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Spherical Ruled Surfaces in S^3 Characterized by the Spherical Gauss Map

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Abstract: The Laplace operator on a Riemannian manifold plays an important role with eigenvalue problems and the spectral theory. Extending such an eigenvalue problem of smooth maps including the Gauss map, the notion of finite-type was introduced. The simplest finite-type is of 1-type. In particular, the spherical Gauss map is defined in a very natural way on spherical submanifolds. In this paper, we study ruled surfaces of the 3-dimensional sphere with generalized 1-type spherical Gauss map which generalizes the notion of 1-type. The classification theorem of ruled surfaces of the sphere with the spherical Gauss map of generalized 1-type is completed.

Keywords: Laplace operator; spherical Gauss map; pointwise 1-type; generalized 1-type

1. Introduction

From the view point of Riemannian geometry, next to Euclidean space, the sphere is the most interesting geometric object. In turn, submanifolds of sphere are also seriously considered as well. Among them, minimal submanifolds are studied in various ways of view point including the stability problem and the spectral problem of the Laplace operator. In [1], it is proved that a Riemannian manifold M immersed in the Euclidean space E^m satisfying $\Delta x = \lambda x$ is either a minimal submanifold of Euclidean space or a minimal submanifold in a hypersphere, where $x : M \rightarrow E^m$ is an isometric immersion of M into E^m and Δ is the Laplace operator of M . Minimal submanifolds of sphere have many interesting geometric characters. For example, a 3-dimensional sphere has been a long time interesting geometric model space together with the Poincaré's conjecture on three-spheres. Among surfaces immersed in a unit sphere $S^3(1)$, there exist infinitely many complete and flat surfaces in $S^3(1)$ such as the tori $S^1(r_1) \times S^1(r_2)$, the product of two plane circles, where $r_1^2 + r_2^2 = 1$. In particular, the Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is minimal and flat in $S^3(1)$ and its closed geodesics are mapped onto closed curves of finite-type in $S^3(1)$. There are many papers devoted to characterize the Clifford torus with different view points by dealing with minimal surfaces of 3-sphere [2–4].

The frame work of finite-type immersion has been introduced and developed since the 1970s in generalizing the theory of minimal submanifolds in Euclidean space [5–7]. The notion of finite-type extended to smooth maps defined on submanifolds in Euclidean space or pseudo-Euclidean space and then many of the results associated with it (in particular, the Gauss map) have been obtained [8–14]. During such studies, Kim, the one of authors, et al. found out the interesting facts that the Gauss maps of the helicoid and the right cone in 3-dimensional Euclidean space look similar to that of 1-type but they are basically different [15]. Such a Gauss map was said to be of pointwise 1-type and then surfaces and submanifolds in Euclidean space or pseudo-Euclidean space with pointwise 1-type Gauss map have been studied [16–19]. Consequently, the minimal submanifolds in Euclidean spaces and spheres were naturally treated with the notion of finite-type and the extended notions of 1-type like pointwise 1-type or generalized 1-type [20–22].

In this article, we investigate and characterize ruled surfaces of the 3-dimensional sphere with the generalized 1-type spherical Gauss map.

2. Preliminaries

Let $x : M \rightarrow S^{m-1}$ be an isometric immersion of an n -dimensional Riemannian manifold M into the unit sphere $S^{m-1} (\subset E^m)$ centered at the origin. We identify x with its position vector field. Let $(x_1, x_2, \dots, x_m)$ be a local coordinate system of M in S^{m-1} . For the components g_{ij} of the Riemannian metric $\langle \cdot, \cdot \rangle$ on M induced from that of S^{m-1} , we denote by (g^{ij}) (respectively, G) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}) . Then the Laplace operator Δ on M is defined by

$$\Delta = -\frac{1}{\sqrt{G}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{G} g^{ij} \frac{\partial}{\partial x_j}). \quad (1)$$

An immersion x of a Riemannian manifold M into S^{m-1} is said to be of finite-type or M is of finite-type if its position vector field x can be expressed as a finite sum of eigenvectors of Δ as follows

$$x = x_0 + x_1 + \dots + x_k$$

for some positive integer k , where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$. If $\lambda_1, \dots, \lambda_k$ are mutually different, the immersion x or M is said to be of k -type. Similarly, a smooth map ϕ on an n -dimensional submanifold M of S^{m-1} is said to be of finite-type if ϕ is a finite sum of E^m -valued eigenfunctions of Δ . In particular, we say that a differential map ϕ is *harmonic* if $\Delta \phi = 0$. In general, a harmonic smooth map is not necessarily of finite-type if M is not compact. (cf. P 146, [23]).

Let Π be an oriented n -plane in E^m and $e_1, \dots, e_n$ an orthonormal basis of Π . If we identify an oriented n -plane Π with a decomposable n vector $e_1 \wedge \dots \wedge e_n$ defined by the exterior product in a natural way, the Grassmann manifold $G(n, m)$ can be regarded as the set of all oriented n -planes in $E^N = \Lambda^n E^m = \{X_1 \wedge X_2 \wedge \dots \wedge X_n | X_i \in E^m\}$, where $i = 1, 2, \dots, n$ and $N = \binom{m}{n}$. We identify each tangent vector X of M in S^{m-1} with the differential $dx(X)$.

For a spherical submanifold M in S^{m-1} , the position vector x of each point p of S^{m-1} and an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space $T_p M$ determine an oriented $(n+1)$ -plane in E^m . Thus, we can have a map

$$G : M \rightarrow G(n+1, m)$$

via $G(p) = x \wedge e_1 \wedge \dots \wedge e_n$. We call G the *spherical Gauss map* of M in S^{m-1} [24]. This map can be viewed as

$$G : M \rightarrow G(n+1, m) \subset S^{\binom{m}{n+1}-1} \subset E^{\binom{m}{n+1}}$$

by considering the norm of vectors. In [22], the authors introduced notion of the pointwise 1-type spherical Gauss map of the spherical submanifold.

Definition 1 ([22]). An oriented n -dimensional submanifold M of S^{m-1} is said to have pointwise 1-type spherical Gauss map G if it satisfies the partial differential equation

$$\Delta G = f(G + C) \quad (2)$$

for a non-zero smooth function f on M and some constant vector C . In particular, if C is zero, the spherical Gauss map G is said to be pointwise 1-type of the first kind. Otherwise, it is said to be of the second kind.

In generalizing the notion of pointwise 1-type spherical Gauss map, we define the generalized 1-type spherical Gauss map of the spherical submanifold.

Definition 2. An oriented n -dimensional submanifold M of S^{m-1} is said to have generalized 1-type spherical Gauss map G if it satisfies the partial differential equation

$$\Delta G = fG + hC \quad (3)$$

for non-zero smooth functions f, h on M and some non-zero constant vector C .

Remark 1. We note that the harmonic spherical Gauss map G , i.e., $\Delta G = 0$ includes the case of the spherical Gauss map of pointwise 1-type of the second kind, for example, the case of $f = h$ and $G = -C$. Without loss of generality, we may assume that $\Delta G \neq 0$ a.e. when we consider generalized 1-type spherical Gauss map.

3. Ruled Surfaces in S^3 with Generalized 1-Type Spherical Gauss Map

Let $\alpha = \alpha(s)$ be a smooth curve in S^3 defined on an open interval I and $\beta = \beta(s)$ a vector field passing through $\alpha(s)$ with $\alpha' \beta \neq 0$. Let M be a ruled surface in the sphere S^3 ($\subset E^4$) parameterized by

$$x = x(s, t) = \cos t \alpha(s) + \sin t \beta(s), \quad s \in I, t \in J, \quad (4)$$

where J is an open interval. Without loss of generality, we may assume that

$$\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \alpha', \alpha' \rangle = 1 \quad \text{and} \quad \langle \alpha, \beta \rangle = \langle \alpha', \beta \rangle = 0, \quad (5)$$

where “ $'$ ” denotes the differentiation with respect to s . From now on, we always assume that the parametrization (4) satisfies condition (5). We put

$$A(s) = (\alpha \wedge \alpha' \wedge \beta)(s) \quad \text{and} \quad B(s) = (\alpha \wedge \beta' \wedge \beta)(s).$$

Then, the spherical Gauss map G of M is defined by

$$G = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|} = \frac{1}{\sqrt{q}} (\cos t A(s) + \sin t B(s)), \quad (6)$$

where the function $q = q(s, t)$ is defined by

$$q = \langle x_s, x_s \rangle = \cos^2 t + 2u(s) \cos t \sin t + w(s) \sin^2 t \quad (7)$$

with $u(s) = \langle \alpha'(s), \beta'(s) \rangle$ and $w(s) = \langle \beta'(s), \beta'(s) \rangle$.

Since the vector fields $\alpha(s)$, $\beta(s)$ and $\alpha'(s)$ in E^4 are mutually orthogonal along s , we can choose a unit vector field $\gamma(s)$ along s such that $\{\alpha(s), \beta(s), \alpha'(s), \gamma(s)\}$ forms an orthonormal frame in E^4 along the curve α .

We note that $\Lambda^3 E^4$ can be identified with E^4 , so that the vectors A and B in $\Lambda^3 E^4$ are regarded as vector fields in E^4 . In a natural way, we define an inner product $(\cdot, \cdot)$ of a vector XYZ in $\Lambda^3 E^4$ and a vector W in E^4 as follows:

$$(X \wedge Y \wedge Z, W) = \det \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix}, \quad (8)$$

where the matrix of the right handed side is a 4×4 -matrix composed of the components of X, Y, Z and W as row vectors. Therefore, by considering the orientation of the orthonormal frame $\{\alpha(s), \beta(s), \alpha'(s), \gamma(s)\}$ together with (8), we have

$$\begin{cases} (\alpha \wedge \beta \wedge \alpha')(s) = \gamma(s), \\ (\alpha \wedge \beta \wedge \gamma)(s) = -\alpha'(s), \\ (\alpha \wedge \alpha' \wedge \gamma)(s) = \beta(s), \\ (\beta \wedge \alpha' \wedge \gamma)(s) = -\alpha(s) \end{cases}$$

for all s . Therefore, the spherical Gauss map G is rewritten of the form

$$G = \frac{1}{\sqrt{q}} \left((\vartheta \sin t) \alpha' - (\cos t + u \sin t) \gamma \right), \quad (9)$$

where we have put

$$\vartheta(s) = \langle \beta', \gamma \rangle.$$

Then, we get

$$w = u^2 + \vartheta^2. \quad (10)$$

By the definition of (1) of the Laplace operator Δ , we get

$$\begin{aligned} \Delta G = & \left\{ -q^{-\frac{7}{2}} q_s^2 + \frac{1}{2} q^{-\frac{5}{2}} q_{ss} - \frac{1}{2} q^{-\frac{5}{2}} q_t^2 + \frac{1}{2} q^{-\frac{3}{2}} q_{tt} + q^{-\frac{1}{2}} \right\} (\cos t A + \sin t B) \\ & + \frac{3}{2} q^{-\frac{5}{2}} q_s (\cos t A' + \sin t B') - q^{-\frac{3}{2}} (\cos t A'' + \sin t B'') \\ & + \frac{1}{2} q^{-\frac{3}{2}} q_t (-\sin t A + \cos t B). \end{aligned} \quad (11)$$

Since $A = -\gamma$, in (11) the terms A', A'', B, B' and B'' can be represented in terms of the orthonormal frame $\{\alpha, \beta, \alpha', \gamma\}$ as

$$\begin{cases} A' = \vartheta \beta + \xi \alpha', \\ A'' = -\xi \alpha + (\vartheta' - u \xi) \beta + (u \vartheta + \xi') \alpha' + (\vartheta^2 + \xi^2) \gamma, \\ B = \vartheta \alpha' - u \gamma, \\ B' = -\vartheta \alpha + (\vartheta' + u \xi) \alpha' + (\vartheta \xi - u') \gamma, \\ B'' = -(2\vartheta' + u \xi) \alpha + (u' \vartheta - u \vartheta' - u^2 \xi - \vartheta^2 \xi) \beta \\ \quad + (\vartheta'' + 2u' \xi + u \xi' - \vartheta - \vartheta \xi^2) \alpha' + (2\vartheta' \xi + u \xi^2 + \vartheta \xi' - u'') \gamma, \end{cases} \quad (12)$$

where the function ξ of s is defined by

$$\xi(s) = \langle \alpha'', \gamma \rangle.$$

From (11) and (12), we see that ΔG is a vector field expressed by α, β, α' and γ along s with functions of s and t as coefficients.

Now, we will examine a ruled surface M in S^3 parameterized by (4) with generalized 1-type spherical Gauss map. That is, the spherical Gauss map G of M satisfies

$$\Delta G = fG + hC$$

for some non-zero smooth functions f, h of s and t and a non-zero constant vector C . If the interior $U_0 = \text{int}(U)$ of a set $U = \{(s, t) \in I \times J \mid h(s, t) = 0\}$ is non-empty, then the spherical Gauss map G is nothing but of pointwise 1-type of the first kind on U_0 . In [22], the authors constructed ruled

surfaces in S^3 with pointwise 1-type spherical Gauss map. Thus, we may assume that $h \neq 0$ and $f \neq h$ everywhere on M . Then, the constant vector $\mathbf{C}$ can be put as

$$\mathbf{C} = \frac{\Delta G - fG}{h} \quad (13)$$

which yields

$$h(\Delta G - fG)_t = h_t(\Delta G - fG) \quad (14)$$

by differentiating (13) with respect to t . From (11) and (12), we have put as

$$\langle \Delta G - fG, \alpha \rangle = q^{-\frac{5}{2}} \Phi \quad \text{and} \quad \langle \Delta G - fG, \beta \rangle = q^{-\frac{5}{2}} \Psi, \quad (15)$$

where the functions Φ and Ψ are given by

$$\Phi(s, t) = -\frac{3}{2}q_s \vartheta \sin t + q\zeta \cos t + q(u\zeta + 2\vartheta') \sin t \quad (16)$$

and

$$\Psi(s, t) = \frac{3}{2}q_s \vartheta \cos t + q(u\zeta - \vartheta') \cos t - q(u'\vartheta - u\vartheta' - u^2\zeta - \vartheta^2\zeta) \sin t. \quad (17)$$

Lemma 1. Let M be a ruled surface immersed in the unit sphere S^3 with the parametrization (4). If M has generalized 1-type spherical Gauss map G , that is,

$$\Delta G = fG + h\mathbf{C}$$

for some non-zero function f , non-vanishing function h and non-zero constant vector $\mathbf{C}$, then we may assume that the function h_t , the derivative of h with respect to t , is non-vanishing on M , i.e., $h_t \neq 0$ everywhere on M .

Proof. We consider a set $V = \{(s, t) \in I \times J \mid h_t = 0\}$ and let $V_o = \text{int}(V)$, the interior of V . Now, we assume that the set V_o is non-empty. Then, (14) implies

$$(\Delta G - fG)_t = \mathbf{0}$$

and hence $(q^{-\frac{5}{2}}\Phi)_t = 0$ and $(q^{-\frac{5}{2}}\Psi)_t = 0$ on V_o as the coefficients of the vector α and β of $(\Delta G - fG)_t$, respectively, where $\mathbf{0}$ denotes zero vector. Thus, we have

$$-\frac{5}{2}q_t\Phi + q\Phi_t = 0 \quad (18)$$

and

$$-\frac{5}{2}q_t\Psi + q\Psi_t = 0 \quad (19)$$

which are polynomials in $\cos^{5-k}t \sin^k t$, $k = 0, 1, \dots, 5$, with functions of s as coefficients. Now, we note that the trigonometric functions $\cos^5 t$, $\cos^4 t \sin t$, $\dots$, $\cos t \sin^4 t$, $\sin^5 t$ are linearly independent due to the Wronskian of these functions given by

$$W(\cos^3 t, \cos^2 t \sin t, \cos t \sin^2 t, \sin^3 t) = 12$$

for all t . Thus, a straightforward computation with (7) and (16) allows us to get

$$2(u\zeta - \vartheta') = 0 \quad (20)$$

and

$$6u\vartheta + 8u^2\zeta + 3\vartheta^2\zeta - 2u\vartheta' - 2\zeta = 0 \quad (21)$$

as the coefficients of the terms containing ' $\cos^5 t$ ' and ' $\cos^4 t \sin t$ ' of (18), respectively. With the help of (17) and (20), the coefficients of the terms containing ' $\cos^5 t$ ' of (19) give

$$2u'\vartheta + 2u^2\zeta + \vartheta^2\zeta = 0. \quad (22)$$

Combining (20)–(22), we see that

$$\zeta = 0 \quad \text{and hence} \quad \vartheta' = 0 \quad \text{on} \quad V_o.$$

Furthermore, putting $\zeta = 0$ into (22) yields

$$u'\vartheta = 0. \quad (23)$$

Suppose $\vartheta \neq 0$ on V_o . Then, $u' = 0$ and $w' = 0$ on V_o because of (10). That is, $q_s = 0$ on V_o and therefore the functions $\Phi(s, t)$ of (16) and $\Psi(s, t)$ of (17) are vanishing on V_o . From (15), $\Delta G - fG$ becomes

$$\Delta G - fG = \left(q^{-\frac{5}{2}}\Lambda_1(s, t)\right)\alpha' + \left(q^{-\frac{5}{2}}\Lambda_2(s, t)\right)\gamma, \quad (24)$$

where Λ_1 and Λ_2 are the functions of s and t given by

$$\Lambda_1 = \left\{ -\frac{1}{2}(q_t)^2 + \frac{1}{2}qq_{tt} + (1-f)q^2 \right\} \vartheta \sin t - \vartheta q(u \cos t - \sin t) + \frac{1}{2}qq_t \vartheta \cos t$$

and

$$\Lambda_2 = \left\{ \frac{1}{2}(q_t)^2 - \frac{1}{2}qq_{tt} - (1-f)q^2 \right\} (\cos t + u \sin t) - \vartheta^2 q \cos t + \frac{1}{2}qq_t(\sin t - u \cos t),$$

respectively. By direct computation, the functions Λ_1 and Λ_2 are simplified as

$$\Lambda_1(s, t) = (-fq^2 + 2\vartheta^2)\vartheta \sin t \quad (25)$$

and

$$\Lambda_2(s, t) = (fq^2 - 2\vartheta^2)(\cos t + u \sin t). \quad (26)$$

Since $(\Delta G - fG)_t = 0$ on V_o , (24) yields

$$\left(q^{-\frac{5}{2}}\Lambda_1\right)_t = 0 = \left(q^{-\frac{5}{2}}\Lambda_2\right)_t$$

which implies

$$q^{-\frac{5}{2}}\Lambda_1 = l_1(s) \quad \text{and} \quad q^{-\frac{5}{2}}\Lambda_2 = l_2(s) \quad (27)$$

for some functions l_1 and l_2 of a single variable s . By putting (25) and (26) into (27), the formulas for f are given by

$$f = \frac{2\vartheta^2}{q^2} - \frac{q^{\frac{1}{2}}l_1(s)}{\vartheta \sin t} = \frac{2\vartheta^2}{q^2} - \frac{q^{\frac{1}{2}}l_2(s)}{(\cos t + u \sin t)},$$

respectively, and therefore we see that

$$\frac{q^{\frac{1}{2}}l_1(s)}{\vartheta \sin t} = \frac{q^{\frac{1}{2}}l_2(s)}{(\cos t + u \sin t)},$$

or, equivalently,

$$l_1(s)(\cos t + u \sin t) = l_2(s)\vartheta \sin t.$$

By the linear independence of the trigonometric functions, it is obvious that

$$l_1 = 0 \quad \text{and} \quad l_2 = 0$$

on V_0 , which indicate that $\Delta G - fG = 0$ on V_0 because of (24) and (27). It contradicts $h \neq 0$ and thus

$$\vartheta = 0 \quad \text{on } V_0.$$

Together with (10), the function q of (7) and the spherical Gauss map G of (9) are then given by

$$q = (\cos t + u \sin t)^2 \quad \text{and} \quad G = -\gamma$$

on V_0 . Since $\xi = 0$ and $\vartheta = 0$, we can see easily that the spherical Gauss map G is constant on V_0 . Thus, the spherical Gauss map is harmonic on V and hence V is the empty set. Therefore, we may assume that $h_t \neq 0$ everywhere on M . $\square$

With the help of (15) and Lemma 1, the coefficient functions of the vectors α and β of (14) are automatically given by

$$h(q^{-\frac{5}{2}}\Phi)_t = h_t(q^{-\frac{5}{2}}\Phi) \quad \text{and} \quad h(q^{-\frac{5}{2}}\Psi)_t = h_t(q^{-\frac{5}{2}}\Psi), \quad (28)$$

respectively, or, equivalently,

$$h\left(-\frac{5}{2}q_t\Phi + q\Phi_t\right) = qh_t\Phi \quad \text{and} \quad h\left(-\frac{5}{2}q_t\Psi + q\Psi_t\right) = qh_t\Psi$$

because of $q \neq 0$.

Lemma 2. Let M be a ruled surface in the unit sphere S^3 parameterized by (4) with generalized 1-type spherical Gauss map. Then we have

$$\alpha' \wedge \beta' = 0 \quad \text{on } M.$$

Proof. We will prove this lemma in the following steps.

Step 1. Let $W = \{(s, t) \in I \times J \mid \vartheta(s) \neq 0\}$. Then, we will show that if $W \neq \emptyset$, both $\Phi(s, t)$ and $\Psi(s, t)$ are non-vanishing on W .

First, we suppose that at least one of two equations $\Phi(s, t)$ and $\Psi(s, t)$ are vanishing on some subset of W , say $\Phi(s, t) = 0$. If $W_0 = \text{int}\{(s, t) \in W \mid \Phi(s, t) = 0\}$ is non-empty, (16) gives

$$\begin{aligned} & \xi \cos^3 t \\ & + (3u\xi + 2\vartheta') \cos^2 t \sin t \\ & - (3u'\vartheta - 3u^2\xi - \vartheta^2\xi - 4u\vartheta') \cos t \sin^2 t \\ & - (3uu'\vartheta + \vartheta^2\vartheta' - u^3\xi - 2u^2\vartheta' - u\vartheta^2\xi) \sin^3 t = 0 \end{aligned} \quad (29)$$

which implies

$$\begin{cases} \xi = 0, \\ \vartheta' = 0, \\ 3u'\vartheta = 0 \end{cases}$$

on W_o by the linear independence of the trigonometric functions of (29). Since $\vartheta \neq 0$ on W , the equations above imply

$$u' = 0, \quad (30)$$

i.e., $q_s = 0$ because of (10). Thus, $\Psi(s, t)$ of (17) must be identically zero on W_o . Similarly, we can show that if $\Psi(s, t) = 0$ on some set of W , so is $\Phi(s, t) = 0$. Therefore, we see that two sets $\{(s, t) \in W | \Phi(s, t) = 0\}$ and $\{(s, t) \in W | \Psi(s, t) = 0\}$ are the same.

Now, we suppose that $\{(s, t) \in W | \Phi(s, t) = 0\}$ is non-empty. If $W_o \neq \emptyset$, both Φ and Ψ are vanishing on W_o and then $\Delta G - fG$ on W_o is given by (24), that is,

$$\Delta G - fG = \left(q^{-\frac{5}{2}}\Lambda_1(s, t)\right)\alpha' + \left(q^{-\frac{5}{2}}\Lambda_2(s, t)\right)\gamma.$$

Equation $h(\Delta G - fG)_t = h_t(\Delta G - fG)$ implies

$$h(q^{-\frac{5}{2}}\Lambda_1)_t = h_t(q^{-\frac{5}{2}}\Lambda_1) \quad \text{and} \quad h(q^{-\frac{5}{2}}\Lambda_2)_t = h_t(q^{-\frac{5}{2}}\Lambda_2).$$

Then, we have

$$\frac{h_t}{h} = \frac{(q^{-\frac{5}{2}}\Lambda_1)_t}{(q^{-\frac{5}{2}}\Lambda_1)} = \frac{(q^{-\frac{5}{2}}\Lambda_2)_t}{(q^{-\frac{5}{2}}\Lambda_2)}$$

which yields that

$$\Lambda_1(s, t) = z(s)\Lambda_2(s, t) \quad (31)$$

by taking the integrand with respect to t , where z is some function of s . By substituting (25) and (26) into (31), the function f becomes of the form

$$f = \frac{2\vartheta^2}{q^2}$$

and hence $\Lambda_1(s, t)$ of (25) and $\Lambda_2(s, t)$ of (26) have to be identically zero on W_o . It is obvious that

$$\Delta G - fG = 0 \quad \text{on} \quad W_o,$$

or,

$$h = 0 \quad \text{on} \quad W_o,$$

a contradiction. Therefore, we conclude that W_o is empty and hence we may assume that the set $\{(s, t) \in W | \Phi(s, t) = 0\}$ is empty, which means that both $\Phi(s, t)$ and $\Psi(s, t)$ are non-vanishing functions on W .

Step 2. We claim $W = \emptyset$ and hence $\beta' = u\alpha'$ on M .

Suppose $W \neq \emptyset$. Since the functions $\Phi(s, t)$ and $\Psi(s, t)$ are non-vanishing on W , we have

$$\frac{h_t}{h} = \frac{(q^{-\frac{5}{2}}\Phi)_t}{(q^{-\frac{5}{2}}\Phi)} = \frac{(q^{-\frac{5}{2}}\Psi)_t}{(q^{-\frac{5}{2}}\Psi)}$$

from (28). It produces

$$h = z_1(s)\left(q^{-\frac{5}{2}}\Phi\right) = z_2(s)\left(q^{-\frac{5}{2}}\Psi\right) \quad (32)$$

for some non-vanishing functions z_1 and z_2 of s . Together with (16) and (17), (32) gives

$$\begin{aligned} & z_1(s)\left(-\frac{3}{2}\vartheta q_s \sin t + \xi q \cos t + (u\xi + 2\vartheta')q \sin t\right) \\ &= z_2(s)\left(\frac{3}{2}\vartheta q_s \cos t + (u\xi - \vartheta')q \cos t - (u'\vartheta - u\vartheta' - u^2\xi - \vartheta^2\xi)q \sin t\right) \end{aligned}$$

which implies

$$z_1 \xi = z_2 (u \xi - \vartheta'), \quad (33)$$

$$z_1 (3u \xi + 2\vartheta') = z_2 (2u' \vartheta - u \vartheta' + 3u^2 \xi + \vartheta^2 \xi), \quad (34)$$

$$z_1 (-3u' \vartheta + 3u^2 \xi + \vartheta^2 \xi + 4u \vartheta') = z_2 (uu' \vartheta + 2\vartheta^2 \vartheta' + u^2 \vartheta' + 3u^3 \xi + 3u \vartheta^2 \xi) \quad (35)$$

and

$$\begin{aligned} & z_1 (-3uu' \vartheta - \vartheta^2 \vartheta' + u^3 \xi + 2u^2 \vartheta' + u \vartheta^2 \xi) \\ &= z_2 (-u^2 u' \vartheta + u^3 \vartheta' + u^4 \xi + 2u^2 \vartheta^2 \xi - u' \vartheta^3 + u \vartheta^2 \vartheta' + \vartheta^4 \xi) \end{aligned} \quad (36)$$

as the coefficients of the terms containing ' $\cos^3 t$ ', ' $\cos^2 t \sin t$ ', ' $\cos t \sin^2 t$ ' and ' $\sin^3 t$ ', respectively. Putting (33) into (34), we get

$$z_1 \vartheta' = z_2 (u' \vartheta + u \vartheta' + \frac{1}{2} \vartheta^2 \xi). \quad (37)$$

Substituting (33) and (37) into (35), we obtain

$$z_1 u' \vartheta = z_2 (uu' \vartheta - \vartheta^2 \vartheta'). \quad (38)$$

Finally, if we put all of (33), (37) and (38) into (36), we have

$$\frac{3}{2} \vartheta^4 \xi z_2 = 0$$

which indicates

$$\xi = 0$$

because of $\vartheta \neq 0$ and $z_2 \neq 0$ on W . From (33), we see that $\vartheta' = 0$, i.e., ϑ is a non-zero constant on W . Equation (34) implies $u' = 0$. Consequently, we get

$$q_s = 0 \quad \text{on } W.$$

Thus, the functions $\Phi(s, t)$ of (16) and $\Psi(s, t)$ of (17) are identically zero on W , it is a contradiction. Consequently, we conclude that $W = \emptyset$, which means that the function ϑ is vanishing on M . $\square$

By Lemma 2, the function q and the spherical Gauss map G on M are given by

$$q = (\cos t + u \sin t)^2 \quad \text{and} \quad G = -\gamma,$$

respectively. By a straightforward computation, $\Delta G - fG$ gets the form

$$\begin{aligned} \Delta G - fG &= -\frac{u' \sin t}{(\cos t + u \sin t)^3} \gamma' + \frac{1}{(\cos t + u \sin t)^2} \gamma'' + f\gamma \\ &= \frac{\xi}{(\cos t + u \sin t)^2} \alpha + \frac{u\xi}{(\cos t + u \sin t)^2} \beta \\ &\quad + \left(\frac{u' \xi \sin t}{(\cos t + u \sin t)^3} - \frac{\xi'}{(\cos t + u \sin t)^2} \right) \alpha' \\ &\quad + \left(f - \frac{\xi^2}{(\cos t + u \sin t)^2} \right) \gamma \end{aligned} \quad (39)$$

which implies

$$\Phi(s, t) = \xi (\cos t + u \sin t)^3 \quad \text{and} \quad \Psi(s, t) = u\xi (\cos t + u \sin t)^3.$$

From (39) we may assume that ξ is non-vanishing on M . Then, by following the arguments to get (32), the function h of (32) can be put as

$$h(s, t) = \frac{u(s)\xi(s)z_2(s)}{(\cos t + u(s)\sin t)^2}, \quad (40)$$

where z_2 is the non-vanishing function of s mentioned in the proof of Lemma 2. With the help of (39) and (40), the constant vector $\mathbf{C}$ is given by

$$\begin{aligned} \mathbf{C} = & \frac{1}{uz_2}\alpha + \frac{1}{z_2}\beta + \left(\frac{u'\sin t}{uz_2(\cos t + u\sin t)} - \frac{\xi'}{u\xi z_2} \right) \alpha' \\ & + \left(\frac{f(\cos t + u\sin t)^2}{u\xi z_2} - \frac{\xi}{uz_2} \right) \gamma, \end{aligned} \quad (41)$$

from which,

$$0 = \left(\frac{u'\sin t}{uz_2(\cos t + u\sin t)} \right)_t \alpha' + \left(\frac{f(\cos t + u\sin t)^2}{u\xi z_2} \right)_t \gamma,$$

or, equivalently,

$$\left(\frac{u'\sin t}{uz_2(\cos t + u\sin t)} \right)_t = 0 = \left(\frac{f(\cos t + u\sin t)^2}{u\xi z_2} \right)_t$$

by the orthogonality of the vector fields α' and γ along s . Thus we can put

$$\frac{u'\sin t}{uz_2(\cos t + u\sin t)} = y_1(s) \quad \text{and} \quad \frac{f(\cos t + u\sin t)^2}{u\xi z_2} = y_2(s) \quad (42)$$

for some functions y_1 and y_2 of s . The linear independence of ' $\cos t$ ' and ' $\sin t$ ' of the first equation of (42) enables us to get

$$uy_1 = 0 \quad \text{and} \quad u' = 0$$

because of $z_2 \neq 0$. If $u = 0$, the function h of (40) becomes identically zero on M , a contradiction. Therefore, u is a non-zero constant on M .

Since the function f in the second equation of (42) is given by

$$f = \frac{u\xi z_2 y_2}{(\cos t + u\sin t)^2}, \quad (43)$$

the constant vector $\mathbf{C}$ of (41) is simplified as

$$\mathbf{C} = \frac{1}{uz_2}\alpha + \frac{1}{z_2}\beta - \frac{\xi'}{u\xi z_2}\alpha' + \left(y_2 - \frac{\xi}{uz_2} \right) \gamma$$

which implies that

$$\begin{aligned} 0 = & \left\{ \left(\frac{1}{z_2} \right)' + \frac{\xi'}{\xi z_2} \right\} \left(\frac{1}{u}\alpha + \beta \right) + \left\{ \frac{1+u^2+\xi^2}{uz_2} - \xi y_2 - \left(\frac{\xi'}{u\xi z_2} \right)' \right\} \alpha' \\ & + \left\{ -\frac{\xi'}{uz_2} + y_2' - \left(\frac{\xi}{uz_2} \right)' \right\} \gamma \end{aligned}$$

by differentiating with respect to s . Therefore, we have

$$\left(\frac{1}{z_2} \right)' + \frac{\xi'}{\xi z_2} = 0, \quad (44)$$

$$\frac{1+u^2+\xi^2}{uz_2} - \xi y_2 - \left(\frac{\xi'}{u\xi z_2} \right)' = 0, \quad (45)$$

$$-\frac{\xi'}{uz_2} + y_2' - \left(\frac{\xi}{uz_2} \right)' = 0 \quad (46)$$

because the vector fields α, β, α' and γ are orthogonal each other along s . Equation (44) yields

$$\frac{z_2}{z_2'} = \frac{\xi}{\xi'}$$

which gives us

$$z_2(s) = a\xi(s) \quad (47)$$

for some non-zero constant $a \in \mathbb{R}$. By using (47), (46) is reduced to

$$y_2' = \frac{1}{au} \left(\frac{\xi'}{\xi} \right) \quad (48)$$

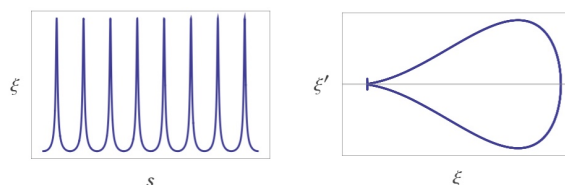
and thus we have

$$y_2 = \frac{1}{au} \ln |b\xi| \quad (49)$$

for some positive constant $b \in \mathbb{R}$. Putting (49) into (45), we can get a non-linear ordinary differential equation for $\xi(s)$ such as

$$\xi^4 \ln |b\xi| = (1+u^2)\xi^2 + \xi^4 + 2(\xi')^2 - \xi\xi''. \quad (50)$$

Among solutions of (50), under the initial conditions of $\xi(0) = 1$ and $\xi'(0) = 0$ with $u = b = 1$, a solution of (50) is given by



For the parametrization for M , we naturally obtain the function $\xi(s) = \langle \alpha'', \gamma \rangle$ which is a solution of (50). Then, the functions f of (43) and h of (40) are then given by

$$f = \frac{\xi^2 \ln |b\xi|}{(\cos t + u \sin t)^2}$$

and

$$h = \frac{au\xi^2}{(\cos t + u \sin t)^2},$$

respectively. In particular, if $f = h$, the spherical Gauss map G is of pointwise 1-type of the second kind. In this case, the parametrization for M is characterized in [22].

Remark 2. If $f = h$, the function ξ is a constant function. Then, the constant vector C is given by

$$C = \frac{1}{au\xi} (\alpha + u\beta + \xi(au - 1)\gamma).$$

We note that the function f and the constant vector C in this case are identical with those in the form $\Delta G = f(G + C)$ which were presented in [22].

In [22], we investigated ruled surfaces in S^3 with pointwise 1-type spherical Gauss map and completed the classification of them. Together with the result of [22], we have

Theorem 1. *Let M be a ruled surface in the unit sphere S^3 with generalized 1-type spherical Gauss map. Then, M is part of the ruled surface parameterized by*

$$M : x(s, t) = \cos t \alpha(s) + \sin t \beta(s) = (\cos t + u \sin t) \alpha(s) + \sin t \mathbf{N} \quad (51)$$

satisfying the function $\xi(s) = \langle \alpha'', \gamma \rangle$ is a solution of the non-linear differentiable equation

$$\xi^4 \ln |b\xi| = (1 + u^2)\xi^2 + \xi^4 + 2(\xi')^2 - \xi\xi''.$$

Proof. Let M be a ruled surface in S^3 parameterized by (4). We suppose that M has generalized 1-type spherical Gauss map. Then, Lemma 2 implies that

$$\beta(s) = u\alpha(s) + \mathbf{N}$$

for some non-zero constant u and some constant vector field $\mathbf{N}$ along s satisfying

$$\langle \alpha, \mathbf{N} \rangle = -u \quad \text{and} \quad \langle \mathbf{N}, \mathbf{N} \rangle = 1 + u^2.$$

Using the orthonormal frame $\{\alpha, \beta, \alpha', \gamma\}$ obtained from the parametrization of M , it is obvious that the function $\xi(s) = \langle \alpha'', \gamma \rangle$ satisfies (50). Therefore, we see that a ruled surface M in S^3 can be given by (51).

Conversely, we consider a ruled surface M in S^3 parameterized by (51). By computation, it follows that

$$\Delta G = \frac{\xi^2 \ln |c\xi|}{(\cos t + u \sin t)^2} G + \frac{\xi^2}{(\cos t + u \sin t)^2} \left(\frac{1}{\xi} \alpha + \frac{u}{\xi} \beta - \frac{\xi'}{\xi^2} \alpha' + (\ln |c\xi| - 1) \gamma \right)$$

for some positive constant c . If we put

$$\mathbf{D} = \left(\frac{1}{\xi} \alpha + \frac{u}{\xi} \beta - \frac{\xi'}{\xi^2} \alpha' + (\ln |b\xi| - 1) \gamma \right),$$

Equation (50) yields that the vector field $\mathbf{D}$ along s is constant, which means that a ruled surface M has generalized 1-type spherical Gauss map G . Thus, this proof is complete. $\square$

By Remark 2, we easily see

Corollary 1. *Let M be a ruled surface in the unit sphere S^3 parameterized by (51). If the function $\xi(s) = \langle \alpha'', \gamma \rangle$ is constant of the form*

$$\xi = \frac{1}{b} e^{au}$$

for some non-zero constant a and positive constant b , then M has pointwise 1-type spherical Gauss map of the second kind.

4. Conclusions

In this paper, we construct ruled surfaces in the 3-dimensional sphere S^3 with spherical Gauss map satisfying some second-order partial differential equation, so called the generalized 1-type spherical Gauss map. The results are more generalized ones including the case of ruled surfaces in S^3 with pointwise 1-type spherical Gauss map and they serve as criteria for classifying and characterizing ruled surfaces in S^3 .

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