation

In Reference [18,19], the AC-Laplace transform, which is an analytic continuation of the Laplace transform, is introduced. The AC-Laplace transform of $u_\mu(t)$ given by (71) is defined by $\hat{u}_\mu(s) = s^{-\mu-1}$ for $\mu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. When $\mu = -n \in \mathbb{Z}_{<0}$, $u_{-n}(t)$ is not defined. In Section 5, we consider $u_{\epsilon-n}(t)$ expressed by (76), in its place. Now, the AC-Laplace transform of this function is given by $\hat{u}_{\epsilon-n}(s) = s^{-\epsilon+n-1}$.

In place of (77), we have

$$(-\frac{d}{ds}s-\mu)^m\hat{u}(s) = (-s\frac{d}{ds}-1-\mu)^m\hat{u}(s) = 0, \quad m \in \mathbb{Z}_{>1},\ \mu \in \mathbb{C}\backslash\mathbb{Z}_{<0}. \tag{85}$$

Its solutions are given by

$$\hat{u}_\mu(s) = s^{-\mu-1},\ \mu(s)\log_e s,\ \hat{u}_\mu(s)(\log_e s)^2,\ \cdots,\ \hat{u}_\mu(s)(\log_e s)^{m-1}. \tag{86}$$

When $\mu = -n \in \mathbb{Z}_{<0}$, in place of (79), we consider

$$(-\frac{d}{ds}s-\mu-\epsilon)^m u(s) = (-s\frac{d}{ds}-1-\mu-\epsilon)^m u(s) = 0, \quad m \in \mathbb{Z}_{>0}. \tag{87}$$

We now obtain the solutions:

$$\hat{u}_{\epsilon-n}(s) = s^{-\epsilon+n-1},\ \hat{u}_{\epsilon-n}(s)\log_e s,\ \hat{u}_{\epsilon-n}(s)(\log_e s)^2,\ \cdots,\ \hat{u}_{\epsilon-n}(s)(\log_e s)^{m-1}. \tag{88}$$

Theorem 8. *Let the condition in Theorem 1 be satisfied. Then, the Laplace transform* $\hat{u}(s)$ *of a solution of Equation* (7) *satisfies*

$$\prod_{k=1}^{k_x}(-\frac{d}{ds}s-\alpha_k)^{m_k}\hat{u}(s) = \prod_{k=1}^{k_x}(-s\frac{d}{ds}-1-\alpha_k)^{m_k}\hat{u}(s) = 0, \tag{89}$$

and we have n solutions of Equation (89)*, which are classified into* k_x *series.* (i) *If* $\alpha_k \notin \mathbb{Z}_{<0}$,

$$\hat{u}_{\alpha_k}(s) = s^{-\alpha_k-1} \tag{90}$$

is a solution in the kth series, and if $m_k > 1$,

$$\hat{u}_{\alpha_k}(s)\log_e s,\ \hat{u}_{\alpha_k}(s)(\log_e s)^2,\ \cdots,\ \hat{u}_{\alpha_k}(s)(\log_e s)^{m_k-1} \tag{91}$$

are also solutions in the kth series. (ii) *If $\alpha_k \in \mathbb{Z}_{<0}$,*

$$\hat{u}_{\alpha_k+\epsilon}(s) = s^{-\epsilon-\alpha_k-1} \tag{92}$$

is a solution in the kth series, and if $m_k > 1$,

$$\hat{u}_{\alpha_k+\epsilon}(s)\log_e s,\ \hat{u}_{\alpha_k+\epsilon}(s)(\log_e s)^2, \cdots, \hat{u}_{\alpha_k+\epsilon}(s)(\log_e s)^{m_k-1}, \tag{93}$$

are also solutions in the kth series. We obtain n solutions of Equation (7) by the inverse Laplace transform of the n solutions of Equation (89).

Remark 13. *In Reference [22], Ghil and Kim adopt that the inverse Laplace transform of s^{n-1} for $n \in \mathbb{Z}_{>0}$ gives Ct^{-n}, which is justified by the present study, where we obtain solution Ct^{-n} and the Laplace transform s^{n-1} for $\alpha_k = -n$, where C is a constant, with the aid of nonstandard analysis.*

6. Conclusions

In recent papers [5–7], theorems are presented to give solutions of Equation (53). That differential equation in distribution theory corresponds to Euler's differential equation given by (7). In Reference [1], a theorem is given on the solution of Equation (7). In Section 2, we present a theorem which provides n solutions of Equation (7) of order n, by modifying the theorem given in Reference [1], and also two theorems which are related with theorems given in Reference [5–7]. In Section 4, we give the corresponding theorems for the solution of Equation (53) in distribution theory. It is shown that the results in Reference [5–7] are obtained as special results of the theorems given in Section 4.

Author Contributions: Author K.-i.S. showed T.M. the paper [5]. Then, T.M. wrote a theorem in distribution theory, which corresponds to the theorem on the Euler's differential equation, written in the paper [1]. Since then, both authors collaborated to complete this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors are grateful to the reviewers of this paper. Following their suggestions and advice, the authors improved the descriptions.

Conflicts of Interest: Authors have declared that no competing interests exist.

Appendix A. Solution of a Linear Differential Equation with Constant Coefficients

A study is given on the solution of a linear differential equation of order $n \in \mathbb{Z}_{>0}$ with constant coefficients:

$$p_n(D_R) := D_R^n u(t) + \sum_{l=0}^{n-1} a_l D_R^l u(t) = f(t), \quad t > 0, \tag{A1}$$

where $D_R = \frac{d}{dt}$ and $f(t)H(t)$ is locally integrable on $\mathbb{R}$.

With the aid of Lemma 11, we write the equation in distribution theory, which corresponds to Equation (A1):

$$p_n(D) := D^n \tilde{u}(t) + \sum_{l=0}^{n-1} a_l D^l \tilde{u}(t) = \tilde{f}(t) + \sum_{l=0}^{n} a_l \sum_{k=0}^{l-1} u_k D^{l-1-k}\delta(t), \tag{A2}$$

where $a_n = 1$, $\tilde{u}(t) = u(t)\tilde{H}(t)$, $\tilde{f}(t) = f(t)\tilde{H}(t)$ and $u_k = u^{(k)}(0)$. We note that the solution of this equation is a linear combination of particular solutions for each of the inhomogeneous terms $\tilde{f}(t)$ and t^k for $k \in \mathbb{Z}_{[0,n-1]}$.

In this Appendix A, we give a proof of the following theorem.

Theorem A1. *Let the Green's function $G(t)$ satisfy*

$$p_n(D)G(t)\tilde{H}(t) = \delta(t). \tag{A3}$$

Then,

$$G(t) = \frac{t^{n-1}}{(n-1)!}(1 + O(t)), \tag{A4}$$

n complementary solutions of (A1) are given by $G^{(m)}(t)$ for $m \in \mathbb{Z}_{[0,n-1]}$, and there exists no other complementary solutions of (A1).

When we have only one term $\tilde{f}(t)$ on the righthand side of Equation (A2), by applying D^{-n} to this equation, we change this equation to an integral equation:

$$\tilde{u}(t) = D^{-n}\tilde{f}(t) - [\int_0^t \sum_{l=0}^{n-1} a_l \frac{(t-t')^{n-l-1}}{(n-l-1)!} u(t')dt']\tilde{H}(t). \tag{A5}$$

where $D^{-n}\tilde{f}(t) = [\int_0^t \frac{(t-t')^{n-1}}{(n-1)!} f(t')dt']\tilde{H}(t)$. We note that this equation can be expressed by

$$\tilde{u}(t) = D^{-n}\tilde{f}(t) + [\int_0^t K(t-t')u(t')dt']\tilde{H}(t). \tag{A6}$$

Here, we put $\tilde{f}(t) = \delta(t-x)$, and then we obtain $\tilde{u}(t)$, which is a function of $t-x$. We write this function by $G(t-x)$, which satisfies

$$G(t-x)\tilde{H}(t-x) = D^{-n}\delta(t-x) + [\int_x^t K(t-t')G(t'-x)dt']\tilde{H}(t-x). \tag{A7}$$

We now put $\tilde{f}(t) = [\int_0^\infty \delta(t-x)f(x)dx]H(t)$ and $\tilde{u}(t) = [\int_0^t G(t-x)f(x)dx]\tilde{H}(t)$, and then (A6) is satisfied.

Equation (A7) shows that (A4) and (A3) are satisfied. The inhomegeneous term of Equation (A2) is a linear combination of $\tilde{f}(t)$ and $D^m\delta(t)$ for $m \in \mathbb{Z}_{[0,n-1]}$. The particular solution of (A2) for the term $D^m\delta(t)$ is given by

$$D^m G(t)\tilde{H}(t) = G^{(m)}(t)\tilde{H}(t) = \frac{t^{n-1-m}}{(n-1-m)!}(1 + O(t))\tilde{H}(t), \quad m \in \mathbb{Z}_{[0,n-1]}. \tag{A8}$$

This shows that we have n complementary solutions of (A1), which are $G^{(m)}(t)$.

We, finally, show that there exists no complementary solution of (A2), which satisfies $\tilde{u}(t) = u(t)\tilde{H}(t)$ and

$$u(t) = \int_0^t K(t-t')u(t')dt'. \tag{A9}$$

We assume that a solution of this equation takes finite values in a finite interval $(0, b)$ for $b > 0$. We put $|u| = \sup_{0<t<b} |u(t)|$ and $|K| = \sup_{0<t'<t<b} |K(t-t')|$, and then by using (A9), we have $|u(t)| \leq |K||u|t$, $|u(t)| \leq \frac{1}{2!}|K|^2|u|t^2, \cdots,$

$$|u(t)| \leq \frac{1}{k!}|K|^k|u|t^k, \quad k \in \mathbb{Z}_{>0},\ 0 < t < b. \tag{A10}$$

This tends to 0 as $k \to \infty$. This shows that there exists no complementary solution of (A2) and no complementary solution of (A1) other than $G^{(m)}(t)$ for $m \in \mathbb{Z}_{[0,n-1]}$.

References

1. Morita, T.; Sato, K. A study on the Solution of Linear Differential Equations with Polynomial Coefficients. *J. Adv. Math. Comput. Sci.* **2018**, *28*, 1–15. [CrossRef]
2. Morita, T.; Sato, K. On the Solution of Linear Differential Equations with Polynomial Coefficients near the Origin and Infinity. *J. Adv. Math. Comput. Sci.* **2018**, *29*, 1–17. [CrossRef]
3. Ince, E.L. *Ordinary Differential Equations*; Dover Publ. Inc.: New York, NY, USA, 1956.
4. Poole, E.G.C. *Introduction to the Theory of Linear Differential Equations*; Oxford U. P.: London, UK, 1936.
5. Sangsuwan, A.; Nonlaopon, K.; Orakitjaroen, S. The Generalized Solutions of a Certain nth Order Cauchy-Euler Equation. *Asian-Eur. J. Math.* **2020**, *13*, 2050047. [CrossRef]
6. Jhanthanam, S.; Nonlaopon, K.; Orakitjaroen, S. Generalized Solutions of the Third-order Cauchy-Euler Equation in the Space of Right-sided Distributions via Laplace Transform. *Mathematics* **2019**, *7*, 376. [CrossRef]
7. Sangsuwan, A.; Nonlaopon, K.; Orakitjaroen, S.; Mirumbe, I. The Generalized Solutions of the nth Order Cauchy-Euler Equation. *Mathematics* **2019**, *7*, 932. [CrossRef]
8. Nishimoto, K. *An Essence of Nishimoto's Fractional Calculus*; Descartes Press: Koriyama, Japan, 1991.
9. Nishimoto, K. Kummer's Twenty-four Functions and N-Fractional Calculus. *Nolinear Anal. Theory Methods Appl.* **1997**, *30*, 1271–1282. [CrossRef]
10. Morita, T.; Sato, K. Asymptotic Expansions of Fractional Derivatives and Their Applications. *Mathematics* **2015**, *3*, 171–189. [CrossRef]
11. Morita, T.; Sato, K. Kummer's 24 Solutions of the Hypergeometric Differential Equations with the Aid of Fractional Calculus. *Adv. Pure Math.* **2016**, *6*, 180–191. [CrossRef]
12. Sneddon, I.N. *Special Functions of Mathematical Physics and Chemistry*; Longman Inc.: New York, NY, USA, 1980.
13. Nishimoto, K. Applications to the Solutions of Linear Second Order Differential Equations of Fuchs Type. In *Fractional Calculus*; McBride, A.C., Roach, G.F., Eds.; Pitman Advanced Publishing Program: Boston, MA, USA, 1985.
14. Schwartz, L. *Théorie des Distributions*; Hermann: Paris, France, 1966.
15. Gelfand, I.M.; Silov, G.E. *Generalized Functions*; Academic Press Inc.: New York, NY, USA, 1964; Volume 1.
16. Vladimirov, V. S. *Methods of the Theory of Generalized Functions*; Taylor & Francis Inc.: New York, NY, USA, 2002.
17. Zemanian, A.H. *Distribution Theory and Transform Analysis*; Dover Publ. Inc.: New York, NY, USA, 1965.
18. Morita, T.; Sato, K. Solution of Differential Equations with the Aid of an Analytic Continuation of Laplace Transform. *Appl. Math.* **2014**, *5*, 1209–1219. [CrossRef]
19. Morita, T.; Sato, K. Solution of Differential Equations with Polynomial Coefficients with the Aid of an Analytic Continuation of Laplace Transform. *Mathematics* **2016**, *4*, 19. [CrossRef]
20. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*; Cambridge U.P.: Cambridge, UK, 1935.
21. Robert, A.M. *Nonstandard Analysis*; Dover Publ. Inc.: New York, NY, USA, 2003.
22. Ghil, B.; Kim, H. The Solution of Euler-Cauchy Equation Using Laplace Transform. *Int. J. Math. Anal.* **2015**, *9*, 2611–2618. [CrossRef]

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.